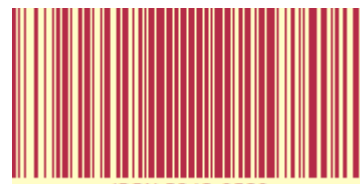




http://www.bomsr.com  
 Email:editorbomsr@gmail.com

RESEARCH ARTICLE

A Peer Reviewed International Research Journal



FURTHER RESULTS ON TOTAL DOMINATING COLOR TRANSVERSAL NUMBER OF GRAPHS

D.K.THAKKAR<sup>1</sup>, A.B.KOTHIYA<sup>2</sup>

<sup>1</sup>Department of Mathematics, Saurashtra University, Rajkot  
 Email: dkthakkar1@yahoo.co.in

<sup>2</sup>G.K.Bharad Institute of Engineering, Kasturbadham, Rajkot  
 Email: kothiyaamin@gmail.com



ABSTRACT

In this contribution, we investigate some relations between Total Dominating Color Transversal number, Chromatic number and Total Domination number of a graph. To be precise, we determine the conditions under which  $\Upsilon_{tstd}$  of a graph becomes equal or unequal to  $\Upsilon_t$  or  $\chi$  of the graph (Where  $\Upsilon_{tstd}$ ,  $\Upsilon_t$  and  $\chi$  are, respectively, the Total Domination Color Transversal number, Total Dominating number and Chromatic number of the graph). We show that when  $\Upsilon_t < \chi$  or  $\chi < \Upsilon_t$  does not imply  $\Upsilon_{tstd} = \chi$  or  $\Upsilon_t$ . We also give different examples to justify our theorems and statements. Also we determine an upper bound of  $\Upsilon_{tstd}$  in terms of  $\chi$  and  $\Upsilon_t$  and prove related results. Additionally, we determine an upper bound of  $\Upsilon_{tstd}$  for perfect graphs.

**Keywords:** Total Dominating Color Transversal number,  $\chi$  – Partition of a graph and Transversal.

**AMS Subject classification code (2010):** 05C15, 05C69

©KY PUBLICATIONS

1. Introduction

In [1], Manoharan introduced the concept of Dominating Color Transversal Set of a graph. We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields  $\chi$  equivalence classes. Transversal of a  $\chi$  - Partition of G is a collection of vertices of G that meets all the color classes of the  $\chi$  - Partition. That is, if T is a subset of V ( the vertex set of G and  $\{V_1, V_2, \dots, V_\chi\}$  is a  $\chi$  - Partition of G then T is called a Transversal of this  $\chi$  - Partition if  $T \cap V_i \neq \emptyset, \forall i \in \{1, 2, \dots, \chi\}$ . Dominating Color Transversal Set of graph G is a Dominating Set with the extra property that it is also Transversal of some such  $\chi$  - Partition of G. Dominating Color Transversal Set of G with minimum cardinality is called Minimum Dominating Color Transversal Set of G and its cardinality, denoted by  $\gamma_{st}(G)$  or just by  $\gamma_{st}$ , is called the Dominating Color Transversal number of G. In [1], Manoharan proved several results regarding this

number. Motivated by this concept, in [2], we analogously defined the concept of Total Dominating Color Transversal Set and Total Dominating Color Transversal number of a graph and proved several results regarding this number.

From the definition it is clear that Total Dominating Color Transversal number must have some relations with Total Domination number and Chromatic number of the graph. We investigate the same in this contribution.

Throughout this paper we assume that graphs are simple, finite, connected and undirected with two or more vertices.

First let us go through some definitions.

## 2. Definitions

**Definition 2.1[4]: (Total Dominating Set)** Let  $G = (V, E)$  be a graph. Then a subset  $S$  of  $V$  (the vertex set of  $G$ ) is said to be a Total Dominating Set of  $G$  if for each  $v \in V$ ,  $v$  is adjacent to some vertex in  $S$ .

**Definition 2.2[4]: (Minimum Total Dominating Set/Total Domination number)** Let  $G = (V, E)$  be a graph. Then a Total Dominating Set  $S$  is said to be a Minimum Total Dominating Set of  $G$  if  $|S| = \text{minimum } \{|D| : D \text{ is a Total Dominating Set of } G\}$ . Here  $S$  is called  $\gamma_t$ -set and its cardinality, denoted by  $\gamma_t(G)$  or just by  $\gamma_t$ , is called the Total Domination number of  $G$ .

**Definition 2.3[2]: ( $\chi$ -partition of a graph)** Proper coloring of vertices of a graph  $G$ , by using minimum number of colors, yields minimum number of independent subsets of vertex set of  $G$  called equivalence classes (also called color classes of  $G$ ). Such a partition of a vertex set of  $G$  is called a  $\chi$ -Partition of the graph  $G$ .

**Definition 2.4[2]: (Transversal of a  $\chi$ -Partition of a graph)** Let  $G = (V, E)$  be a graph with  $\chi$ -Partition  $\{V_1, V_2, \dots, V_\chi\}$ . Then a set  $S \subset V$  is called a Transversal of this  $\chi$ -Partition if  $S \cap V_i \neq \emptyset$ ,  $\forall i \in \{1, 2, 3, \dots, \chi\}$ .

**Definition 2.5[2]: (Total Dominating Color Transversal Set)** Let  $G = (V, E)$  be a graph. Then a Total Dominating Set  $S \subset V$  is called a Total Dominating Color Transversal Set of  $G$  if it is Transversal of at least one  $\chi$ -Partition of  $G$ .

**Definition 2.6[2]: (Minimum Total Dominating Color Transversal Set/Total Dominating Color Transversal number)** Let  $G = (V, E)$  be a graph. Then a Total Dominating Color Transversal Set  $S$  of  $G$  is called a Minimum Total Dominating Color Transversal Set of  $G$  if  $|S| = \text{minimum } \{|D| : D \text{ is a Total Dominating Color Transversal Set of } G\}$ . Here  $S$  is called  $\gamma_{tstd}$ -Set and its cardinality, denoted by  $\gamma_{tstd}(G)$  or just by  $\gamma_{tstd}$ , is called the Total Dominating Color Transversal number of  $G$ .

**Definition 2.7: (Clique)** Let  $G = (V, E)$  be a graph. A clique is a subset of vertices,  $S \subset V$ , such that every two distinct vertices are adjacent. This is equivalent to a condition that the sub graph induced by  $S$ , denoted by  $\langle S \rangle$ , is complete.

**Definition 2.8: (Maximum Clique/ Clique number)** Let  $G = (V, E)$  be a graph. A clique  $S$  is called Maximum clique of  $G$  if its cardinality is maximum among all cliques that the graph  $G$  contains. Here cardinality of  $S$  is called Clique number of  $G$  and it is denoted by  $\omega(G)$  or just by  $\omega$ .

**Definition 2.9: (Perfect Graph)** Let  $G = (V, E)$  be a graph. Then  $G$  is said to be perfect if for every subset  $S$  of  $V$ ,  $\chi(\langle S \rangle) = \omega(\langle S \rangle)$ .

**Definition 2.10: (Pendant Vertex)** Let  $G = (V, E)$  be a graph. Then a vertex  $v$  of  $G$  is called pendant if its degree is one.

**Definition 2.11: (Support Vertex)** Let  $G = (V, E)$  be a graph. Then a vertex  $v$  of  $G$  is called support if it is adjacent to a pendant vertex.

**Definition 2.12: (Isolated Vertex)** Let  $G = (V, E)$  be a graph. Then a vertex  $v$  of  $G$  is isolated vertex if its degree is 0.

**Definition 2.13: (Isolate vertex with respect to a set)** Let  $G = (V, E)$  be a graph and  $S$  be a subset of  $V$ . Then a vertex  $v$  in  $S$  is called isolate with respect to  $S$  if  $v$  is not adjacent to any vertex in  $S$ .

3. Main results

**Result 3.1:** For any graph  $G$ ,  $\chi(G) \leq \gamma_{tstd}(G)$  and  $\gamma_t(G) \leq \gamma_{tstd}(G)$ .

**Theorem 3.2 [2]:** If  $\chi(G) = 2$  then  $\gamma_{tstd}(G) = \gamma_t(G)$ .

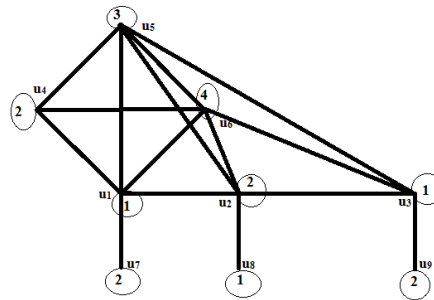
**Theorem 3.3 [2]:** If  $\gamma_t(G) = 2$  then  $\gamma_{tstd}(G) = \chi(G)$ .

First let us note down one basic theorem which will prove very important elementary result later on.

**Result 3.4:** If a graph  $G$  has  $k$  distinct support vertices then  $\gamma_{tstd}(G) \geq k$ .

**Proof:** As any Total Dominating Set must contain all the support vertices of the graph  $G$ ,  $\gamma_t \geq k$  and hence  $\gamma_{tstd}(G) \geq k$ .

**Example 3.5:** For the below given graph  $G$ ,  $\gamma_t(G) < \chi(G)$  but  $\gamma_{tstd}(G) \neq \chi(G)$ .

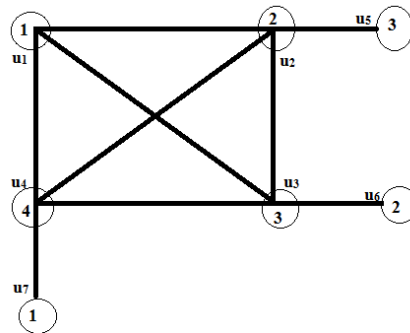


G

Fig.1

$\gamma_t(G) = 3$  and  $\chi(G) = 4$  but  $\gamma_{tstd}(G) = 5 \neq 4 = \chi(G)$ .

**Example 3.6:** For the below given graph  $G$ ,  $\gamma_t(G) < \chi(G)$  and  $\gamma_{tstd}(G) = \chi(G)$ .



G

Fig. 2

$\gamma_t(G) = 3$ ,  $\chi(G) = 4$  and  $\gamma_{tstd}(G) = 4 = \chi(G)$ .

**Theorem 3.7:** Let  $G$  be a graph. Then  $\gamma_{tstd}(G) = \chi(G) > \gamma_t(G)$  if and only if following two conditions hold:

- (1) There exists a Total Dominating Set  $D$  of  $G$  and a  $\chi$  – Partition  $\{V_1, V_2, \dots, V_\chi\}$  of  $G$  such that  $|D \cap V_i| = 1, \forall i = 1, 2, \dots, \chi$ .
- (2) No  $\gamma_t$  - Set of  $G$  is a transversal of any  $\chi$  – Partition of  $G$ .

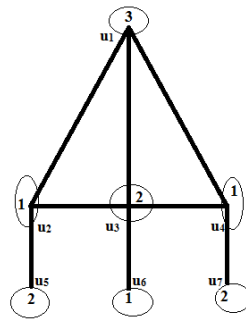
**Proof:** We know that  $\gamma_{tstd}(G) \geq \chi(G)$ .

Suppose (1) is not true. Then  $\gamma_{tstd}(G) > \chi(G)$ , which is contradiction to  $\gamma_{tstd}(G) = \chi(G)$ . Hence (1) is true.

Suppose (2) is not true. Then there exists a  $\gamma_t$  - Set of  $G$  which is a transversal of some  $\chi$  – Partition of  $G$ . Therefore  $\gamma_{tstd}(G) = \gamma_t(G)$  which is contradiction to  $\gamma_{tstd}(G) > \gamma_t(G)$ . Hence (2) is true.

Conversely assume (1) and (2). (1) implies that  $\gamma_{tstd}(G) \leq \chi(G)$  which implies that  $\gamma_{tstd}(G) = \chi(G)$  and (2) implies that  $\gamma_{tstd}(G) > \gamma_t(G)$ . So  $\chi(G) = \gamma_{tstd}(G) > \gamma_t(G)$ .

**Example 3.8:** For the below given graph  $G$ ,  $\gamma_t(G) = \chi(G) = k$  but  $\gamma_{tstd}(G) \neq k$ .

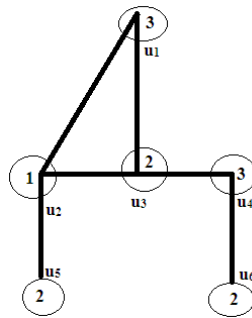


G

Fig.3

$$\Upsilon_t(G) = 3, \chi(G) = 3 \text{ but } \Upsilon_{tstd}(G) = 4 \neq 3 = \chi(G) = \Upsilon_t(G).$$

**Example 3.9:** For the below given graph G,  $\Upsilon_{tstd}(G) = \chi(G) = \Upsilon_t(G)$ .



G

Fig.4

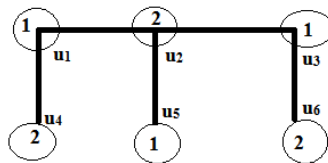
$$\Upsilon_{tstd}(G) = \chi(G) = \Upsilon_t(G) = 3.$$

**Theorem 3.10:** Let G be a graph. Then  $\Upsilon_{tstd}(G) = \chi(G) = \Upsilon_t(G)$  if and only if there exists a Minimum Total Dominating Set D of G and a  $\chi$ -Partition  $\{V_1, V_2, \dots, V_\chi\}$  of G such that  $|D \cap V_i| = 1, \forall i = 1, 2, \dots, \chi$ .

**Proof:** Obvious.

**Remark 3.11:** Let G be a graph. If there exists a  $\Upsilon_t$ -Set of G such that it is a transversal of some  $\chi$ -Partition of G then it is not necessary that  $\Upsilon_{tstd}(G) = \chi(G) = \Upsilon_t(G)$ . We give the following example to justify this.

**Example 3.12:**



G

Fig. 5

Here  $\{u_1, u_2, u_3\}$  is a  $\Upsilon_t$ -set of G. Clearly it is a transversal of the defined  $\chi$ -Partition of G.

Also  $\Upsilon_{tstd}(G) = 3 = \Upsilon_t(G) \neq \chi(G) = 2$ .

**Example 3.13:** For the below given graph G,  $\chi(G) < \Upsilon_t(G)$  but  $\Upsilon_{tstd}(G) \neq \Upsilon_t(G)$ .

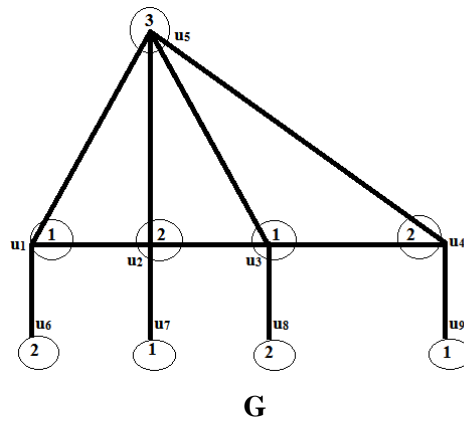


Fig. 6

$$\chi(G) = 3, Y_t(G) = 4 \text{ but } Y_{tstd}(G) = 5 \neq Y_t(G).$$

**Example 3.14:** For the below given graph G,  $\chi(G) < Y_t(G)$  and  $Y_{tstd}(G) = Y_t(G)$ .

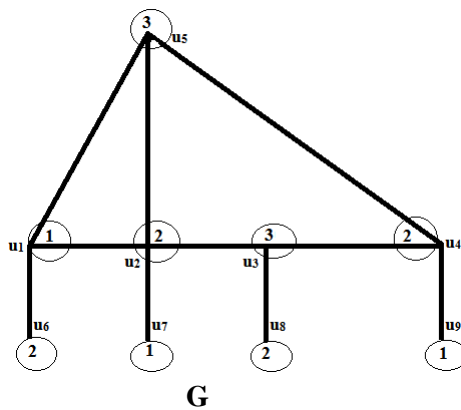


Fig. 7

$$\chi(G) = 3, Y_t(G) = 4 \text{ and } Y_{tstd}(G) = 4 = Y_t(G).$$

**Theorem 3.15:** Let G be a graph. Then  $Y_{tstd}(G) = Y_t(G) > \chi(G)$  if and only if following two conditions holds:

- (1) There exists a  $Y_t$  - Set of G such that it is a transversal of some  $\chi$  - Partition  $\{V_1, V_2, \dots, V_\chi\}$  of G.
- (2) For every  $Y_t$  - Set D of G satisfying (1),  $|D \cap V_i| > 1$  for some  $i \in \{1, 2, 3, \dots, \chi\}$ .

**Proof:** Obvious.

Now we discuss upper bound of  $Y_{tstd}$  in terms of  $Y_t$  and  $\chi$ . We provide different examples to justify our results.

**Theorem 3.16:** For every graph G,  $Y_{tstd}(G) \leq Y_t(G) + \chi(G) - 2$ .

**Proof:** Let S be a  $Y_t$  - set of G. Then S has at least two vertices that are adjacent and so they are in different color classes for every  $\chi$  - Partition of G. Adding, to S, one vertex from each remaining  $\chi - 2$  color classes yields a Total Dominating Color Transversal Set of G. Hence  $Y_{tstd}(G) \leq Y_t(G) + \chi(G) - 2$ .

**Remark 3.17:** Above given upper bound is sharp. Following example 3.18 justifies this.

**Example 3.18:** Let G be a graph. If  $\chi(G) = 2$  or  $Y_t(G) = 2$  then  $Y_{tstd}(G) = Y_t(G) + \chi(G) - 2$ .

**Proof:** If  $\chi(G) = 2$  then  $Y_{tstd}(G) = Y_t(G)$  by theorem 3.2. So  $Y_{tstd}(G) = Y_t(G) + 2 - 2 = Y_t(G) + \chi(G) - 2$ . If  $Y_t(G) = 2$  then  $Y_{tstd}(G) = \chi(G)$  by theorem 3.3. So  $Y_{tstd}(G) = \chi(G) + 2 - 2 = Y_t(G) + \chi(G) - 2$ .

**Remark 3.19:** There are graphs G for which  $\chi(G) > 2$  and  $Y_t(G) > 2$  but still  $Y_{tstd}(G) = Y_t(G) + \chi(G) - 2$ . We give the following example 3.20 to justify this.

**Example 3.20:** Consider graph G as in Fig. 3.  $Y_{tstd}(G) = 4$ ,  $Y_t(G) = 3$  and  $\chi(G) = 3$ . Trivially  $Y_{tstd}(G) = 4 = 3 + 3 - 2 = Y_t(G) + \chi(G) - 2$ .

**Theorem 3.21:** Let G be a graph. Then  $Y_{tstd}(G) = Y_t(G) + \chi(G) - 2$  if and only if following two conditions are satisfied:

- (1) Every  $Y_t$  - Set of G intersects exactly two color classes of every  $\chi$  - Partition of G.
- (2) Every  $Y_t$  - Set of G is contained in a  $Y_{tstd}$  - Set of G.

**Proof:** Assume  $Y_{tstd}(G) = Y_t(G) + \chi(G) - 2$ .

(1) Suppose there exists a  $Y_t$  - Set D of G such that it intersects at least three color classes of some  $\chi$  - Partition  $\{V_1, V_2, \dots, V_\chi\}$  of G. Then adding, to D, one vertex from each remaining at most  $\chi - 3$  color classes, yields a Total Dominating Color Transversal Set of G. Hence  $Y_{tstd}(G) \leq |D| + \chi(G) - 3 < Y_t(G) + \chi(G) - 2$ . So we get contradiction. Hence (1) is true.

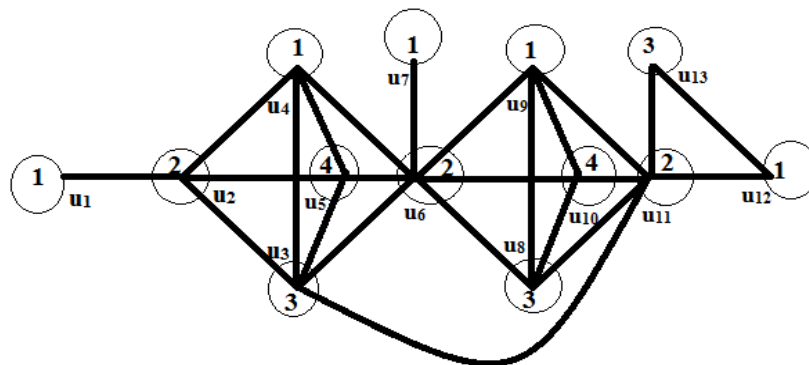
(2) Consider a  $Y_t$  - Set D of G. We know, by (1), that every  $Y_t$  - Set meet exactly two color classes of every  $\chi$  - Partition of G. So by adding, to D, one vertex from each remaining  $\chi - 2$  color classes yields a Total Dominating Color Transversal Set of G. Note that this set is minimum with this property. So every  $Y_t$  - Set of G is contained in a  $Y_{tstd}$  - Set of G.

Conversely, assume (1) and (2). Clearly  $\chi(G) - 2 \leq Y_{tstd}(G) - Y_t(G)$ . So  $Y_{tstd}(G) \geq Y_t(G) + \chi(G) - 2$ . Therefore by theorem 3.16,  $Y_{tstd}(G) = Y_t(G) + \chi(G) - 2$ .

**Corollary 3.22:** Let G be a graph. If there exists  $Y_t$  - Set of G that intersects k color classes of some  $\chi$  - Partition of G then  $Y_{tstd}(G) \leq Y_t(G) + \chi(G) - k$ .

**Remark 3.23:** Let G be a graph. If every  $Y_t$  - Set of G intersects exactly two color of every  $\chi$  - Partition of G then it is not necessary that  $Y_{tstd}(G) = Y_t(G) + \chi(G) - 2$ . Following example 3.24 justifies this.

**Example 3.24:**



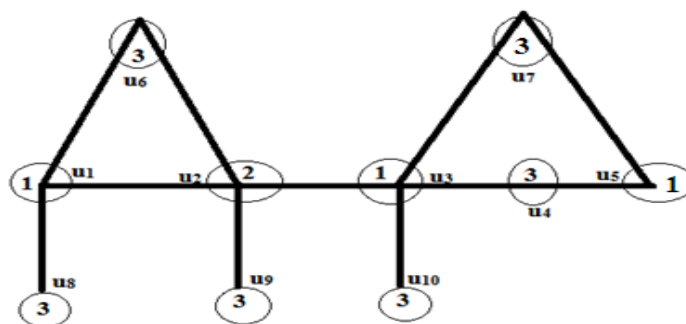
G

Fig. 8

Here we note that  $\{u_2, u_6, u_{11}\}$  is a color class of every  $\chi$  - Partition of G. Also G has unique  $Y_t$  - Set  $D = \{u_2, u_3, u_6, u_{11}\}$  and it intersects exactly two color classes of every  $\chi$  - Partition of G. Clearly,  $\{u_2, u_5, u_6, u_{12}, u_{13}\}$  is a  $Y_{tstd}$  - Set of G. Also  $\chi(G) = 4$ . Hence  $Y_{tstd}(G) = 5 \neq 6 = Y_t(G) + \chi(G) - 2 = 4 + 4 - 2$ .

**Remark 3.25:** If every  $Y_t$  - Set of  $G$  is contained in a  $Y_{tstd}$  - Set of  $G$ , then it is not necessary that  $Y_{tstd}(G) = Y_t(G) + \chi(G) - 2$ . See the following example 3.26.

**Example 3.26:** Consider the graph  $G$  given in Fig. 9.



G

Fig. 9

Graph  $G$  has exactly two  $Y_t$  - Sets. They are  $D_1 = \{u_1, u_2, u_3, u_4\}$  and  $D_2 = \{u_1, u_2, u_3, u_7\}$ . Note that both are  $Y_{tstd}$  - Sets of  $G$ . So we can say that both  $D_1$  and  $D_2$  are contained a  $Y_{tstd}$  - Set of  $G$ . But  $Y_{tstd}(G) = 4 \neq 5 = 4 + 3 - 2 = Y_t(G) + \chi(G) - 2$ .

Now we obtain an upper bound of  $Y_{tstd}$  number for Perfect graphs. We assume that complement graph  $\bar{G}$  have no isolated vertex.

**Theorem 3.27 [8]:** A graph  $G$  is Perfect if and only if  $\bar{G}$  is Perfect.

**Theorem 3.28:** If  $G$  is a Perfect graph then  $Y_{tstd}(G) \leq Y_t(G) + \omega(G) - 2$ .

**Proof:** Obvious as for a Perfect  $G$ ,  $\chi(G) = \omega(G)$  and by theorem 3.16.

**Theorem 3.29:** If  $G$  is a Perfect graph then  $Y_{tstd}(\bar{G}) \leq 2\omega(G) + \omega(\bar{G}) - 2$ .

**Proof:** It is obvious to say that the theorem 3.16 is also true for a disconnected graph  $G$ .

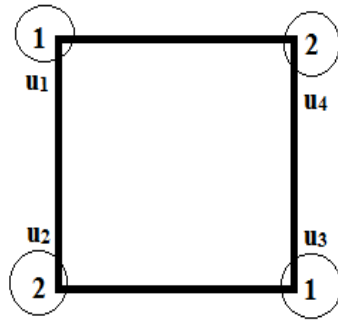
By theorem 3.16,  $Y_{tstd}(\bar{G}) \leq Y_t(\bar{G}) + \chi(\bar{G}) - 2$ . By theorem 3.27,  $\chi(\bar{G}) = \omega(\bar{G})$ . And hence  $Y_{tstd}(\bar{G}) \leq Y_t(\bar{G}) + \omega(\bar{G}) - 2$ .

Claim:  $Y_t(\bar{G}) \leq 2\omega(G)$

Suppose  $S \subset V$  is a maximum clique of  $G$ . Hence  $|S| = \omega(G)$ . Also  $S$  is a maximum independent set in  $\bar{G}$ . Therefore  $S$  is a dominating set in  $\bar{G}$ . If  $S$  is not a total dominating set in  $\bar{G}$  then  $S$  has at most  $|S|$  isolates. As  $\bar{G}$  is a graph with no isolated vertex, each vertex of  $S$  has adjacent vertex in  $\bar{G}$ . So adding at most  $|S|$  vertices the resultant set becomes a total dominating set of  $\bar{G}$ . So  $Y_t(\bar{G}) \leq 2|S| = 2\omega(G)$ . Hence  $Y_{tstd}(\bar{G}) \leq 2\omega(G) + \omega(\bar{G}) - 2$ .

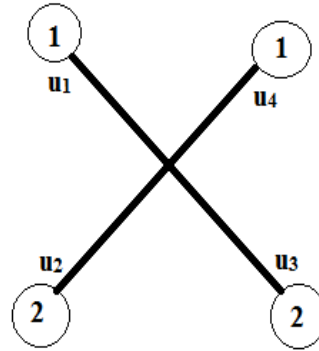
**Remark 3.30:** Bound of  $Y_{tstd}(\bar{G})$  in theorem 3.29 is sharp. Following example 3.31 justifies this.

**Example 3.31:** We know that every bipartite graph is Perfect.



$C_4$

Fig. 10



$\overline{C_4}$

Fig. 11

$\gamma_{tstd}(\overline{C_4}) = 4$ ,  $\omega(C_4) = 2$  and  $\omega(\overline{C_4}) = 2$ . So  $\gamma_{tstd}(\overline{C_4}) = 4 = 2\omega(C_4) + \omega(\overline{C_4}) - 2$ .

**Theorem 3.32:** If  $G$  is a Perfect graph then  $\gamma_{tstd} = k - 2$ , where

$$k = \min \{ \gamma_t(G) + \omega(G), 2\omega(\overline{G}) + \omega(G) \}.$$

**Proof:** By theorem 3.16,  $\gamma_{tstd}(G) \leq \gamma_t(G) + \chi(G) - 2 = \gamma_t(G) + \omega(G) - 2$  ..... (1)

By theorem 3.29,  $\gamma_{tstd}(\overline{G}) \leq 2\omega(G) + \omega(\overline{G}) - 2$ .....(2).

As  $G$  is perfect,  $\overline{G}$  is also Perfect (by theorem 3.27). Hence applying inequality (2) to  $\overline{G}$ , we obtain the following inequality:  $\gamma_{tstd}(G) = \gamma_{tstd}(\overline{\overline{G}}) \leq 2\omega(\overline{G}) + \omega(G) - 2$ .....(3)

From (1) and (3),  $\gamma_{tstd} = k - 2$ , where  $k = \min \{ \gamma_t(G) + \omega(G), 2\omega(\overline{G}) + \omega(G) \}$ .

**4. References**

- [1]. R. L. J. Manoharan, Dominating colour transversals in graphs, Bharathidasan University, September, 2009.
- [2]. D. K. Thakkar and A. B. Kothiya, Total Dominating Color Transversal number of Graphs, Annals of Pure and Applied Mathematics, Vol. 11(2), 2016, 39 – 44.
- [3]. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of domination in graphs, Marcel Dekker, NewYork, 1998.
- [4]. Michael A. Henning and Andres Yeo, Total Domination in Graphs, Springer, 2013.
- [5]. Sandi Klavzar, “Coloring Graph Products – A survey”, Elsevier Discrete Mathematics 155 (1996), 135- 145.
- [6]. R. Balakrishnan and K. Ranganathan, “A Textbook of Graph Theory, Springer”, New York, 2000.
- [7]. Goksen BACAk, “Vertex Color of a Graph”, Master of Science Thesis, ‘IZM’IR, December, 2004.
- [8]. R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Springer, New York, 2000.