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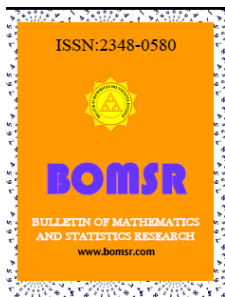
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CELLULAR FOLDING OF SOME NEW CW-COMPLEXES

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ABSTRACT

In this paper we obtained the conditions satisfied by a cellular folding of a given CW-complex to be able to cellular fold some new CW-complexes generated by some known operations like quotient, suspension of a regular CW-complex, Cartesian product, join product, and wedge sum of two CW-complexes.

Keywords: Cellular folding, quotient, suspension, join product, and wedge sum

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1. INTRODUCTION

A cellular folding is a folding defined on regular CW-complexes first defined by, E. El-Kholy and H. Al-Khurasani, [1] and various properties of this type of folding are also studied by them.

Let  $K$  and  $L$  be complexes, a continuous map  $f : K \rightarrow L$  is called cellular if  $f(K^n) \subset L^n$  for  $n = 0, 1, 2, \dots, n$ , where  $K^n$  and  $L^n$  denote the  $n$ -skeletons of  $K$  and  $L$  respectively.

Now, let  $K$  and  $L$  be regular CW-complexes of the same dimension  $n$ , a cellular map  $f : K \rightarrow L$  is a cellular folding if and only if  $f$  satisfies the following:

- (i) For each  $i$ -cells  $e^i \in K$ ,  $f(e^i)$  is an  $i$ -cell in  $L$ , i.e.,  $f$  maps  $i$ -cells to  $i$ -cells;
- (ii) If  $\bar{e}$  contains  $n$  vertices, then  $\overline{f(e)}$  must contains  $n$  distinct vertices, [1]. The set of regular CW-complexes together with cellular folding form a category denoted by  $C(K, L)$ . If  $f \in C(K, L)$ , then  $x \in K$  is said to be a singularity of  $f$  iff  $f$  is not a local homeomorphism at  $x$ . The set of all singularities of  $f$  is denoted by  $\sum f$ . This set corresponds to the "folds" of the map. It is noticed that for a cellular folding  $f$ , the set  $\sum f$  of singularities of  $f$  is a proper subset of the union of cells of dimension  $\leq n - 1$ . Thus, when we consider any  $f \in C(K, L)$ , where  $K$  and  $L$  are connected regular CW-complexes of dimension 2, the set  $\sum f$  will consist of 0-cells, and 1-cells, each of 0-cells (vertices) has

even valences [2]. Of course  $\sum f$  need not be connected. Thus in this case  $\sum f$  has the structure of a locally finite graph  $\Gamma_f$  embedded in  $K$ , for which every vertex has an even valency.

From now on by a complex we mean regular CW-complexes.

**2. Cellular folding of the Cartesian product of complexes:**

If  $X$  and  $Y$  are cell complexes, then  $X \times Y$  has the structure of a cell complex with cell products  $e_\alpha^m \times e_\beta^n$  where  $e_\alpha^m$  ranges over the cells of  $X$  and  $e_\beta^n$  ranges over the cells of  $Y$  [3].

**2.1. Theorem**

Let  $K$  and  $X$  be complexes of the same dimension  $n$ ,  $L$  and  $Y$  be complexes of the same dimension  $m$ . Let  $f : K \rightarrow X$  and  $g : L \rightarrow Y$  be cellular maps. Then  $f \times g \in C(K \times L, X \times Y)$  if and only if  $f$  and  $g$  are cellular foldings.

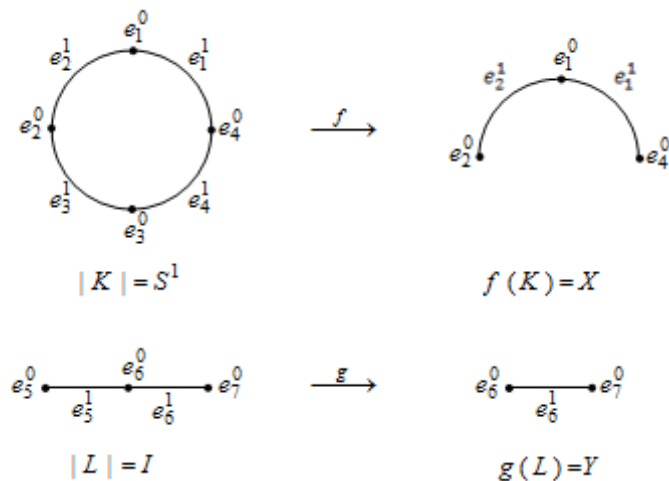
**Proof:**

If  $f$  and  $g$  are cellular foldings, then each will maps cells to cells of the same dimension hence do  $f \times g$ . Also  $\bar{e}$  and  $\overline{f \times g(e)}$  contains the same number of vertices because each of  $f$  and  $g$  are cellular foldings.

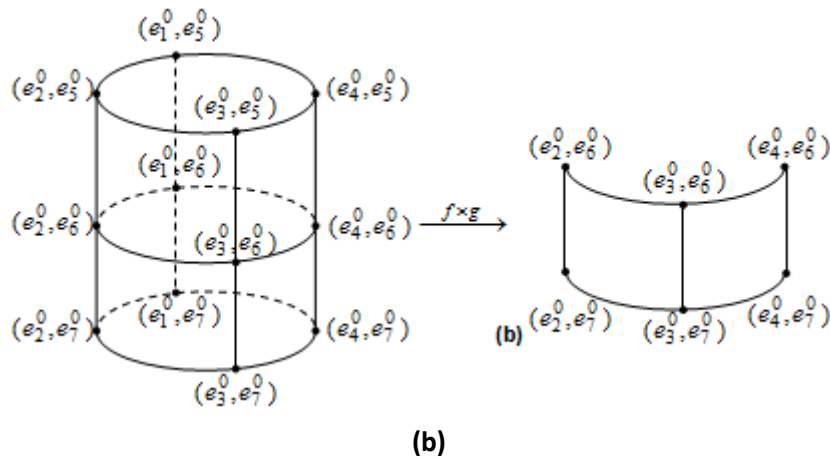
Suppose now  $f \times g$  is a cellular folding, then  $f \times g$  maps p-cells to p-cells, i.e., if  $(e, e')$  is a p-cell in  $K \times L$ , then  $(f \times g)(e, e') = (f(e), g(e'))$  is a p-cell in  $X \times Y$ . Let  $e$  be an  $i$ -cell in  $K$  and  $e'$  be a  $(p-i)$ -cell in  $L$ . The cellular map must maps  $i$ -cells to  $j$ -cells such that  $j \leq i$ . If  $j = i$  nothing to prove, so let  $i > j$ . In this case  $g$  will maps  $(p-i)$ -cells to  $(p-j)$ -cells and hence is not a cellular map. This is a contradiction and hence  $i = j$  is the only possibility. The second condition of cellular folding certainly satisfied in this case.

**2.2. Example:**

Suppose that  $K$  and  $L$  are complexes such that  $|K| = S^1, |L| = I$  with cell decomposition shown in Fig. (1-a). Let  $f : K \rightarrow K$  be a cellular folding defined by  $f(e_3^0) = e_1^0, f(e_3^1, e_4^1) = (e_2^1, e_1^1)$ , where the omitted cells through the paper are mapped into themselves. The image of  $f$  is a complex consisting of three vertices and two 1-cells. Let  $g : L \rightarrow L$  be a cellular folding defined by  $(e_5^0) = e_7^0, g(e_5^1) = e_6^1$ . Then  $f \times g : K \times L \rightarrow K \times L$  is a cellular folding. The cell decomposition of  $K \times L$  and  $(f \times g)(K \times L)$  are shown in Fig.(1-b).



(a)



(b)  
Fig. (1)

**3. Cellular folding of the quotient of a complex:**

If  $(X, A)$  is a pair consisting of a cell complex  $X$  and a subcomplex  $A$ , then the quotient space  $X/A$  inherits a natural cell complex structure from  $X$ . The cells of  $X/A$  are the cells of  $X - A$  plus one new 0-cell, the image of  $A$  in  $X/A$ , [3].

**3.1. Example:**

If  $X = D^2 = \{(x, y) \in E^2 : x^2 + y^2 \leq 1\}$  is a disc with the cell structure consisting of two 0-cells, two 1-cells and one 2-cell, and let  $A = S^1 = \partial D^2$ . Then  $D^2/A$  is a sphere  $S^2$  with one 2-cell and one 0-cell, see Fig. (2).

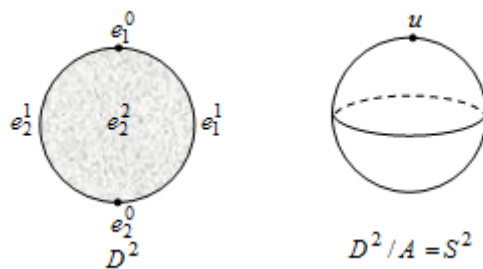


Fig. (2)

Generally, if we give  $S^{n-1}$  any cell structure and build  $D^n$  from  $S^{n-1}$  by attaching an  $n$ -cell, then the quotient  $D^n/S^{n-1}$  is  $S^n$  with its usual cell structure [3].

**3.2. Theorem**

Let  $X$  be a complex,  $A \subset X$  a subcomplex,  $f : X \rightarrow X$  a cellular map. Let  $g : X/A \rightarrow X/A$  be defined by, for each  $i$ -cell  $e$  in  $X - A$ ,  $g(e) = f(e)$ ,  $g(e^0) = e^0$ , where  $e^0$  is the new 0-cell of  $X/A$ . Then  $g$  is a cellular folding if and only if both  $f$  and  $f|_A$  is a cellular folding. In this case  $g(X/A) = f(X)/f(A)$ .

**Proof:**

Let  $f : X \rightarrow X$  be a cellular folding,  $e$  an  $i$ -cell in  $X/A$  such that  $\bar{e}$  has  $n$  distinct vertices  $g(e) = f(e)$  is an  $i$ -cell such that  $\overline{g(e)} = \overline{f(e)}$  has  $n$  distinct vertices,  $g(e^0) = e^0$ . Thus  $g : X/A \rightarrow X/A$  is a cellular folding.

Now suppose  $g : X / A \rightarrow X / A$  is a cellular folding and  $f : X \rightarrow X$  is a cellular map, if  $e$  is an  $i$ -cell in  $X - A$ , then  $f(e) = g(e)$  is an  $i$ -cell in  $X$ , but if  $e$  is an  $i$ -cell in  $A$ , then  $f(e)$  might be a  $j$ -cell in  $X$ ,  $j < i$  while if  $f|_A : A \rightarrow A$  is a cellular folding, then for any  $i$ -cell in  $X$ ,  $f(e)$  is an  $i$ -cell in  $X$  and consequently  $f$  is a cellular folding.

**3.3. Example**

Let  $X = D^2$  be a disc with cellular subdivision consisting of two 0-cells, three 1-cells and two 2-cells, and let  $A = S^1 = \partial D^2$ ,  $f : D^2 \rightarrow D^2$  be a cellular folding defined as follows :  $f(e_1^0, e_2^0) = (e_1^0, e_2^0)$ ,  $f(e_2^1) = (e_1^1)$ ,  $f(e_2^2) = f(e_1^2)$ .

The map  $f|_A$  is the cellular folding shown in Fig.(3). Now  $g : X / A \rightarrow X / A$  is a cellular folding defined by,  $g(e^0) = e^0$ ,  $g(e_3^1) = e_1^1$ ,  $g(e_2^2) = e_1^2$ , see Fig. (3).

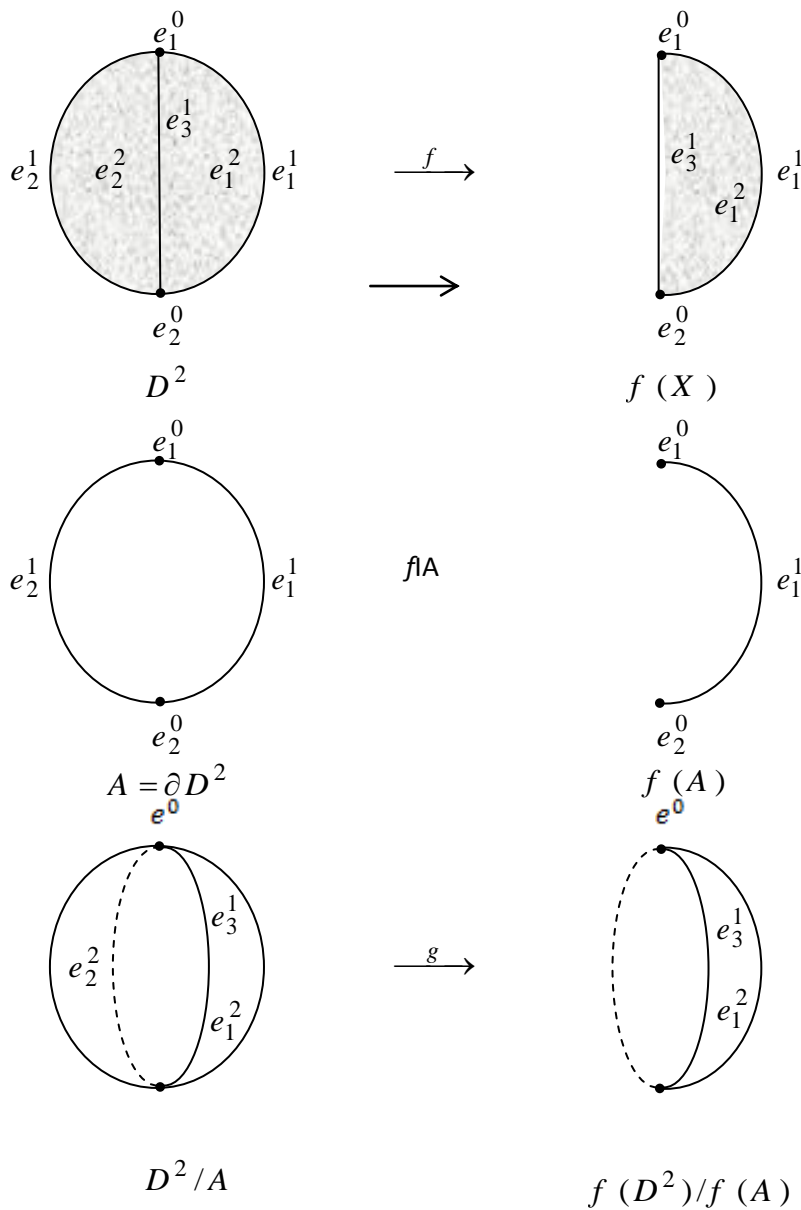
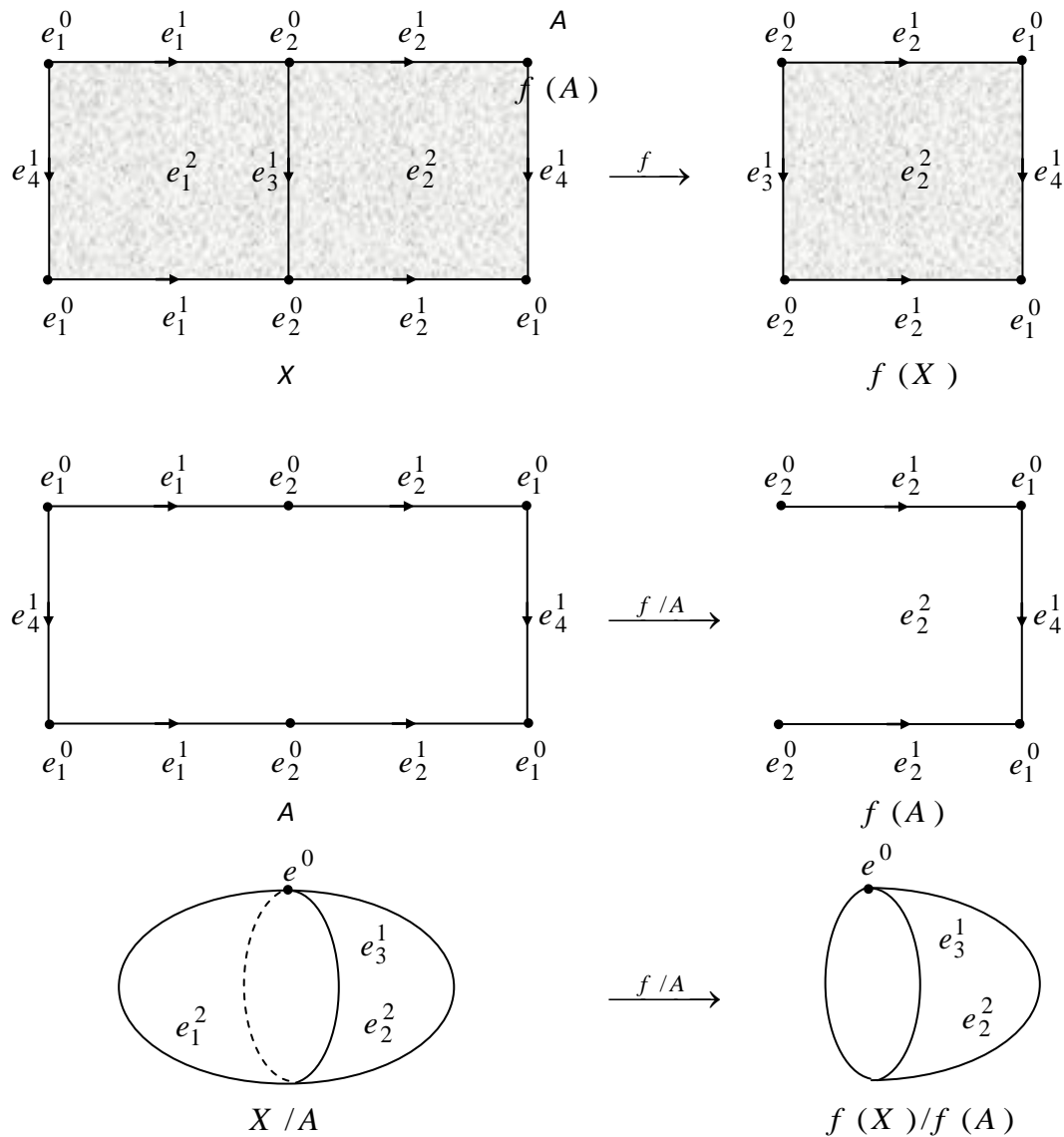


Fig. (3)

**3.4. Example**

Let  $X$  be a complex, such that  $|X| = T$  is a torus, with cellular subdivision consisting of two 0-cells, four 1-cells, and two 2-cells,  $A \subset X$  be the subcomplex shown in Fig. 4. Let  $f : X \rightarrow X$  be a cellular folding defined as follows:  $f(e_i^0) = e_i^0$ ,  $i = 1, 2$ ;  $f(e_1^1) = (e_2^1)$ ,  $f(e_2^1) = (e_1^1)$ ,  $f(e_1^2) = (e_2^2)$ . The map  $f|_A$  is the cellular folding shown in Fig. (3).

Now  $g : X/A \rightarrow X/A$  is a cellular folding defined by,  $g(e^0) = e^0$ ,  $g(e_3^1) = e_3^1$ ,  $g(e_1^2) = e_2^2$ , see Fig.(4).



**Fig.(4)**

**4. Cellular folding of the suspension**

For a space  $X$ , the suspension  $SX$  is the quotient of  $X \times I$  obtained by collapsing  $X \times \{0\}$  to one point and  $X \times \{1\}$  to another point. If  $X$  is a complex, so is  $SX$  as quotient of  $X \times I$  with its product cell structure,  $I$  being given the standard cell structure of two 0-cells joined

by a 1-cell [3]. Thus we can define the suspension  $S X$  as the union of all line segments joining points of  $X$  to two external vertices called "suspension points".

Useful property of suspension is that not only spaces but also maps can be suspended, a map  $f : X \rightarrow Y$  suspends to  $S f : S X \rightarrow S Y$ , the quotient map of  $f \times I : X \times I \rightarrow Y \times I$ , [3].

**4.1. Example**

If  $X = S^1$ , circle, then  $S(S^1) = S^2$ , see Fig.(5)

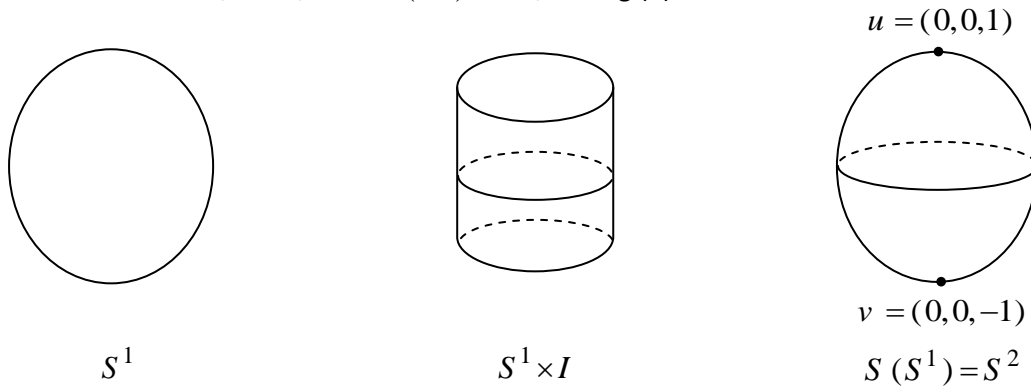


Fig. (5)

Generally  $S(S^n) = S^{n+1}$ .

**4.2. Theorem**

Let  $X$  and  $Y$  be complexes of the same dimension  $n$ , let  $f : X \rightarrow Y$  be a cellular map. Then  $g = S f : S X \rightarrow S Y$  mapping suspension points (vertices)  $u, v$  into itself, and for each  $i$ -cell  $(e, e') \in S X$ ,  $g(e, e') = (f(e), e')$ , where  $e'$  is a zero or a one-cell of  $I$ , is a cellular folding if and only if  $f$  is a cellular folding.

**Proof:**

If  $f$  is a cellular folding, then it will maps cells to cells of the same dimension, and hence does  $g$ . Also  $\overline{(e, e')}$  and  $\overline{g(e, e')} = \overline{(f(e), e')}$  contains the same number of vertices because  $f$  is a cellular folding.

Suppose now  $g$  is a cellular folding, then  $g$  maps  $i$ -cell to  $i$ -cell, i.e., if  $(e, e')$  is an  $i$ -cell in  $S X$ , then  $g(e, e') = (f(e), e')$  is an  $i$ -cell in  $S Y$ . Let  $e$  be a  $j$ -cell in  $X$ , and  $e'$  be an  $(i - j)$ -cell in  $I$ . The cellular map must maps  $j$ -cells to  $k$ -cells such that  $k \leq j$ . If  $k = j$  nothing to prove, so let  $k < j$ . In this case  $g$  will maps  $(i - j)$ -cells to  $(i - k)$ -cells and hence is not a cellular map. This is a contradiction, and hence  $k = j$  is only possibility. The second condition of cellular folding certainly satisfied in this case.

**4.3. Example:**

Let  $X = S^1$  be a complex with cellular subdivision shown in Fig.(6-a), and  $f : X \rightarrow X$  be a cellular folding defined by:  $f(e_3^0) = (e_1^0)$ ,  $f(e_2^1, e_3^1) = (e_1^1, e_4^1)$ .

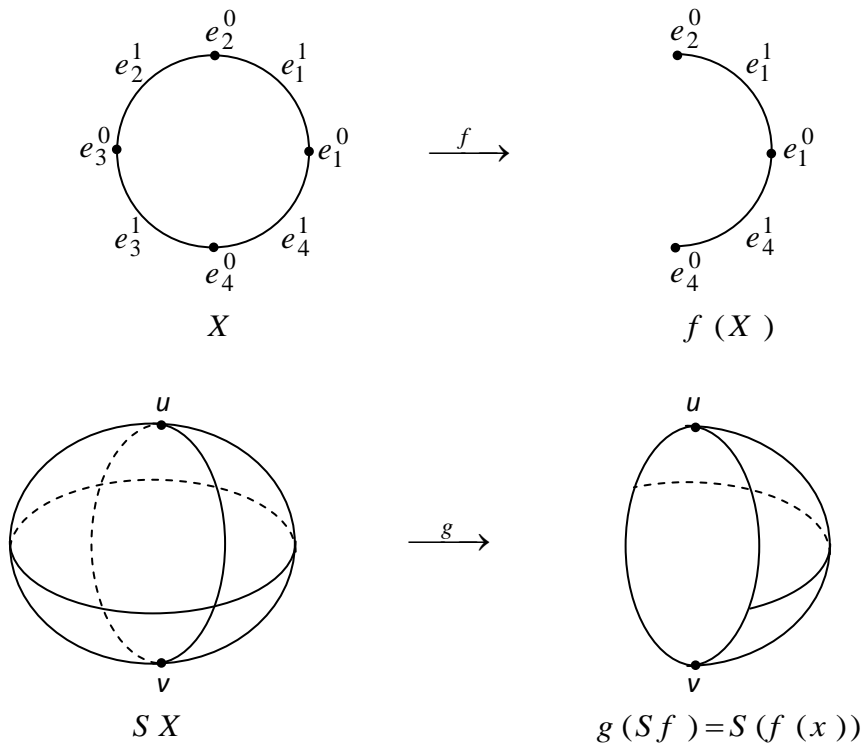


Fig. (6)

$$g(u, v) = (u, v), g(e, e') = (f(e), e'), e \in X, e' \in I$$

Then  $g \circ Sf: SX \rightarrow SY$  is a cellular folding defined by  $g(u, v) = (u, v), (e, e') = (f(e), e'), e \in X, e' \in I$ .

**5. Cellular folding of the join of complexes**

The join  $X * Y$  of the two spaces  $X$  and  $Y$  is the quotient space  $X \times Y \times I$  under the identification  $(x, y_1, 0) \sim (x, y_2, 0)$  and  $(x_1, y, 0) \sim (x_2, y, 0)$ . Thus we are collapsing the subcomplex  $X \times Y \times \{0\}$  to  $X$  and  $X \times Y \times \{1\}$  to  $Y$ , [3]. One can define this space as the space of all line segments joining points in  $X$  to points in  $Y$ .

Note that if  $X$  and  $Y$  are complexes, then there is a natural CW structure on  $X * Y$  having the subspaces  $X$  and  $Y$  as a subcomplexes, with the remaining cells being the produce cells of  $X \times Y \times (0, 1)$ .

**5.1. Example**

If  $X$  and  $Y$  are both closed intervals, then we are collapsing two opposite faces of a cube onto line segments so that the cube becomes a tetrahedron, see Fig.(7).

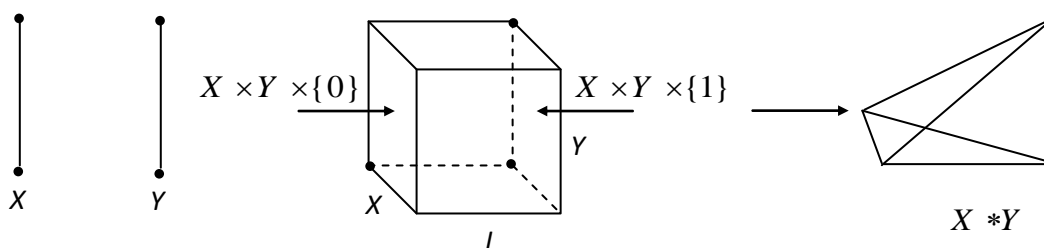


Fig. (7)

**5.2. Theorem**

Let  $X$  and  $Y$  be complexes of the same dimension  $n$ , let  $f : X \rightarrow X$ ,  $g : Y \rightarrow Y$  be cellular maps. Then  $h = f * g : X * Y \rightarrow X * Y$  defined as the quotient map of  $f \times g \times I : X \times Y \times I \rightarrow X \times Y \times I$  under the identifications  $(x, y_1, 0) \sim (x, y_2, 0)$  and  $(x_1, y, 0) \sim (x_2, y, 0)$  is a cellular folding if and only if  $f$  and  $g$  are both cellular foldings.

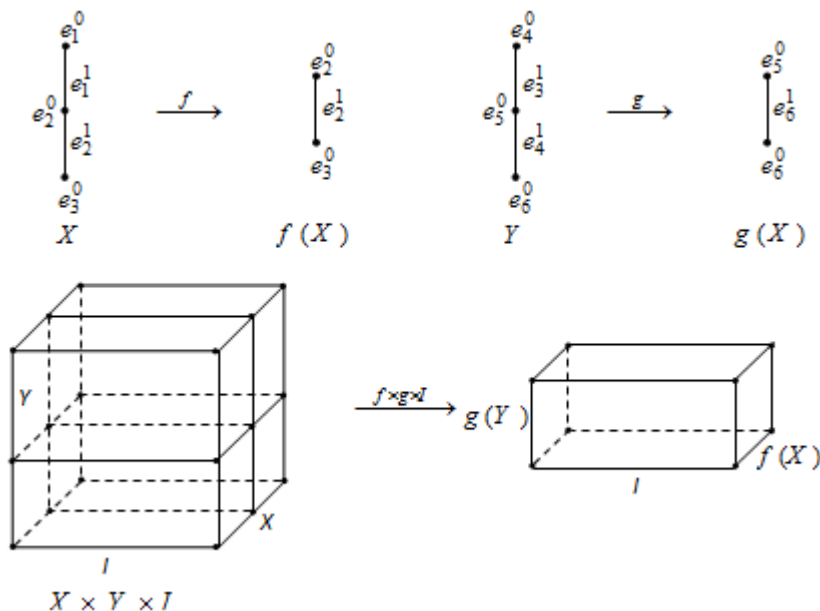
**Proof**

Suppose that  $f$  and  $g$  are cellular foldings. Let  $e$  be an  $i$ -cell in  $X$  and  $\sigma$  be a  $j$ -cell in  $Y$ . Then  $(e, \sigma)$  is an  $(i + j + 1)$ -cell in  $X * Y$ . Now  $(f * g)(e, \sigma) = (f(e), g(\sigma))$ , but since each of  $f$  and  $g$  are cellular foldings, then  $f(e)$  is an  $i$ -cell in  $f(X)$  and  $g(\sigma)$  is a  $j$ -cell in  $g(Y)$ . Thus  $(f * g)(e, \sigma)$  is an  $(i + j + 1)$ -cell in  $f(X) * g(Y)$ , i.e.,  $f * g$  sends cells to cells of the same dimension. Also  $(e, \sigma)$  and  $f * g(e, \sigma)$  contains the same number of vertices because each of  $f$  and  $g$  is a cellular folding.

To prove the converse, suppose  $f * g$  is a cellular folding, then  $f * g$  maps cells of  $X * Y$  to cells of the same dimension, so if  $(e, \sigma)$  is a  $p$ -cell in  $X * Y$ , then  $(f * g)(e, \sigma) = (f(e), g(\sigma))$  is a  $p$ -cell in  $f(X) * g(Y)$ . Now let  $e$  be an  $i$ -cell in  $X$ , then  $\sigma$  is a  $(p - i - 1)$ -cell in  $Y$ . But any cellular map maps  $i$ -cells to  $j$ -cells where  $j \leq i$ . If  $i = j$ , then nothing to prove, so let  $i > j$ . In this case  $g$  will maps a  $(p - i - 1)$ -cell to  $(p - i - 1)$ -cell and hence it is not a cellular folding, which is a contradiction and hence  $i = j$  is the only possibility. The second condition of cellular folding is certainly satisfied in this case, then  $f, g$  are cellular foldings.

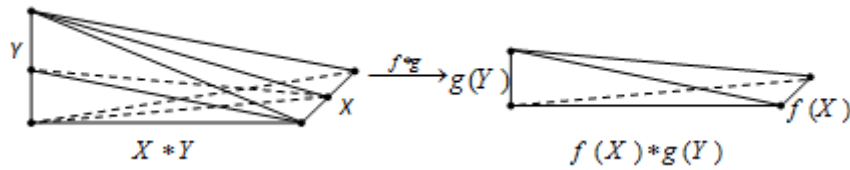
**5.3. Example**

Let  $X$  and  $Y$  be complexes such that  $|X| = |Y| = I$  with cellular divisions shown in Fig. (8), and  $f : X \rightarrow X$ ,  $g : Y \rightarrow Y$  be cellular foldings defined as follows:  $f(e_1^0) = e_3^0$ ,  $f(e_1^1) = e_2^1$  and  $g(e_4^0) = e_6^0$ ,  $g(e_3^1) = e_5^1$



(a)





(b)

Fig. (8)

Then the map  $f * g : X * Y \rightarrow X * Y$  is a cellular folding, see Fig.(8-b).

**6. Cellular folding of the wedge sum of two complexes**

Given two complexes  $X$  and  $Y$  with chosen zero cells  $u \in X$  and  $v \in Y$ , then the wedge sum  $X \vee Y$  is the quotient of the disjoint union  $X \cup Y$  obtained by identifying  $u$  and  $v$  to a single 0-cell, [3]. We will call this 0-cell, the identifying 0-cell.

Note that for any cell complex  $X$ , the quotient  $X^n / X^{n-1}$  is a wedge sum of  $n$ -spheres  $V_\alpha S_\alpha^n$ , with one sphere for each  $n$ -cell of  $X$ .

**6.1. Example**

Let  $X, Y$  be two complexes such that  $|X| = |Y| = S^1$ . Then  $X \vee Y = S^1 \vee S^1$  is the figure eight (8), see Fig. (9).

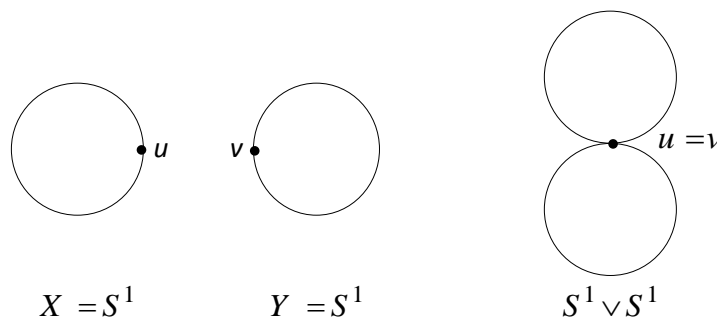


Fig. (9)

More generally one could form the wedge sum  $V_\alpha X_\alpha$  of an arbitrary collection of spaces  $X_\alpha$  by starting with the disjoint  $\cup_\alpha X_\alpha$  and identifying points  $x_\alpha \in X_\alpha$  to a single point. In case the spaces  $X_\alpha$  are cell complexes and the points  $x_\alpha$  are 0-cells, then  $V_\alpha X_\alpha$  is a cell complex since it is obtained from the cell complex  $\cup_\alpha X_\alpha$  by collapsing a subcomplex to a point.

**6.2. Theorem**

Let  $X$  and  $Y$  be complexes of the same dimension  $n$ , let  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be cellular maps. Let  $h = f \vee g : X \vee Y \rightarrow X \vee Y$  be defined as follows: for each  $i$ -cell  $e$ ,

$$h(e) = \begin{cases} f(e), & e \in X \\ g(e), & e \in Y \end{cases}$$

$f(e^0) = g(e^0) = e^0$ ,  $e^0$  is the identifying 0-cell. Then  $h$  is a cellular folding if and only if  $f$  and  $g$  are cellular foldings.

**Proof**

Suppose  $f$  and  $g$  are cellular foldings. Let  $e$  be an  $i$ -cell of  $X \vee Y$  such that  $\bar{e}$  has  $r$  distinct vertices, then we have:

- (i) If  $e \in X$ , then  $h(e) = f(e)$  is an  $i$ -cell in  $Y$ ,  $\overline{f(e)}$  has  $r$  distinct vertices, since  $f$  is a cellular folding.
- (ii) If  $e \in X$ , then  $h(e) = g(e)$  is an  $i$ -cell in  $X$ ,  $\overline{g(e)}$  has  $r$  distinct vertices, since  $g$  is a cellular folding. Thus  $h = f \vee g$  is a cellular folding.

Conversely, let  $h = f \vee g$  be a cellular folding, then  $f \vee g$  maps  $p$ -cells to  $p$ -cells. Let  $e$  be an  $i$ -cell in  $X$  and  $f$  a cellular map, then it will maps  $i$ -cells to  $j$ -cells such that,  $j \leq i$ . If  $j = i$  nothing to prove, so let  $j < i$ . In this case  $h = f \vee g$  will maps  $i$ -cells to  $j$ -cells and hence it is not a cellular folding. Which is a contradiction and hence  $j = i$  is the only possibility. The second condition of cellular foldings is certainly satisfied in this case, then  $f, g$  are cellular foldings.

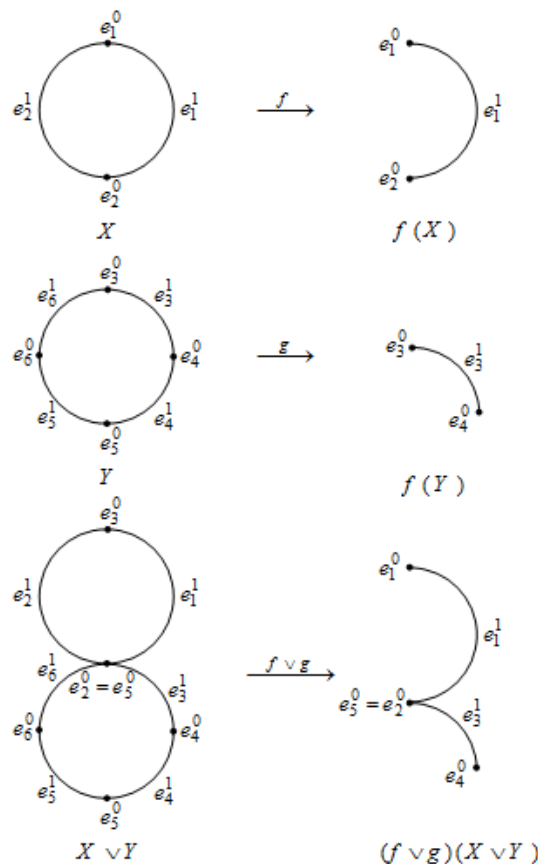
**6.3. Examples**

- (1) Let  $X$  and  $Y$  be two complexes such that  $|X| = |Y| = S^1$ , and  $f : X \rightarrow X, g : Y \rightarrow Y$  be cellular foldings defined as follows:

$f(e_i^0) = (e_i^0), i = 1, 2; f(e_2^1) = (e_1^1), g(e_5^0, e_6^0) = (e_3^0, e_4^0), g(e_i^1) = e_3^1, i = 3, 4, 5$ . See Fig. (10)

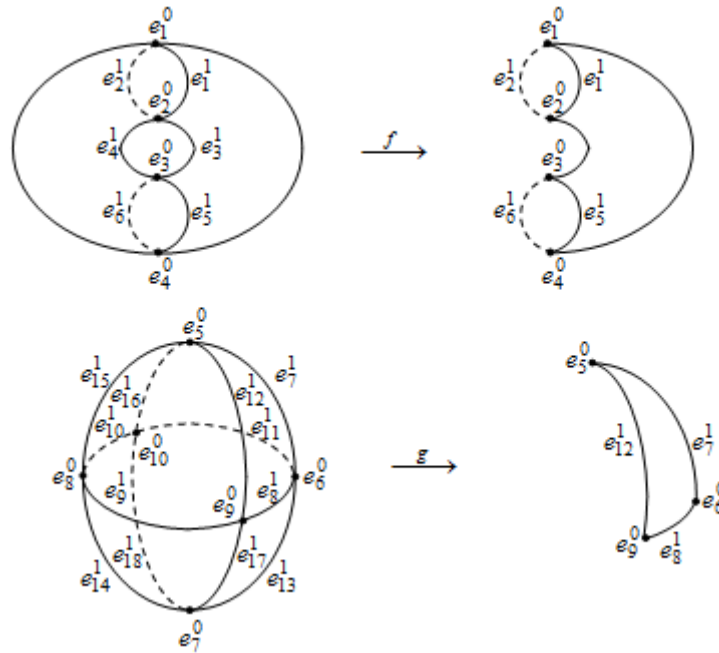
Then the map  $f \vee g : X \vee Y \rightarrow X \vee Y$  is defined by:

$(f \vee g)(e_5^0, e_6^0) = (e_3^0, e_4^0), (f \vee g)(e_2^1, e_4^1, e_5^1, e_6^1) = (e_1^1, e_3^1, e_3^1, e_3^1)$  is a cellular folding, see Fig. (10)

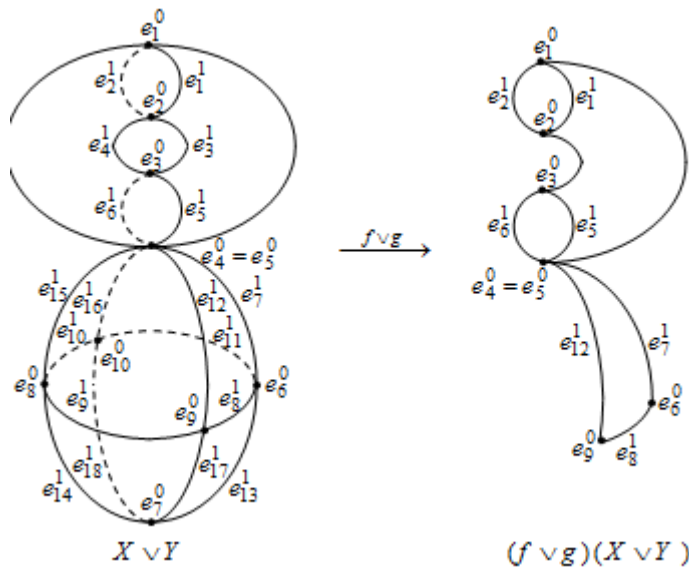


**Fig. (10)**

(2) Let  $X$  and  $Y$  be two complexes such that  $|X| = T^2$ ,  $|Y| = S^2$  with the cellular subdivision shown in Fig. (11-a). Let  $f : X \rightarrow X$ ,  $g : Y \rightarrow Y$  be cellular foldings defined as follows:  
 $f(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_3^0, e_4^0)$ ,  $f(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1) = (e_1^1, e_2^1, e_3^1, e_3^1, e_5^1, e_6^1)$   
 $g(e_5^0, e_6^0, e_7^0, e_8^0, e_9^0, e_{10}^0) = g(e_5^0, e_6^0, e_5^0, e_9^0, e_9^0, e_6^0)$ ,  $g(e_7^1, e_8^1, e_9^1, \dots, e_{18}^1) = (e_7^1, e_8^1, e_{12}^1)$



(a)



(b)

**Fig. (11)**

Then the map  $f \vee g : X \vee Y \rightarrow X \vee Y$  defined by  $f \vee g(e_1^0, e_2^0, e_3^0) = (e_1^0, e_2^0, e_3^0)$ ,  $e_4^0 = e_5^0$   
 $f \vee g(e_6^0, e_7^0, e_8^0, e_9^0, e_{10}^0) = (e_6^0, e_9^0, e_6^0, e_9^0, e_6^0)$ ,  
 $f \vee g(e_1^1, e_2^1, e_3^1, \dots, e_{18}^1) = (e_1^1, e_2^1, e_3^1, e_3^1, e_5^1, e_6^1, e_7^1, e_8^1, e_{12}^1)$   $f \vee g(e_i^2) = e_i^2, i = 1, \dots, 8$  is a cellular folding, see Fig.(11-b).

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