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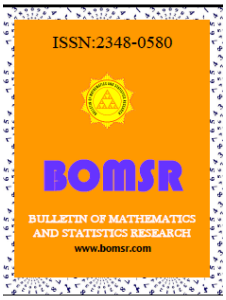
RESEARCH ARTICLE

COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS WITHOUT CONTINUITY IN METRIC SPACES

SUSHIL SHARMA[1], SERVET KUTUKCU[2*], ATUL PATHAK[3]

[1]Department of Mathematics Madhav Science College Ujjain-456010, India
E-mail : sksharma2005@yahoo.com
[2]Department of Mathematics Faculty of Science and Arts Ondokuz Mayis University Kurupelit, Samsun, Turkey
*E-mail: skutukcu@yahoo.com
[3]Department of Mathematics Mahakal Institute of Technology Ujjain, India



ABSTRACT

The aim of this paper is to prove some common fixed point theorem in metric spaces by removing the assumption of continuity, relaxing the condition of compatibility of type (B) to weak compatibility and replacing the completeness of the space with a set of alternative conditions.

Keywords: Metric space, Cauchy sequence, common fixed point.
AMS(2010) Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 1976, Jungck [3] proved a common fixed point theorem for commuting maps, generalizing the Banach's fixed point theorem. Sessa [11] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [4] introduced more generalized commutativity so called compatibility , which is more general than that of weak commutativity. The concept has been used by several authors to prove common fixed point theorems and in the study of periodic points (see e.g. [4],[5],[7],[9],[10])

Jungck, Murthy and Cho [6] defined compatible mappings of type (A) and pointed out that under some conditions these two concept are equivalent, and proved common fixed point theorems. Pathak and Khan [8] defined compatible mappings of type (B) as a generalization of compatible mappings of type (A). The same authors remarked that under some conditions, compatible mappings, compatible mappings of type (A) and compatible mappings of type (B) are equivalent. They derived relations between these mappings and proved a fixed point theorem of Gregus type for compatible mappings of type (B) in Banach spaces.

In 1998, Jungck and Rhoades [2] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true .

The aim of this paper is to prove some common fixed point theorems in metric spaces by removing the assumption of continuity, replacing the condition of compatibility of type (B) by weak compatibility and replacing the completeness of the space with a set of alternative conditions. We improve results of Djoudi [1].

2. PRELIMINARIES

Throughout this paper , X denotes a metric space (X,d) with the metric d.

Definition 2.1. A sequence $\{x_n\}$ in a metric space (X,d) is said to be convergent to a point x in X if

$lim_{n\to\infty} d(x_n, x) = 0.$

Definition 2.2. A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy sequence if

$lim_{m,n\to\infty} d(x_m, x_n) = 0.$

Definition 2.3. A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.4. [4] Let S and T be mappings from a metric space (X,d) into itself. Then S and T are said to be compatible, if

$lim_{n\to\infty} d(STx_n, TSx_n) = 0$

where $\{x_n\}$ is a sequence in X such that

$lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 2.5. [2] Two mappings S and T are said to be weakly compatible if they commute at their coincidence points.

Example 2.1 : Define S,T : [0,3] $\to$ [0,3] by

$$S(x) = \begin{cases} x, & x \in [0,1) \\ 3, & x \in [1,3] \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 3-x, & x \in [0,1) \\ 3, & x \in [1,3] \end{cases}$$

Then for any x$\in$ [1,3], STx = TSx and hence S,T are weakly compatible maps on [0,3].

Example 2.2. Let X = R and define S,T: R $\to$ R by Sx = x/3, x$\in$R and Tx = x^2, x$\in$R. Hence 0 and 1/3 are two coincidence points for the maps S and T. Note that S and T commute at 0, i.e. ST(0) = TS(0) = 0, but ST(1/3) = S(1/9) = 1/27 and TS(1/3) = T(1/9) = 1/81 and so S and T are not weakly compatible maps on R.

Remark 2.1. Weakly compatible maps need not be compatible . Let X = [2,20] and d be the usual metric on X. Define mappings S,T : X $\to$ X by Sx = x if x = 2 or x > 5, Sx = 6 if 2 < x $\le$ 5, Tx = x if x = 2, Tx = 12 if 2 < x $\le$ 5, Tx = x – 3 if x > 5. The mappings S and T are non-compatible since sequence $\{x_n\}$ defined by $x_n = 5 + (1/n)$, n $\ge$ 1. Then $Tx_n = 2$, $Sx_n = 2$, $TSx_n = 2$ and $STx_n = 6$, as n $\to \infty$. But they are weakly cmpatible since they commute at coincidence point at x = 2

Example 2.3. Let X = [0, 2] with the metric d, defined by d(x, y) = |x – y| for all x,y $\in$ X. Clearly (X,d) is a metric space . Define S, T : X $\to$ X by Sx = x if x $\in$ [0, ⅓) , S(x) = ⅓ if x $\ge$ ⅓ and Tx = x/(x+1) for all x $\in$ [0, 2]. Consider the sequence $\{x_n = (½) + (1/n) : n \ge 1\}$ in X. Then $lim_{n\to\infty} Sx_n$ = ⅓ , $lim_{n\to\infty} Tx_n$ = ⅓. But $lim_{n\to\infty} d(STx_n, TSx_n)$ = |1/3 – 1/4| $\neq$ 0. Thus S and T are non compatible. But S and T are commuting at their coincidence point x = 0, that is, weakly compatible at x = 0. $lim_{n\to\infty} d(STx_n, TTx_n)$ = |1/3 – 1/4| $\neq$ 0 and $lim_{n\to\infty} d(TSx_n, SSx_n)$ = |1/4 – 1/3| $\neq$ 0. Thus A and B are not compatible of type (A). Further, $lim_{n\to\infty} d(SSx_n, TTx_n)$ = |1/3 – 1/4| $\neq$ 0. Thus A and B are not compatible of type (B).

In view of the above examples, we observe that

(i) weakly compatible maps need not be compatible,

(ii) weakly compatible maps need not be compatible of type (A),

(iii) weakly compatible maps need not be compatible of type (B).

3. MAIN RESULTS

Let R_+ be the set of non-negative real numbers and let $\varphi : (R_+)^5 \to R_+$ be a function satisfying the following conditions: for any $t > 0$, φ is upper semi continuous in each coordinate variable and non-decreasing, and $\phi(t) = \max\{ \varphi(0,t,0,0,t), \varphi(t,0,0,t,t), \varphi(t,t,t,2t,0), \varphi(0,0,t,t,0) \}$

For our main result we need the following lemmas ;

Lemma 3.1. [12] For any $t > 0$, $\phi(t) < t$ if and only if $\lim_{n\to\infty} \phi^n(t) = 0$ where ϕ^n denotes the n-times repeated composition of ϕ with itself.

Lemma 3.2. [1] Let I, J, S and T be mappings from a metric space (X,d) into itself satisfying (1) and (2). Then $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, where $\{y_n\}$ is the sequence constructed in X and described by (7).

Lemma 3.3. [1] Let I, J, S and T be mappings from a metric space (X,d) into itself satisfying the conditions (1) and (2). Then the sequence $\{y_n\}$ defined by (7) is a Cauchy sequence in X.

Djoudi [1] proved the following

Theorem A. Let I,J,S and T be mappings from a complete metric space (X,d) into itself satisfying

(1) $S(X) \subset J(X)$ and $T(X) \subset I(X)$,

(2) $d(Sx,Ty) \leq \varphi(d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx))$ for all $x,y \in X$,

(3) one of I,J,S or T is continuous,

(4) the pairs {S,I} and {J,T} are compatible of type (B).

Then I,J,S and T have a unique common fixed point z.

Now, we prove the following

Theorem 3.1. Let I, J, S and T be mappings from a metric space (X,d) into itself satisfying the conditions (1) and (2) and

(5) one of I(X), J(X), S(X) or T(X) is a complete subspace of X, then

(i) S and I have a coincidence point,

(ii) J and T have a coincidence point.

Further if

(6) The pairs {S,I} and {J,T} are weakly compatible,

(iii) I, J, S and T have a unique common fixed point in X.

Proof: By assumption (1), since $S(X) \subset J(X)$, for an arbitrary $x_0 \in X$ there exists a point $x_1 \in X$ such that $Sx_0 = Jx_1$. Since $T(X) \subset I(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Tx_1 = Ix_2$. Continuing this way, we can construct a sequence $\{y_n\}$ in X such that

(7) $y_{2n} = Jx_{2n+1} = Sx_{2n}$ and $y_{2n+1} = Ix_{2n+2} = Tx_{2n+1}$,

for every $n = 0,1,2,\ldots$

Thus in the view of Theorem 2 in [1], $\{y_n\}$ is a Cauchy sequence in X. Now suppose that I(X) is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in I(X) and has a limit in I(X). Call it z. Let u $= I^{-1}z$. Then $Iu = z$. We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z. By (2), we have

$$d(Su, y_{2n+1}) = d(Su, Tx_{2n+1})$$
$$\leq \varphi(d(Iu, Jx_{2n+1}), d(Iu,Su), d(Jx_{2n+1},Tx_{2n+1}), d(Iu, Tx_{2n+1}), d(Jx_{2n+1},Su))$$
$$= \varphi(d(z, y_{2n}), d(z,Su), d(y_{2n},y_{2n+1}), d(z, y_{2n+1}), d(y_{2n},Su))$$

which implies that as $n \to \infty$,

$$d(Su,z) \leq \phi(0,d(z,Su),0,0,d(z,Su)) < d(Su,z),$$

which is a contradiction . Thus we have $Su = z$. Since $Iu = z$ thus $Su = z = Iu$, i.e. u is a coincidence point of S and I. This proves (i), since $S(X) \subset J(X)$, Su = z implies that $z \in J(X)$. Let $v \in J^{-1}z$. Then Jv = z. It can be easily verified by using similar arguments of the previous part of the proof that Tv = z.

If we assume J(X) is complete, then argument analogous to the previous completeness argument establishes (i) and (ii). The remaining two cases pertain essentially to the previous cases. Indeed, if T(X) is complete, then by (1), $z \in T(X) \subset I(X)$.

Similarly if S(X) is complete, then $z \in S(X) \subset J(X)$. Thus (i) and (ii) are completely established.

Since the pair {S,I} is weakly compatible therefore S and I commute at their coincidence point i.e. SIu = ISu or Sz = Iz. Similarly JTv = TJv or Tz = Jz. Now we prove that Sz = z.

By (2), we have

$d(Sz, y_{2n+1}) = d(Sz, Tx_{2n+1})$

$\leq \varphi\,(d(Iz, Jx_{2n+1}), d(Iz,Sz), d(Jx_{2n+1},Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Jx_{2n+1},Sz))$

$= \varphi\,(d(Sz, y_{2n}), d(Sz,Sz), d(y_{2n},y_{2n+1}), d(Sz, y_{2n+1}), d(y_{2n},Sz))$

Proceeding limit as $n \to \infty$, we have

$d(Sz,z) \leq \phi(d(Sz,z),0,0,d(Sz,z),d(z,Sz)) < d(Sz,z),$

which is a contradiction. Thus we have Sz = z and therefore Sz = z = Iz.

Similarly , we have Tz = z = Jz. This means that z is a common fixed point of I, J, S and T.

For uniqueness of common fixed point let w ($w \neq z$) be another common fixed point of I, J, S and T. Then by (2), we have

$d(w,z) = d(Sw, Tz)$

$\leq \varphi\,(d(Iw, Jz), d(Iw,Sw), d(Jz,Tz), d(Iw, Tz), d(Jz,Sw))$

$\leq \varphi\,(d(w, z),0,0, d(w, z), d(z,w))$

$< d(w, z),$

hence w = z. This completes the proof of the theorem.

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