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RESEARCH ARTICLE

OPTIMAL CONVEX COMBINATION BOUNDS OF HARMONIC AND CENTROIDAL MEANS FOR NEUMAN-SÁNDOR MEAN

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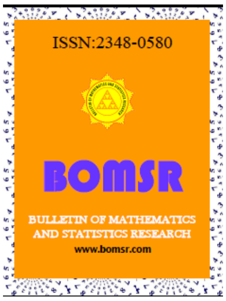

ABSTRACT

In this paper, we present the least value α and the greatest value β such that the double inequality

$$\alpha H(a,b)+(1-\alpha)\overline{C}(a,b)<M(a,b)<\beta H(a,b)+(1-\beta)\overline{C}(a,b)$$

holds for all $a,b>0$ with $a\neq b$. Here $H(a,b)$, $\overline{C}(a,b)$ and $M(a,b)$ denote the harmonic, centroidal and Neuman-Sándor means of two positive numbers a and b, respectively.

Keywords: Inequality, Neuman-Sándor mean, Harmonic mean, Arithmetic mean, Centroidal mean

1. INTRODUCTION

For $a,b>0$ with $a\neq b$, the Neuman-Sándor mean $M(a,b)$ [1] was defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1}x=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for $M(a,b)$ can be found in the literature [1, 2]. Let $H(a,b)=(2ab)/(a+b)$, $G(a,b)=\sqrt{ab}$,

$L(a,b)=(a-b)/(\log a-\log b)$, $N(a,b)=\left[(\sqrt{a}+\sqrt{b})/2\right]^2$, $P(a,b)=(a-b)/(4\tan^{-1}\sqrt{a/b}-\pi)$,

$A(a,b)=(a+b)/2$,

$T(a,b)=(a-b)/[2\tan^{-1}(a-b)/(a+b)]$, $\overline{C}(a,b)=2/3\cdot(a^2+ab+b^2)/(a+b)$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, square-root, first Seiffert, arithmetic, second Seiffert, centroidal, quadratic and contra-harmonic harmonic, geometric, logarithmic,, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\min\{a,b\}<H(a,b)<G(a,b)<L(a,b)<N(a,b)<P(a,b)<A(a,b)$$
$$<M(a,b)<T(a,b)<\overline{C}(a,b)<Q(a,b)<C(a,b)<\max(a,b) \quad (1.2)$$

hold for all $a,b>0$ with $a\neq b$.

In [3], Neuman proved that the double inequalities

$$\alpha Q(a,b)+(1-\alpha)A(a,b)<M(a,b)<\beta Q(a,b)+(1-\beta)A(a,b) \quad (1.3)$$

and

$$\lambda C(a,b)+(1-\lambda)A(a,b)<M(a,b)<\mu C(a,b)+(1-\mu)A(a,b) \quad (1.4)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq[1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]=0.3249\cdots$, $\beta\geq 1/3$,

$\lambda\leq[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$ and $\mu\geq 1/6$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b)<M(a,b)<L_2(a,b) \quad (1.5)$$

holds for all $a,b>0$ with $a\neq b$, where $L_p(a,b)=[(a^{p+1}-b^{p+1})/(p+1)(a-b)]^{1/p}$ $(p\neq -1,0)$, $L_0(a,b)=1/e[(a^a)/b^b]^{1/(a-b)}$ and $L_{-1}(a,b)=(a-b)/(\log a-\log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0=1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p}=\log(1+\sqrt{2})$.

In [5], Chu etc proved that the double inequalities

$$\alpha_1 L(a,b)+(1-\alpha_1)Q(a,b)<M(a,b)<\beta_1 L(a,b)+(1-\beta_1)Q(a,b) \quad (1.6)$$

and

$$\alpha_2 L(a,b)+(1-\alpha_2)C(a,b)<M(a,b)<\beta_2 L(a,b)+(1-\beta_2)C(a,b) \quad (1.7)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\geq 2/5$, $\beta_1\leq 1-1/[\sqrt{2}\log(1+\sqrt{2})]=0.1977\cdots$, $\alpha_2\geq 5/8$ and $\beta_2\leq 1-1/[2\log(1+\sqrt{2})]=0.4327\cdots$.

The main purpose of this paper is to found the least value α and the greatest value β such that the double inequality

$$\alpha H(a,b)+(1-\alpha)\overline{C}(a,b)<M(a,b)<\beta H(a,b)+(1-\beta)\overline{C}(a,b)$$

holds for all $a,b>0$ with $a\neq b$.

2. MAIN RESULTS

THEOREM 2.1. The double inequality

$$\alpha H(a,b)+(1-\alpha)\overline{C}(a,b)<M(a,b)<\beta H(a,b)+(1-\beta)\overline{C}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\geq 1-3/[4\log(1+\sqrt{2})]=0.1490\cdots$ and $\beta\leq 1/8$.

Proof. Without loss of generality, we assume that $a>b>0$.

Let $x=(a-b)/(a+b)\in(0,1)$, $\lambda=1-3/[4\log(1+\sqrt{2})]=0.1490\cdots$ and $p\in\{1/8,\lambda\}$. Then

$$\frac{H(a,b)}{A(a,b)} = 1 - x^2,\ \frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1} x},\ \frac{\overline{C}(a,b)}{A(a,b)} = 1 + \frac{x^2}{3}. \tag{2.1}$$

Firstly, we prove that

$$M(a,b) < \frac{1}{8}H(a,b) + \frac{7}{8}\overline{C}(a,b), \tag{2.2}$$

and

$$\lambda A(a,b) + (1-\lambda)\overline{C}(a,b) < M(a,b), \tag{2.3}$$

From (2.1) we have

$$\frac{pH(a,b) + (1-p)\overline{C}(a,b) - M(a,b)}{A(a,b)} = \frac{3 + (1-4p)x^2}{3\log(x + \sqrt{1+x^2})} D_p(x), \tag{2.4}$$

where

$$D_p(x) = \log(x + \sqrt{1+x^2}) - \frac{3x}{3 + (1-4p)x^2}. \tag{2.5}$$

(2.5) lead to

$$\lim_{x \to 0^+} D_p(x) = 0, \tag{2.6}$$

$$\lim_{x \to 1^-} D_p(x) = \log(1+\sqrt{2}) - \frac{3}{4(1-p)}, \tag{2.7}$$

and

$$D_p'(x) = \frac{1}{\left[3 + (1-4p)x^2\right]^2} F_p(x), \tag{2.8}$$

where

$$F_p(x) = \frac{(1-4p)^2 x^4 + 6(1-4p)x^2 + 9}{\sqrt{1+x^2}} + 3(1-4p)x^2 - 9. \tag{2.9}$$

Let $x = \sqrt{t}$, $t \in (0,1)$, then

$$F_p(x) = \frac{(1-4p)^2 t^2 + 6(1-4p)t + 9}{\sqrt{1+t}} + 3(1-4p)t - 9 = G_p(t). \tag{2.10}$$

Now we distinguish between two cases:

Case 1. $p = 1/8$. From (2.10) one has

$$G_{1/8}(t) = \frac{(t+6)^2}{4\sqrt{1+t}} - \left(9 - \frac{3}{2}t\right) = \frac{\left[\frac{(t+6)^2}{4\sqrt{1+t}}\right]^2 - \left(9 - \frac{3}{2}t\right)^2}{\frac{(t+6)^2}{4\sqrt{1+t}} + \left(9 - \frac{3}{2}t\right)} = \frac{t^2[(t-6)^2 + 576]}{16(1+t)\left[\frac{(t+6)^2}{4\sqrt{1+t}} + \left(9 - \frac{3}{2}t\right)\right]} > 0 \tag{2.11}$$

for $t \in (0,1)$. From (2.8), (2.10) and (2.11) we clearly see that $D_{1/8}(x)$ is strictly increasing in $(0,1)$. Therefore the inequality (2.2) follows from (2.4) and (2.6) together with the monotonicity of $D_{1/8}(x)$.

Case 2. $p = \lambda$. (2.10) leads to

$$G_\lambda(t) = \frac{(1-4\lambda)^2 t^2 + 6(1-4\lambda)t + 9}{\sqrt{1+t}} + 3(1-4\lambda)t - 9, \tag{2.12}$$

Simple computations yield

$$\lim_{t \to 0^+} G_\lambda(t) = 0, \tag{2.13}$$

$$\lim_{t\to\frac{1}{4}} G_\lambda(t) = \frac{2}{\sqrt{5}}\left(\frac{13}{4} - \lambda\right)^2 - 3\left(\frac{11}{4} + \lambda\right) < \frac{2}{\sqrt{5}}\left(\frac{13}{4} - \frac{7}{50}\right)^2 - 3\left(\frac{11}{4} + \frac{7}{50}\right)$$

$$= \frac{96721\sqrt{5} - 216750}{25000} < 0, \tag{2.14}$$

$$\lim_{t\to 1^-} G_\lambda(t) = 8\sqrt{2}(1-\lambda)^2 - 6(1+2\lambda) > 8\sqrt{2}\left(1-\frac{3}{20}\right)^2 - 6\left(1+2\times\frac{3}{20}\right) = \frac{289\sqrt{2} - 390}{50} > 0, \tag{2.15}$$

$$G_\lambda'(t) = \frac{3(1-4\lambda)t^2 + 2(32\lambda^2 - 28\lambda + 5)t + 3(1-16\lambda)}{2(1+t)^{3/2}} + 3(1-4\lambda), \tag{2.16}$$

and

$$G_\lambda''(t) = \frac{3(1-4\lambda)t^2 + 2(64\lambda^2 - 20\lambda + 1)t + 128\lambda^2 + 32\lambda + 11}{4(1+t)^{5/2}} > 0 \tag{2.17}$$

for $t \in (0,1)$. From (2.17) we clearly see that $G_\lambda(t)$ is a strictly convex function in $(0,1)$. It follows from (2.13)- (2.15) and convexity of $G_\lambda(t)$ that there exists $t_0 \in (0,1)$ such that $G_\lambda(t) < 0$ for $t \in (0,t_0)$ and $G_\lambda(t) > 0$ for $t \in (t_0,1)$, this fact together with (2.8) and (2.10) result in the conclusion that $D_\lambda'(x) < 0$ for $x \in (0,x_0)$ and $D_\lambda'(x) > 0$ for $x \in (x_0,1)$, where $x_0 = \sqrt{t_0}$, hence $D_\lambda(x)$ is strictly decreasing in $(0,x_0)$ and strictly increasing in $(x_0,1)$.

Notice that (2.7) becomes

$$\lim_{x\to 1^-} D_\lambda(x) = 0. \tag{2.18}$$

Therefore the inequality (2.3) follows from (2.4), (2.6) and (2.18) together with the monotonicity of $D_\lambda(x)$.

Finally we prove that $\lambda H(a,b) + (1-\lambda)\overline{C}(a,b)$ is the best possible lower convex combination bound and $1/8H(a,b) + 7/8\overline{C}(a,b)$ is the best possible upper convex combination bound of the harmonic and centroidal means for the Nueman-Sándor mean.

(2.1) leads to

$$\frac{\overline{C}(a,b) - M(a,b)}{\overline{C}(a,b) - H(a,b)} = \frac{\left(1+\frac{x^2}{3}\right) - \frac{x}{\sinh^{-1}x}}{\left(1+\frac{x^2}{3}\right) - (1-x^2)} = B(x), \tag{2.19}$$

From (2.19) one has

$$\lim_{x\to 1^-} B(x) = \lambda, \tag{2.20}$$

and

$$\lim_{x\to 0^+} B(x) = \frac{1}{8}. \tag{2.21}$$

If $\alpha < \lambda$, then (2.19) and (2.20) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $\alpha H(a,b) + (1-\alpha)\overline{C}(a,b) > M(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-\delta_1, 1)$.

If $\beta > 1/8$, then (2.19) and (2.21) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \beta H(a,b) + (1-\beta)\overline{C}(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0,\delta_2)$.

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