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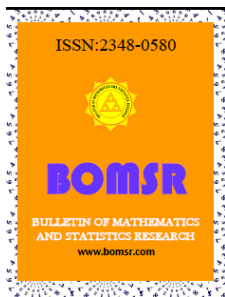
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CELLULAR FOLDING OF SOME NEW CW-COMPLEXES

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ABSTRACT

In this paper we obtained the conditions satisfied by a cellular folding of a given CW-complex to be able to cellular fold some new CW-complexes generated by some known operations like quotient, suspension of a regular CW-complex, Cartesian product, join product, and wedge sum of two CW-complexes.

Keywords: Cellular folding, quotient, suspension, join product, and wedge sum

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1. INTRODUCTION

A cellular folding is a folding defined on regular CW-complexes first defined by, E. El-Kholy and H. Al-Khurasani, [1] and various properties of this type of folding are also studied by them.

Let K and L be complexes, a continuous map $f : K \rightarrow L$ is called cellular if $f(K^n) \subset L^n$ for $n = 0, 1, 2, \dots, n$, where K^n and L^n denote the n -skeletons of K and L respectively.

Now, let K and L be regular CW-complexes of the same dimension n , a cellular map $f : K \rightarrow L$ is a cellular folding if and only if f satisfies the following:

- (i) For each i -cells $e^i \in K$, $f(e^i)$ is an i -cell in L , i.e., f maps i -cells to i -cells;
- (ii) If \bar{e} contains n vertices, then $\overline{f(e)}$ must contains n distinct vertices, [1]. The set of regular CW-complexes together with cellular folding form a category denoted by $C(K, L)$. If $f \in C(K, L)$, then $x \in K$ is said to be a singularity of f iff f is not a local homeomorphism at x . The set of all singularities of f is denoted by $\sum f$. This set corresponds to the "folds" of the map. It is noticed that for a cellular folding f , the set $\sum f$ of singularities of f is a proper subset of the union of cells of dimension $\leq n - 1$. Thus, when we consider any $f \in C(K, L)$, where K and L are connected regular CW-complexes of dimension 2, the set $\sum f$ will consist of 0-cells, and 1-cells, each of 0-cells (vertices) has

even valences [2]. Of course $\sum f$ need not be connected. Thus in this case $\sum f$ has the structure of a locally finite graph Γ_f embedded in K , for which every vertex has an even valency.

From now on by a complex we mean regular CW-complexes.

2. Cellular folding of the Cartesian product of complexes:

If X and Y are cell complexes, then $X \times Y$ has the structure of a cell complex with cell products $e_\alpha^m \times e_\beta^n$ where e_α^m ranges over the cells of X and e_β^n ranges over the cells of Y [3].

2.1. Theorem

Let K and L be complexes of the same dimension n , X and Y be complexes of the same dimension m . Let $f : K \rightarrow X$ and $g : L \rightarrow Y$ be cellular maps. Then $f \times g \in C(K \times L, X \times Y)$ if and only if f and g are cellular foldings.

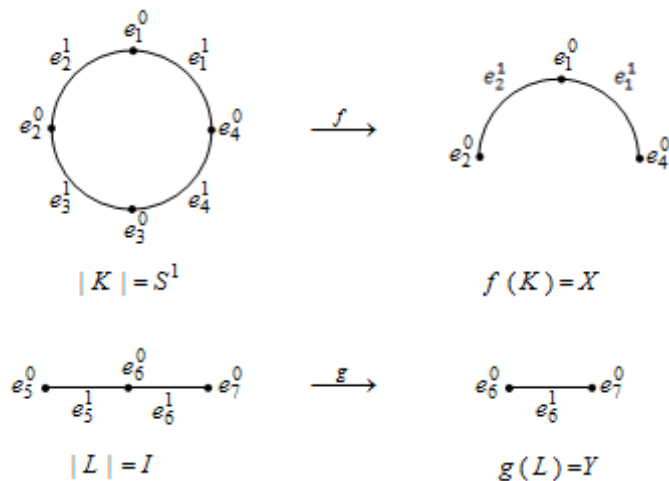
Proof:

If f and g are cellular foldings, then each will map cells to cells of the same dimension hence do $f \times g$. Also \bar{e} and $\overline{f \times g(e)}$ contains the same number of vertices because each of f and g are cellular foldings.

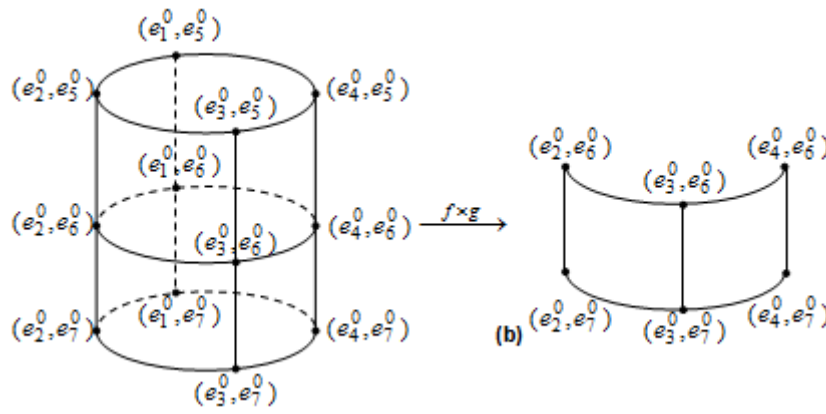
Suppose now $f \times g$ is a cellular folding, then $f \times g$ maps p -cells to p -cells, i.e., if (e, e') is a p -cell in $K \times L$, then $(f \times g)(e, e') = (f(e), g(e'))$ is a p -cell in $X \times Y$. Let e be an i -cell in K and e' be a $(p-i)$ -cell in L . The cellular map must map i -cells to j -cells such that $j \leq i$. If $j = i$ nothing to prove, so let $i > j$. In this case g will map $(p-i)$ -cells to $(p-j)$ -cells and hence is not a cellular map. This is a contradiction and hence $i = j$ is the only possibility. The second condition of cellular folding certainly satisfied in this case.

2.2. Example:

Suppose that K and L are complexes such that $|K| = S^1, |L| = I$ with cell decomposition shown in Fig. (1-a). Let $f : K \rightarrow K$ be a cellular folding defined by $f(e_3^0) = e_1^0, f(e_3^1, e_4^1) = (e_2^1, e_1^1)$, where the omitted cells through the paper are mapped into themselves. The image of f is a complex consisting of three vertices and two 1-cells. Let $g : L \rightarrow L$ be a cellular folding defined by $(e_5^0) = e_7^0, g(e_5^1) = e_6^1$. Then $f \times g : K \times L \rightarrow K \times L$ is a cellular folding. The cell decomposition of $K \times L$ and $(f \times g)(K \times L)$ are shown in Fig.(1-b).



(a)



(b)

Fig. (1)

3. Cellular folding of the quotient of a complex:

If (X, A) is a pair consisting of a cell complex X and a subcomplex A , then the quotient space X/A inherits a natural cell complex structure from X . The cells of X/A are the cells of $X - A$ plus one new 0-cell, the image of A in X/A , [3].

3.1. Example:

If $X = D^2 = \{(x, y) \in E^2 : x^2 + y^2 \leq 1\}$ is a disc with the cell structure consisting of two 0-cells, two 1-cells and one 2-cell, and let $A = S^1 = \partial D^2$. Then D^2/A is a sphere S^2 with one 2-cell and one 0-cell, see Fig. (2).

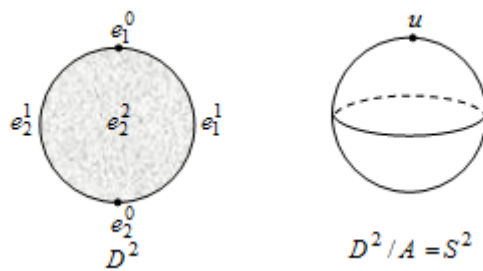


Fig. (2)

Generally, if we give S^{n-1} any cell structure and build D^n from S^{n-1} by attaching an n -cell, then the quotient D^n/S^{n-1} is S^n with its usual cell structure [3].

3.2. Theorem

Let X be a complex, $A \subset X$ a subcomplex, $f : X \rightarrow X$ a cellular map. Let $g : X/A \rightarrow X/A$ be defined by, for each i -cell e in $X - A$, $g(e) = f(e)$, $g(e^0) = e^0$, where e^0 is the new 0-cell of X/A . Then g is a cellular folding if and only if both f and $f|A$ is a cellular folding. In this case $g(X/A) = f(X)/f(A)$.

Proof:

Let $f : X \rightarrow X$ be a cellular folding, e an i -cell in X/A such that \bar{e} has n distinct vertices $g(e) = f(e)$ is an i -cell such that $\overline{g(e)} = \overline{f(e)}$ has n distinct vertices, $g(e^0) = e^0$. Thus $g : X/A \rightarrow X/A$ is a cellular folding.

Now suppose $g : X / A \rightarrow X / A$ is a cellular folding and $f : X \rightarrow X$ is a cellular map, if e is an i -cell in $X - A$, then $f(e) = g(e)$ is an i -cell in X , but if e is an i -cell in A , then $f(e)$ might be a j -cell in X , $j < i$ while if $f|_A : A \rightarrow A$ is a cellular folding, then for any i -cell in X , $f(e)$ is an i -cell in X and consequently f is a cellular folding.

3.3. Example

Let $X = D^2$ be a disc with cellular subdivision consisting of two 0-cells, three 1-cells and two 2-cells, and let $A = S^1 = \partial D^2$, $f : D^2 \rightarrow D^2$ be a cellular folding defined as follows : $f(e_1^0, e_2^0) = (e_1^0, e_2^0)$, $f(e_2^1) = (e_1^1)$, $f(e_2^2) = f(e_1^2)$.

The map $f|_A$ is the cellular folding shown in Fig.(3). Now $g : X / A \rightarrow X / A$ is a cellular folding defined by, $g(e^0) = e^0$, $g(e_3^1) = e_1^1$, $g(e_2^2) = e_1^2$, see Fig. (3).

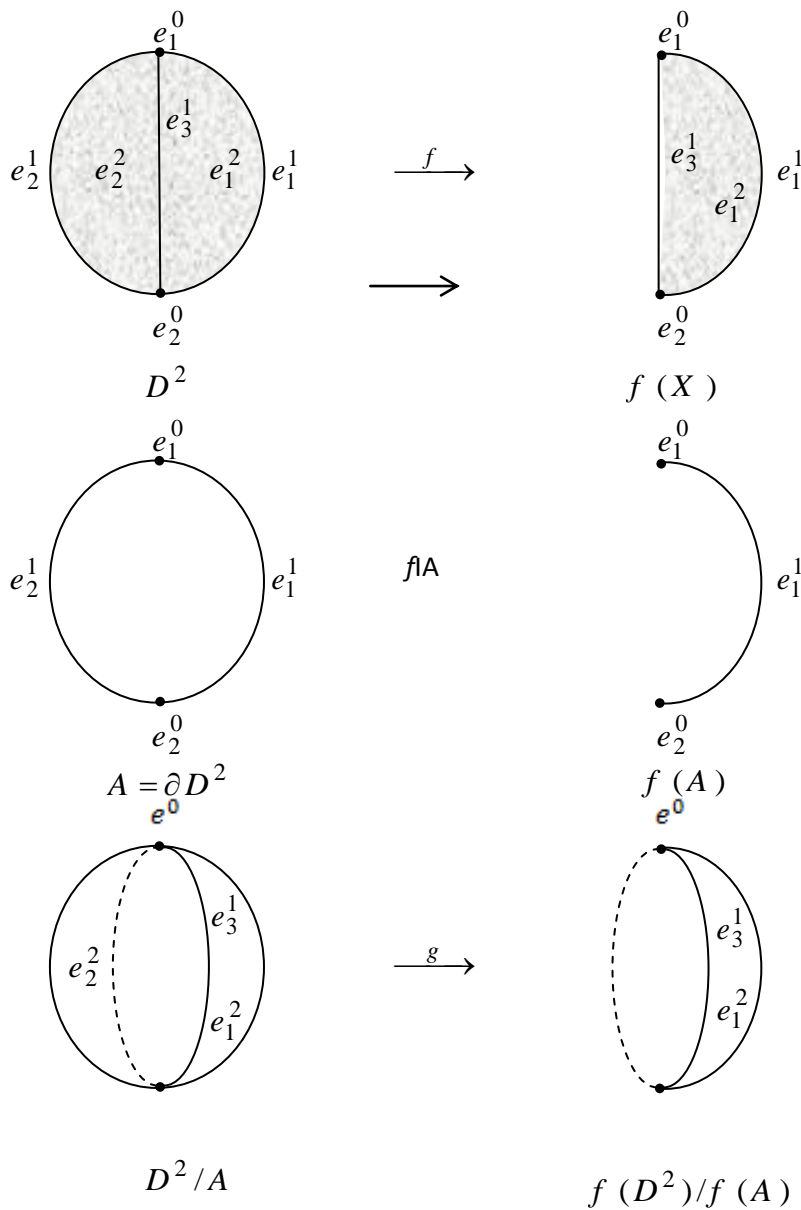


Fig. (3)

3.4. Example

Let X be a complex, such that $|X| = T$ is a torus, with cellular subdivision consisting of two 0-cells, four 1-cells, and two 2-cells, $A \subset X$ be the subcomplex shown in Fig. 4. Let $f : X \rightarrow X$ be a cellular folding defined as follows: $f(e_i^0) = e_i^0$, $i = 1, 2$; $f(e_1^1) = (e_2^1)$, $f(e_2^1) = (e_1^1)$, $f(e_1^2) = (e_2^2)$. The map $f|_A$ is the cellular folding shown in Fig. (3).

Now $g : X/A \rightarrow X/A$ is a cellular folding defined by, $g(e^0) = e^0$, $g(e_3^1) = e_3^1$, $g(e_1^2) = e_2^2$, see Fig.(4).

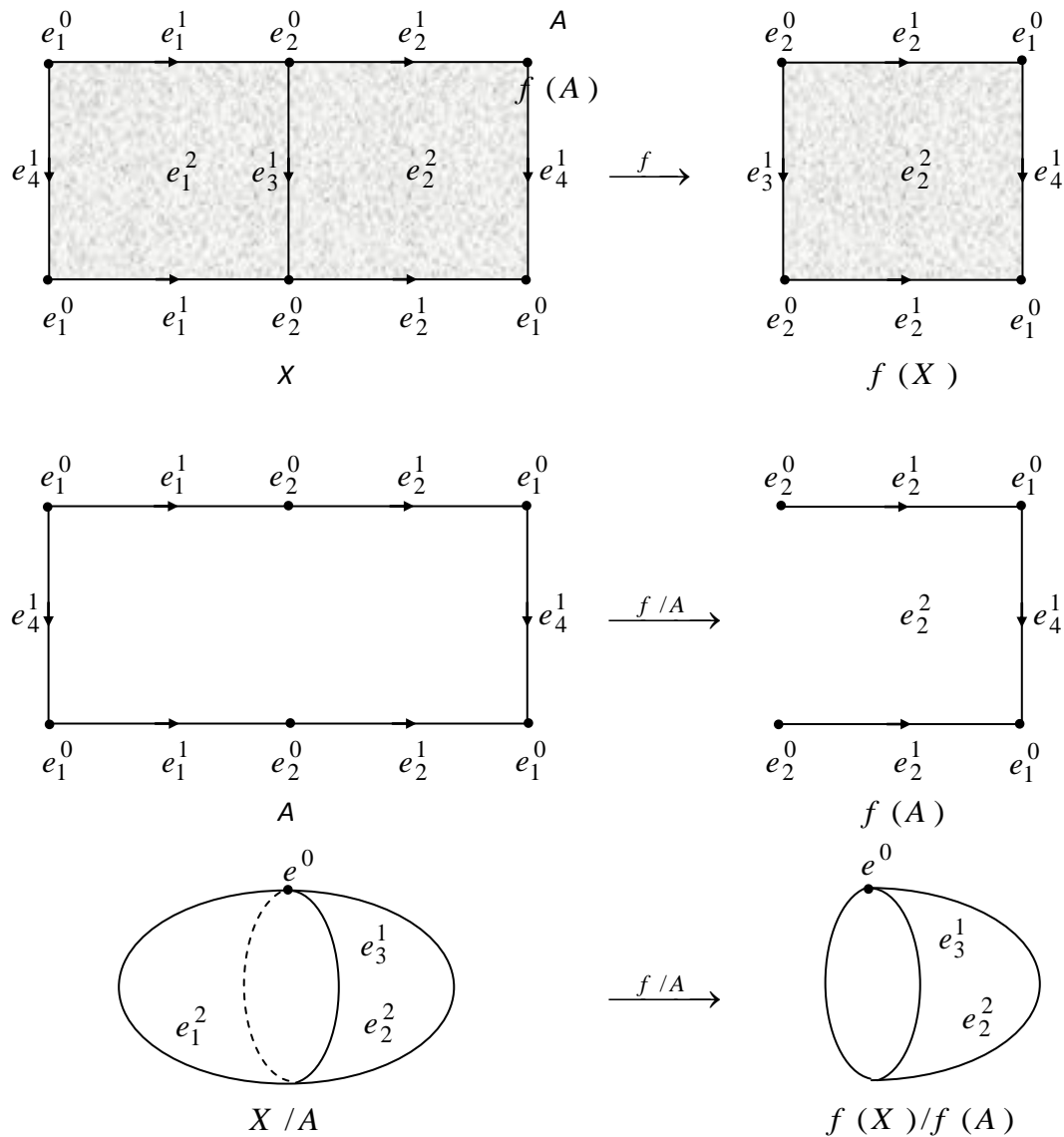


Fig.(4)

4. Cellular folding of the suspension

For a space X , the suspension SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. If X is a complex, so is SX as quotient of $X \times I$ with its product cell structure, I being given the standard cell structure of two 0-cells joined

by a 1-cell [3]. Thus we can define the suspension $S X$ as the union of all line segments joining points of X to two external vertices called "suspension points".

Useful property of suspension is that not only spaces but also maps can be suspended, a map $f : X \rightarrow Y$ suspends to $S f : S X \rightarrow S Y$, the quotient map of $f \times I : X \times I \rightarrow Y \times I$, [3].

4.1. Example

If $X = S^1$, circle, then $S(S^1) = S^2$, see Fig.(5)

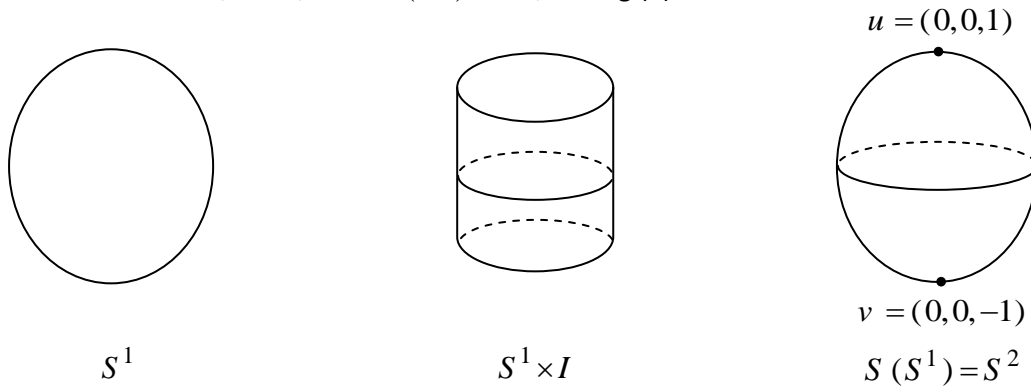


Fig. (5)

Generally $S(S^n) = S^{n+1}$.

4.2. Theorem

Let X and Y be complexes of the same dimension n , let $f : X \rightarrow Y$ be a cellular map. Then $g = S f : S X \rightarrow S Y$ mapping suspension points (vertices) u, v into itself, and for each i -cell $(e, e') \in S X$, $g(e, e') = (f(e), e')$, where e' is a zero or a one-cell of I , is a cellular folding if and only if f is a cellular folding.

Proof:

If f is a cellular folding, then it will maps cells to cells of the same dimension, and hence does g . Also $\overline{(e, e')}$ and $\overline{g(e, e')} = \overline{(f(e), e')}$ contains the same number of vertices because f is a cellular folding.

Suppose now g is a cellular folding, then g maps i -cell to i -cell, i.e., if (e, e') is an i -cell in $S X$, then $g(e, e') = (f(e), e')$ is an i -cell in $S Y$. Let e be a j -cell in X , and e' be an $(i - j)$ -cell in I . The cellular map must maps j -cells to k -cells such that $k \leq j$. If $k = j$ nothing to prove, so let $k < j$. In this case g will maps $(i - j)$ -cells to $(i - k)$ -cells and hence is not a cellular map. This is a contradiction, and hence $k = j$ is only possibility. The second condition of cellular folding certainly satisfied in this case.

4.3. Example:

Let $X = S^1$ be a complex with cellular subdivision shown in Fig.(6-a), and $f : X \rightarrow X$ be a cellular folding defined by: $f(e_3^0) = (e_1^0)$, $f(e_2^1, e_3^1) = (e_1^1, e_4^1)$.

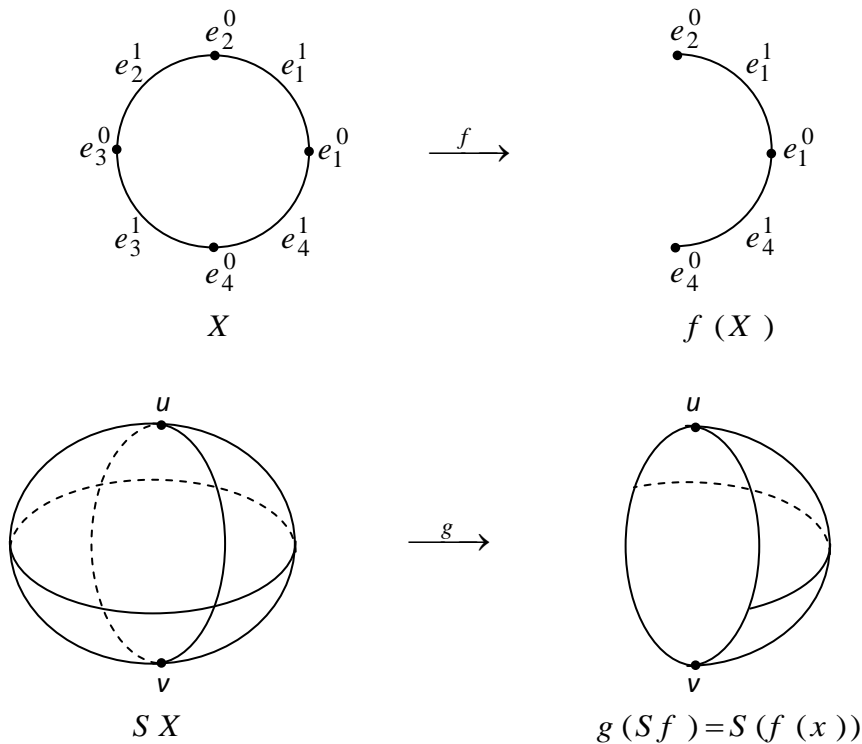


Fig. (6)

$$g(u, v) = (u, v), g(e, e') = (f(e), e'), e \in X, e' \in I$$

Then $g \circ Sf: SX \rightarrow SY$ is a cellular folding defined by $g(u, v) = (u, v), (e, e') = (f(e), e'), e \in X, e' \in I$.

5. Cellular folding of the join of complexes

The join $X * Y$ of the two spaces X and Y is the quotient space $X \times Y \times I$ under the identification $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 0) \sim (x_2, y, 0)$. Thus we are collapsing the subcomplex $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y , [3]. One can define this space as the space of all line segments joining points in X to points in Y .

Note that if X and Y are complexes, then there is a natural CW structure on $X * Y$ having the subspaces X and Y as a subcomplexes, with the remaining cells being the produce cells of $X \times Y \times (0, 1)$.

5.1. Example

If X and Y are both closed intervals, then we are collapsing two opposite faces of a cube onto line segments so that the cube becomes a tetrahedron, see Fig.(7).

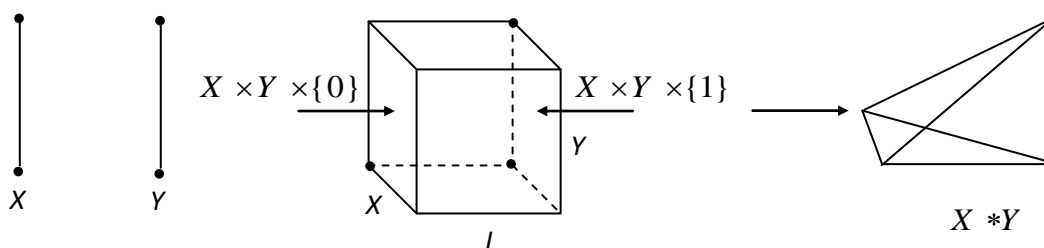


Fig. (7)

5.2. Theorem

Let X and Y be complexes of the same dimension n , let $f : X \rightarrow X$, $g : Y \rightarrow Y$ be cellular maps. Then $h = f * g : X * Y \rightarrow X * Y$ defined as the quotient map of $f \times g \times I : X \times Y \times I \rightarrow X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 0) \sim (x_2, y, 0)$ is a cellular folding if and only if f and g are both cellular foldings.

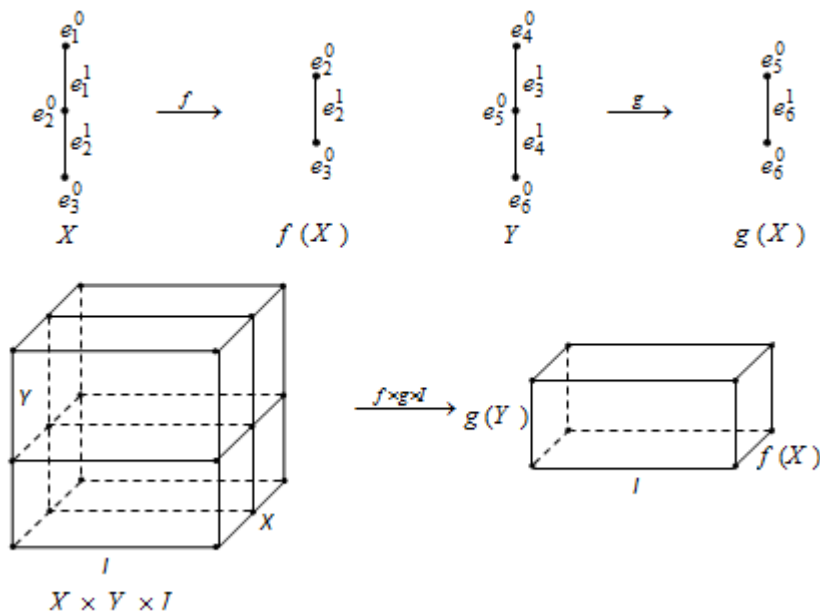
Proof

Suppose that f and g are cellular foldings. Let e be an i -cell in X and σ be a j -cell in Y . Then (e, σ) is an $(i + j + 1)$ -cell in $X * Y$. Now $(f * g)(e, \sigma) = (f(e), g(\sigma))$, but since each of f and g are cellular foldings, then $f(e)$ is an i -cell in $f(X)$ and $g(\sigma)$ is a j -cell in $g(Y)$. Thus $(f * g)(e, \sigma)$ is an $(i + j + 1)$ -cell in $f(X) * g(Y)$, i.e., $f * g$ sends cells to cells of the same dimension. Also $\overline{(e, \sigma)}$ and $\overline{(f * g)(e, \sigma)}$ contains the same number of vertices because each of f and g is a cellular folding.

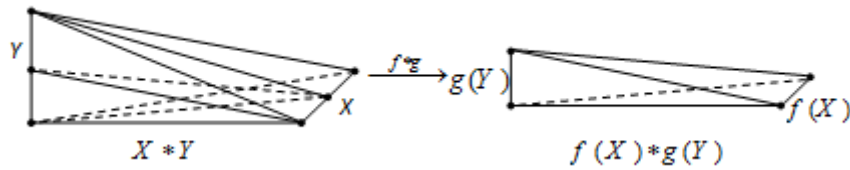
To prove the converse, suppose $f * g$ is a cellular folding, then $f * g$ maps cells of $X * Y$ to cells of the same dimension, so if (e, σ) is a p -cell in $X * Y$, then $(f * g)(e, \sigma) = (f(e), g(\sigma))$ is a p -cell in $f(X) * g(Y)$. Now let e be an i -cell in X , then σ is a $(p - i - 1)$ -cell in Y . But any cellular map maps i -cells to j -cells where $j \leq i$. If $i = j$, then nothing to prove, so let $i > j$. In this case g will maps a $(p - i - 1)$ -cell to $(p - i - 1)$ -cell and hence it is not a cellular folding, which is a contradiction and hence $i = j$ is the only possibility. The second condition of cellular folding is certainly satisfied in this case, then f, g are cellular foldings.

5.3. Example

Let X and Y be complexes such that $|X| = |Y| = I$ with cellular divisions shown in Fig. (8), and $f : X \rightarrow X$, $g : Y \rightarrow Y$ be cellular foldings defined as follows: $f(e_1^0) = e_3^0$, $f(e_1^1) = e_2^1$ and $g(e_4^0) = e_6^0$, $g(e_3^1) = e_5^1$



(a)



(b)

Fig. (8)

Then the map $f * g : X * Y \rightarrow X * Y$ is a cellular folding, see Fig.(8-b).

6. Cellular folding of the wedge sum of two complexes

Given two complexes X and Y with chosen zero cells $u \in X$ and $v \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained by identifying u and v to a single 0-cell, [3]. We will call this 0-cell, the identifying 0-cell.

Note that for any cell complex X , the quotient X^n / X^{n-1} is a wedge sum of n -spheres $V_\alpha S_\alpha^n$, with one sphere for each n -cell of X .

6.1. Example

Let X, Y be two complexes such that $|X| = |Y| = S^1$. Then $X \vee Y = S^1 \vee S^1$ is the figure eight (8), see Fig. (9).

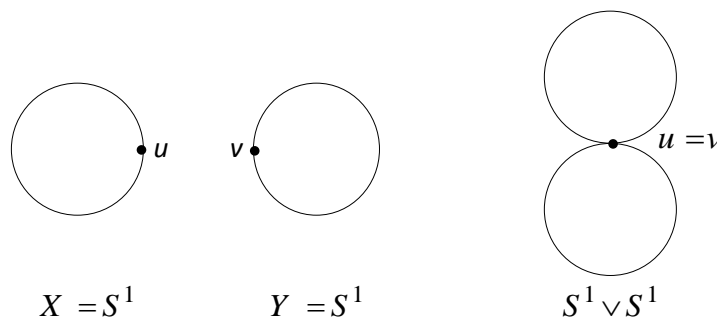


Fig. (9)

More generally one could form the wedge sum $V_\alpha X_\alpha$ of an arbitrary collection of spaces X_α by starting with the disjoint $\bigcup_\alpha X_\alpha$ and identifying points $x_\alpha \in X_\alpha$ to a single point. In case the spaces X_α are cell complexes and the points x_α are 0-cells, then $V_\alpha X_\alpha$ is a cell complex since it is obtained from the cell complex $\bigcup_\alpha X_\alpha$ by collapsing a subcomplex to a point.

6.2. Theorem

Let X and Y be complexes of the same dimension n , let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be cellular maps. Let $h = f \vee g : X \vee Y \rightarrow X \vee Y$ be defined as follows: for each i -cell e ,

$$h(e) = \begin{cases} f(e), & e \in X \\ g(e), & e \in Y \end{cases}$$

$f(e^0) = g(e^0) = e^0$, e^0 is the identifying 0-cell. Then h is a cellular folding if and only if f and g are cellular foldings.

Proof

Suppose f and g are cellular foldings. Let e be an i -cell of $X \vee Y$ such that \bar{e} has r distinct vertices, then we have:

- (i) If $e \in X$, then $h(e) = f(e)$ is an i -cell in Y , $\overline{f(e)}$ has r distinct vertices, since f is a cellular folding.
- (ii) If $e \in X$, then $h(e) = g(e)$ is an i -cell in X , $\overline{g(e)}$ has r distinct vertices, since g is a cellular folding. Thus $h = f \vee g$ is a cellular folding.

Conversely, let $h = f \vee g$ be a cellular folding, then $f \vee g$ maps p -cells to p -cells. Let e be an i -cell in X and f a cellular map, then it will maps i -cells to j -cells such that, $j \leq i$. If $j = i$ nothing to prove, so let $j < i$. In this case $h = f \vee g$ will maps i -cells to j -cells and hence it is not a cellular folding. Which is a contradiction and hence $j = i$ is the only possibility. The second condition of cellular foldings is certainly satisfied in this case, then f, g are cellular foldings.

6.3. Examples

- (1) Let X and Y be two complexes such that $|X| = |Y| = S^1$, and $f : X \rightarrow X, g : Y \rightarrow Y$ be cellular foldings defined as follows:

$f(e_i^0) = (e_i^0), i = 1, 2; f(e_2^1) = (e_1^1), g(e_5^0, e_6^0) = (e_3^0, e_4^0), g(e_i^1) = e_3^1, i = 3, 4, 5$. See Fig. (10)

Then the map $f \vee g : X \vee Y \rightarrow X \vee Y$ is defined by:

$(f \vee g)(e_5^0, e_6^0) = (e_3^0, e_4^0), (f \vee g)(e_2^1, e_4^1, e_5^1, e_6^1) = (e_1^1, e_3^1, e_3^1, e_3^1)$ is a cellular folding, see Fig. (10)

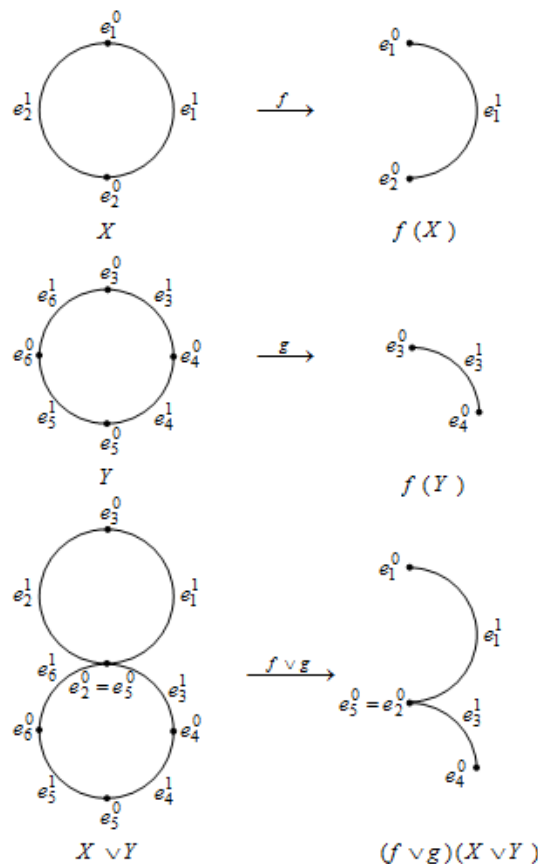
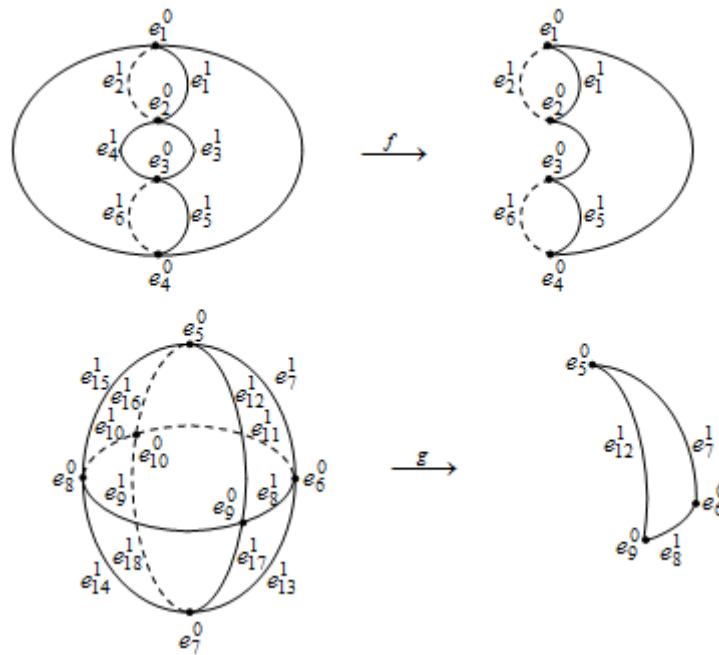
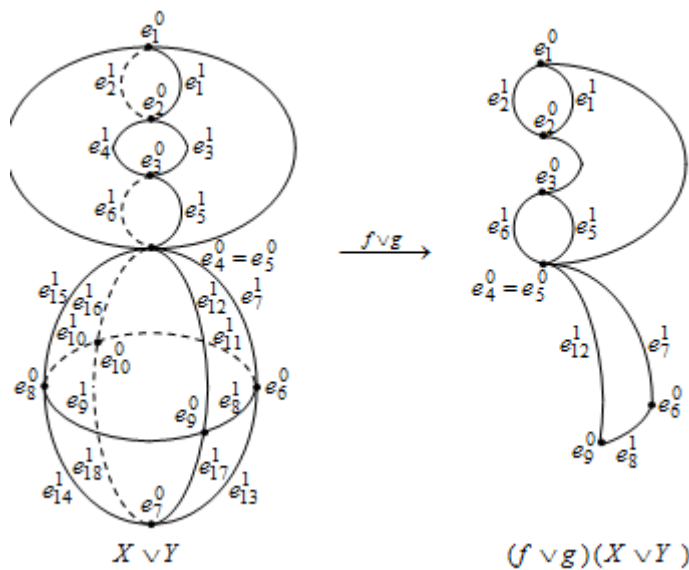


Fig. (10)

(2) Let X and Y be two complexes such that $|X| = T^2$, $|Y| = S^2$ with the cellular subdivision shown in Fig. (11-a). Let $f : X \rightarrow X$, $g : Y \rightarrow Y$ be cellular foldings defined as follows:
 $f(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_3^0, e_4^0)$, $f(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1) = (e_1^1, e_2^1, e_3^1, e_3^1, e_5^1, e_6^1)$
 $g(e_5^0, e_6^0, e_7^0, e_8^0, e_9^0, e_{10}^0) = g(e_5^0, e_6^0, e_5^0, e_9^0, e_9^0, e_6^0)$, $g(e_7^1, e_8^1, e_9^1, \dots, e_{18}^1) = (e_7^1, e_8^1, e_{12}^1)$



(a)



(b)

Fig. (11)

Then the map $f \vee g : X \vee Y \rightarrow X \vee Y$ defined by $f \vee g(e_1^0, e_2^0, e_3^0) = (e_1^0, e_2^0, e_3^0)$, $e_4^0 = e_5^0$
 $f \vee g(e_6^0, e_7^0, e_8^0, e_9^0, e_{10}^0) = (e_6^0, e_9^0, e_6^0, e_9^0, e_6^0)$,
 $f \vee g(e_1^1, e_2^1, e_3^1, \dots, e_{18}^1) = (e_1^1, e_2^1, e_3^1, e_3^1, e_5^1, e_6^1, e_7^1, e_8^1, e_{12}^1)$ $f \vee g(e_i^2) = e_i^2, i = 1, \dots, 8$ is a cellular folding, see Fig.(11-b).

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