Vol.4.Issue.4.2016 (October -December)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

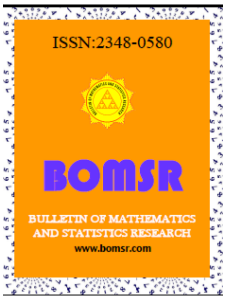
http://www.bomsr.com
Email:editorbomsr@gmail.com

ISSN:2348-0580

RESEARCH ARTICLE

ON CAUCHY SEQUENCES IN INTUITIONISTIC FUZZY METRIC SPACES

SERVET KUTUKCU[1*], SUNEEL KUMAR[2]
[1]Department of Mathematics, Ondokuz Mayis University 55139 Kurupelit, Samsun, Turkey
*E-mail: skutukcu@omu.edu.tr
[2]Goverment Secondary High School Sanyasiowala PO-Jaspur 244712 (U.S. Nagar), Uttarakhand, India



ABSTRACT
Intuitionistic fuzzy metric spaces in literature are different and the conditions of fixed point theorem in one of those spaces are inadequate, and the proof of the theorem need to be corrected.

Keywords: Contraction mapping, Fixed point, Cauchy sequence.
AMS(2010) Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In [1], the authors proved an intuitionistic fuzzy form of Banach contraction theorem in an intuitionistic fuzzy metric space. In fact, there are some errors.

In this paper, to explain those errors, we will make the meanings of Cauchy sequence clear, so point out that the definition of Cauchy sequence given in [1] for an intuitionistic fuzzy metric space is weaker than proposed in [2], and hence it is essential to modify in a particular way that definition to get better results in intuitionistic fuzzy metric spaces. We have obtained some results in this connection.

2. CAUCHY SEQUENCES IN INTUITIONISTIC FUZZY METRIC SPACES

In this paper, the author has adopted the following definition of intuitionistic fuzzy metric space.

Definition 2.1 [1]. A 5-tuble $(X,M,N,*,\diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M,N are fuzzy sets on $X^2\times[0,\infty)$ satisfying the following conditions:

(i) $M(x,y,t) + N(x,y,t) \leq 1$,

(ii) $M(x,y,0) = 0$,

(iii) $M(x,y,t) = 1$ for all $t > 0$ iff $x = y$,

(iv) $M(x,y,t) = M(y,x,t)$,

(v) $M(x,z,t+s) \geq M(x,y,t)*M(y,z,s)$ for all $t,s > 0$,

(vi) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous,

(vii) $\lim_{t\to\infty} M(x,y,t) = 1$,

(viii) $N(x,y,0) = 1$,

(ix) $N(x,y,t) = 0$ for all $t > 0$ iff $x = y$,

(x) $N(x,y,t) = N(y,x,t)$,

(xi) $N(x,z,t+s) \le N(x,y,t) \diamond N(y,z,s)$ for all $t,s > 0$,

(xii) $N(x,y,.) : [0,\infty) \to [0,1]$ is right continuous,

(xiii) $\lim_{t\to\infty} N(x,y,t) = 0$.

Definition 2.2 [1]. A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is called a Cauchy sequence if and only if $\lim_{n\to\infty} M(x_{n+p},x_n,t)=1$ and $\lim_{n\to\infty} N(x_{n+p},x_n,t)=0$, for each t>0 and p>0.

Definition 2.3 [2]. A sequence {x_{n}} in an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is called a Cauchy sequence if and only if for each $\varepsilon\in(0,1)$ and $t>0$ there exists $n_0\in\mathbb{N}$ such that M(x_{n},x_{m},t)>1-ε and N(x_{n},x_{m},t)<ε, for all $n,m\ge n_0$.

Remark 2.1. It is clear from Definitions 2.2 and 2.3 that the notion of Cauchy sequence given in [1] is weaker than the one proposed in [2]. So, it would have been better if the authors have used the terms "weak intuitionistic fuzzy Cauchy sequence" and "weak complete intuitionistic fuzzy metric space" instead of Cauchy sequence and complete intuitionistic fuzzy metric space in [1]. It is also clear from Definitions 2.2 and 2.3 that every complete intuitionistic fuzzy metric space defined in [2] need not to be a complete intuitionistic fuzzy metric space defined in [1]. It can be seen easly from the following illustration.

Example 2.1. Let $X=\mathbb{R}$, the set of all real numbers. For $x,y\in X$ and $t\ge 0$, define

$$M(x,y,t)=\begin{cases}\dfrac{t}{t+|x-y|}, & \text{if } t>0,\\ 0, & \text{if } t=0,\end{cases} \quad \text{and} \quad N(x,y,t)=\begin{cases}\dfrac{|x-y|}{t+|x-y|}, & \text{if } t>0,\\ 1, & \text{if } t=0.\end{cases}$$

It is clear that (M,N) is an intuitionistic fuzzy metric on $\mathbb{R}$.

Let $x_n = 1+(1/2)+(1/3)+...+(1/n)$, for $n\in\mathbb{N}$. Then,

$$M(x_{n+p},x_n,t)=\frac{t}{t+1/(n+1)+...+1/(n+p)}\to 1,$$

$$N(x_{n+p},x_n,t)=\frac{1/(n+1)+...+1/(n+p)}{t+1/(n+1)+...+1/(n+p)}\to 0$$

as $n\to\infty$ for all p>0. Hence, $\{x_n\}$ is a Cauchy sequence in the sense of Definition 2.2 but obviously $\{x_n\}$ is not a Cauchy sequence in the sense of Definition 2.3. In fact, if $\{x_n\}$ is a Cauchy sequence in the sense of Definition 2.3 then it is a Cauchy sequence in the standard metric space $\mathbb{R}$ which gives a contradiction. Further, if $\mathbb{R}$ is intuitionistic fuzzy complete then there exists $x\in\mathbb{R}$ such that $M(x_n,x,t)\to 1$ and $N(x_n,x,t)\to 0$ as $n\to\infty$. It follows that $t/(t+|x_n-x|)\to 1$ and $(|x_n-x|)/(t+|x_n-x|)\to 0$ as $n\to\infty$ and $|x_n-x|\to 0$ as $n\to\infty$ so $x_n\to x$ in $\mathbb{R}$ which is not true. Hence, to make $\mathbb{R}$ complete intuitionistic fuzzy metric space, we should take Definition 2.3 instead of Definition 2.2 or we should modify Definition 2.2 as follows:

Definition 2.4. A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is said to be a Cauchy sequence if and only if $M(x_{n+p},x_n,t)\to 1$ and $N(x_{n+p},x_n,t)\to 0$ as $n\to\infty$ uniformly on $p\in\mathbb{N}$, for each t>0.

Remark 2.2. It is clear that the definition of Cauchy sequence given in [1] for an intuitionistic fuzzy metric space, that is Definition 2.2, is weaker than Definition 2.4. In fact, if $\{x_n\}$ is a convergent sequence in an intuitionistic fuzzy metric space (X,M, N,∗,◇), without the loss of the generality, we can assume $x_n \to x_0$ ($n\to\infty$), then we have,

$$M\left(x_{n+p}, x_n, t\right) \geq M(x_{n+p}, x_0, t/2) * M(x_n, x_0, t/2) \to 1,$$

$$N\left(x_{n+p}, x_n, t\right) \leq N(x_{n+p}, x_0, t/2) \diamond N(x_n, x_0, t/2) \to 0$$

as $n\to\infty$ for any t>0. Therefore, $\lim_{n\to\infty} M(x_{n+p},x_n,t)=1$ and $\lim_{n\to\infty} N(x_{n+p},x_n,t)=0$ (t>0) uniformly with respect to p∈ℕ. This shows that a completely intuitionistic fuzzy metric space does not exist if we consider the Cauchy sequence in an intuitionistic fuzzy metric space according to [1]. That is to say, it is meaningless to discuss the fixed point theorem in an intuitionistic fuzzy metric space if the discuss is set to this situation.

In order to state our results, let us list the main result in [1].

Theorem 2.1 [1]. Let (X,M, N,∗,◇) be a complete intuitionistic fuzzy metric space. Let T : X → X be a mapping satisfying

M(Tx,Ty,kt) ≥ M(x,y,t) and N(Tx,Ty,kt) ≤ N(x,y,t)

for all x,y in X, 0<k<1. Then T has a unique fixed point.

Our aim in this paper is to show that the conditions of above theorem are inadequate. We also would like to point out that a number of results in [1] must be corrected. Similar comments have discussed in fuzzy metric spaces (see [3,5,6]).

In fact, in the hypothesis of Theorem 2.1, we understand that the inequalities are satisfied for all k∈(0,1), but k is a contractive constant so the inequalities must be satisfied by some k∈(0,1). Also the proof of Theorem 2.1 is erroneous because it is based on the following:

$$M(x_n, x_{n+p}, t) \geq \ldots \geq M(x_0, x_1, t/(pk^n)) * \overset{(p)}{\ldots} * M(x_0, x_1, t/(pk^{n+p-1})),$$

$$N(x_n, x_{n+p}, t) \leq \ldots \leq N(x_0, x_1, t/(pk^n)) \diamond \overset{(p)}{\ldots} \diamond N(x_0, x_1, t/(pk^{n+p-1}))$$

and so

$$(2.1) \quad M\left(x_n, x_{n+p}, t\right) \to 1 \text{ and } N\left(x_n, x_{n+p}, t\right) \to 0.$$

Therefore $\{x_n\}$ is a Cauchy sequence in X and hence it converges to x in X.

It should be noted that $\{x_n\}$ satisfies the (2.1), but that $\{x_n\}$ need not be a Cauchy sequence in X. First of all, we point out that the proof is false under ∗=min and ◇=max [4]. In fact, by (2.1), we have

$$(2.2) \quad M(x_n, x_{n+p}, t) \geq M(x_0, x_1, t/(pk^n)) \text{ and } N(x_n, x_{n+p}, t) \leq N(x_0, x_1, t/(pk^n)).$$

Since the positive integer p was arbitrary, choose p such that $p=1/k^n$, we have

$$(2.3) \quad M(x_0, x_1, t/(pk^n)) \leq M(x_0, x_1, t) \text{ and } N(x_0, x_1, t/(pk^n)) \geq N(x_0, x_1, t).$$

From (2.2) and (2.3), it is easy to see that we have not yet shown that $\{x_n\}$ is a Cauchy sequence in X.

Next, the $\{x_n\}$ need not to be a Cauchy sequence under ∗=product and ◇=max. To this end, we give a counterexample.

Example 2.2. Let X=ℝ. For x,y∈X and t∈ℝ, define M(x,y,t)=H(t-|x-y|) and N(x,y,t)=1-H(t-|x-y|), where $H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$

Then, (X,M, N,∗,◇) is a complete intuitionistic fuzzy metric space, where ∗=product and ◇=max are continuous t-norm and continuous t-conorm, respectively. In fact, it is easy to verify that conditions (i)-(iv), (vi)-(x), (xii) and (xiii) of Definition 2.1 are satisfied. For all x,y,z∈X and t,s∈ℝ, by

$$\begin{aligned} M(x,z,t+s) &= H(t+s-|x-z|) \\ &\geq H(t+s-|x-y|-|z-y|) \\ &\geq H(t-|x-y|).H(s-|z-y|), \end{aligned}$$

$$\begin{aligned} N(x,z,t+s) &= 1-H(t+s-|x-z|) \\ &\leq 1-H(t-|x-y|).H(s-|z-y|) \\ &\leq 1-\min\{H(t-|x-y|),H(s-|z-y|)\} \\ &= \max\{1-H(t-|x-y|),1-H(s-|z-y|)\} \end{aligned}$$

and we know that conditions (v) and (xi) of Definition 2.1 hold. Hence, (X,M, N,∗,◇) is an intuitionistic fuzzy metric space. It is easy to see that (X,M, N,∗,◇) is a complete intuitionistic fuzzy metric space. Set

$$x_n = \sum_{i=1}^{n} \frac{1}{i},\ n=1,2,3,\dots .$$

Then $\{x_n\} \subset X$. For every positive integer p and t>0, we have

$$M(x_n, x_{n+p}, t) = H(t-|x_{n+p}-x_n|) = H\left(t - \sum_{i=n+1}^{n+p} \frac{1}{i}\right) \to 1,$$

$$N(x_n, x_{n+p}, t) = 1-H(t-|x_{n+p}-x_n|) = 1-H\left(t - \sum_{i=n+1}^{n+p} \frac{1}{i}\right) \to 0$$

as $n \to \infty$. Obviously, the $\{x_n\}$ is not a Cauchy sequence in X.

This example shows that Theorem 2.1 is not true under ∗=product and ◇=max.

Remark 2.3. Obviously, for every p∈ℕ and t>0, $M(x_n,x_{n+p},t) \to 1$ and $N(x_n,x_{n+p},t) \to 0$ in Theorem 2.1, but convergence is not uniform on p∈ℕ.

For the above reasons, it is easy to see that the conditions of intuitionistic fuzzy Banach contraction theorem in [1] are inadequate, and the proof is false. As an immediate correction of Theorem 2.1, we have

Theorem 2.2. Suppose that all conditions of Theorem 2.1 are satisfied, and t-norm ∗ and t-conorm ◇ satisfy for any a,b∈[0,1], a∗b=min{a,b} and a◇b=max{a,b} (or a∗a≥a and (1-a)◇(1-a)≤(1-a)), respectively. Then conclusion of Theorem 2.1 remains true.

Proof: We need only prove the $M(x_n,x_{n+p},t) \to 1$ and $N(x_n,x_{n+p},t) \to 0$ (∀t>0) as $n \to \infty$ uniformly on p∈ℕ. In fact, by proof of Theorem 2.1, it is easy to know that

$$(2.4)\quad M(x_n, x_{n+1}, t) \geq M(x_0, x_1, t/(k^n)) \text{ and } N(x_n, x_{n+1}, t) \leq N(x_0, x_1, t/(k^n)).$$

For p∈ℕ, by (2.4) and our assumptions, we have

$$\begin{aligned} (2.5)\quad M(x_n, x_{n+p}, t) &\geq M(x_n, x_{n+1}, (1-k)t) * M(x_{n+1}, x_{n+p}, kt) \\ &\geq M(x_0, x_1, (1-k)t/k^n) * M(x_n, x_{n+p-1}, t) \geq \dots \\ &\geq M(x_0, x_1, (1-k)t/k^n) * M(x_0, x_1, (1-k)t/k^n) * \dots \\ &\quad * M(x_0, x_1, (1-k)t/k^n) \\ &\geq M(x_0, x_1, (1-k)t/k^n), \end{aligned}$$

$$(2.6)\quad N(x_n, x_{n+p}, t) \leq N(x_n, x_{n+1}, (1-k)t) \Diamond N(x_{n+1}, x_{n+p}, kt)$$
$$\leq N(x_0, x_1, (1-k)t/k^n) \Diamond N(x_n, x_{n+p-1}, t) \leq \ldots$$
$$\leq N(x_0, x_1, (1-k)t/k^n) \Diamond N(x_0, x_1, (1-k)t/k^n) \Diamond \ldots$$
$$\Diamond N(x_0, x_1, (1-k)t/k^n)$$
$$\leq N(x_0, x_1, (1-k)t/k^n),$$

From (2.5) and (2.6), it is easy to see that $M(x_n,x_{n+p},t)\to 1$ and $N(x_n,x_{n+p},t)\to 0$ ($\forall t>0$) as $n\to\infty$ uniformly on $p\in\mathbb{N}$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. By the same way as in Theorem 2.1, the rest can be proved.

Question: Can we replace in the statement of Theorem 2.2, the conditions on t-norm $*$ and t-conorm $\Diamond$ with $\sup_{a<1} a*a=1$ and $\inf_{a<1} (1-a)\Diamond(1-a)=0$, respectively?

3. REFERENCES

[1] Alaca C., Turkoglu D., Yildiz C., 2006, Fixed points in intuitionistic fuzzy metric spaces, *Chaos, Solitons Fractals* **29**, 1073-1078.

[2] Park J.H., 2004, Intuitionistic fuzzy metric spaces, *Chaos, Solitons Fractals* **22**, 1039-1046.

[3] Romaguera S., Tirado P., 2007, On fixed point theorems in intuitionistic fuzzy metric spaces, *Int. J. Nonlin. Sci. Num*. **8**, 233-238.

[4] Schweizer B., Sklar A., 1960, Statistical metric spaces, *Pacific J. Math.* **10**, 314-334.

[5] Song G., 2003, Comments on "A common fixed point theorem in a fuzzy metric space", *Fuzzy Sets Syst*. **135,** 409-413.

[6] Vasuki R., Veeramani P., 2003, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, *Fuzzy Sets Syst*. **135**, 415-417.