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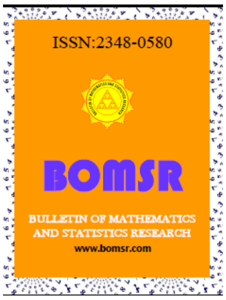
RESEARCH ARTICLE

ON THE CONTRAVARIANT FUNCTOR C^{∞}

ABOUBACAR NIBIRANTIZA

University of Burundi, Institute of Applied Pedagogy, Department of mathematics, B.P 2523, Bujumbura-Burundi



ABSTRACT

In this paper, we study the equivalence between the category of differential manifolds and the category of the associated $\mathbb{R}$ -algebras of smooth functions on those differential manifolds. We show that the functor, denoted by C^{∞}, is contravariant and it establishes the equivalence between those two categories.

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1. INTRODUCTION

The concept of equivalence of categories is very important to do a transport of results of one category to another category. Differential Calculus on smooth manifolds (Classical Physics) can be developed in the language of commutative algebra [1]. Algebraic translation allows to define in an elegant and powerful way differential calculus on more "exotic" spaces. In practice, Differential Calculus is embedded into Commutative algebra according for example to the following dictionary:

$$\begin{pmatrix} \text{smooth manifold } M \\ \text{smooth map } f_{ij}: M_i \to M_j \end{pmatrix} \to \begin{pmatrix} \mathbb{R} - \text{algebra } (M, \mathbb{R}) \\ \mathbb{R} - \text{homomorphism } f_{ij}^{*}: \mathrm{C}^{\infty}(M_j) \to \mathrm{C}^{\infty}(M_i) \end{pmatrix}$$

where f_{ij}^{*} is defined by $f_{ij}^{*}(g) = g \text{ o } f_{ij}$ for all $g \in \mathrm{C}^{\infty}(M_j)$. In this work, we use the abuse of the notation $\mathrm{C}^{\infty}(M)$ instead of $\mathrm{C}^{\infty}(M, \mathbb{R})$

We begin by some definitions and notations [3, 4] which allow us to set our problem of equivalence between categories and we give the criteria of equivalence of categories. In our work, the notion of smooth functions means differentiable functions between smooth manifolds.

2. Notation and problem setting

We begin by general notions about categories and we give some notations.

2.1. Category. *First, we define a category.*

Definition 2.1. *A category C is a data of*

(1) *a collection $Ob(C)$ of objects A,B,...,*

(2) *a collection Hom(A, B) of morphisms for every pair of objects A and B such that for every object A and for all morphisms f and g,*

- *(i) it exists the identity morphism, i.e,* $\mathrm{id}_A : A \to A$
- *(ii) and the composition f o g is associative, i.e, for $f: A \to B$ and $g: B \to C$ and $h: C \to D$, we have $(h \circ g) \circ f = h \circ (g \circ f)$;*
- *(iii) and the identity morphisms are such that for every morphism $f: A \to B$, one has* $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

In a category, the morphism $f \in Hom(X; Y)$ is called an isomorphism if it exists a morphism $g \in Hom(Y;X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

For example, an isomorphism in the category of sets is a bijection and an isomorphism in the category of topological spaces is an homeomorphism

2.2. Functor between categories. A functor generalizes the notion of application between two sets.

Definition 2.2. *A contravariant functor (resp.covariant)* $F : C \to \mathcal{D}$ *of a category C to the category $\mathcal{D}$ is a data of*

I. *an application which maps on any object A of C to the object $F(A)$ of $\mathcal{D}$;*

II. *an application which maps on any morphism $f : A \to B$ in C to the morphism* $F(f) : F(B) \to F(A)$ of $\mathcal{D}$ (resp. $F(f): F(A) \to F(B)$ of $\mathcal{D}$);

such that

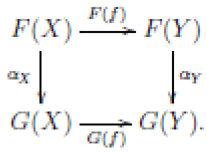
- *for every object A of C one has $F(id_A) = id_{F(A)}$,*
- *for all objects A, B, C and morphisms $f : A \to B$ and $g : B \to C$ of C, we have* $F(g \circ f) = F(f) \circ F(g)$ (resp. $F(g \circ f) = F(g) \circ F(f)$.

Remark 2.3. *The definitions of contravariant functor and covariant functor are analogue but the contravariant functor reverses the morphisms.*

The natural transformations between two functors $F,G : C \to \mathcal{D}$ are the maps

$$\alpha : X \in Ob\,(C) \to \alpha_X \in Hom_D(F(X), G\,(X))$$

Such that, for all $f \in Hom_D\,(X,Y)$, the following diagram is commutative,



We call natural isomorphism when the all $\alpha_X(F(X) \to G\,(X)$ are the isomorphism.

2.3. Equivalence of categories. We give the definition of the equivalence of categories and the criteria of equivalence between two categories.

Definition 2.4. An equivalence of categories is a functor $F : C \to \mathcal{D}$ that is invertible up to natural isomorphisms.

Explicitly, this definition means that there is a functor $G : D \to C$ such that

I. $G \circ F(X) \cong X$,

II. and $F \circ G\,(Y) \cong Y$,

The following proposition gives us the criteria of equivalence between categories.

Proposition 2.5. *A functor F is an equivalence of categories if and only if it is faithful and essentially surjective, i.e.,*

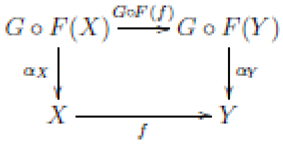
(FF) *The functor F is bijective on morphisms, i.e. the application*

$$Hom_C(X,Y) \to Hom_D\big(F(X),F\ (Y)\big): f \to F(f)$$

is bijective,

$$(ES)\ Any\ object\ A\ in\ D\ is\ isomorphic\ to\ some\ F(X), i.e. for\ all$$
$$A \in \mathrm{O}b(D), it\ exists\ an\ object\ X \in\ \mathrm{O}b(C)\ such\ that\ A\ \cong F\ (X)$$

Proof. Suppose that the functor $\mathrm{F} : C \to \mathcal{D}$ is the equivalence between the categories C and $\mathcal{D}$. Let $G: \mathcal{D} \to C$ be a functor such that $G\ o\ F(X)\ \cong X$ and $F\ o\ G\ (Y) \cong Y$, for all $X\ \in\ \mathrm{O}b(C)$ and $Y\ \in\ \mathrm{O}b(\mathcal{D})$. The functor F is essentially surjective: for all $\mathrm{A} \in \mathrm{O}b(D)$, one has $A\ \cong F\ (G(A))$. The functor F is faith: Every morphism $f \in Hom_D(X,Y)$ is determined by its image $F\ (f) \in Hom_D\big(F(X),F\ (Y)\big)$ because the commutative diagram



gives the relation

(1) $\qquad f = \alpha_Y \circ (G \circ F(f)) \circ \alpha_X^{-1}.$

The functor F is full: Let consider a morphism $g \in Hom_D\big(F(X),F\ (Y)\big)$ The formula (1) can be written by $G \circ F(f) = \alpha_Y^{-1} \circ f \circ \alpha_X.$

We can deduce that the correspondence $f\ \to G\ o\ F(f)$ induces a bijection

$$Hom_C(X,Y) \to Hom_C(G(F(X)),G(F(Y))).$$

So, since $G\ (f) \in Hom_C\ (G\ (F(X), (G\ \big(F(Y)\big))$, it must exist a morphism $f\ \in Hom_C\ (X,Y)$ such that $G\ (g) = G\ o\ F(f)$. And because G is faith by symmetry, one obtains $g\ =\ F(f)$. Suppose now that the functor $\mathrm{F} : C \to \mathcal{D}$ is essentially surjective and faithful. Because of the property of essentially surjective, for all object $A \in \mathrm{O}b(\mathcal{D})$ we can find an object $X \in \mathrm{O}b(C)$ such that $A \cong\ F\ (X)$. Then we set $G\ (A) \coloneqq X$ and we choose the isomorphism $\beta_A : A \to F\ \big(G(A)\big) = F\ (X)$. For every morphism $g \in Hom_D\ (A,B)$, one considers the composite

$$\beta_B \circ g \circ \beta_A^{-1} \in Hom_{\mathcal{D}}(F(G(A), F(G(B))$$

Since F is faithful, $\beta_B\ o\ g\ o\ \beta_A^{-1}$ is the image by F of the unique morphism $f \in Hom_C\ (G(A), G\ (B))$ and ones sets $G\ (g) \coloneqq f$. Finally, we can see that $F\ \big(G(A)\big) \cong A$ and $(G\ \big(F(X)\big) \cong X$,

3. CATEGORY OF SMOOTH MANIFOLDS AND CATEGORY OF ASSOCIATED $\mathbb{R}$-ALGEBRAS OF SMOOTH FUNCTIONS

We describe the category of algebras of smooth functions associated to any category of smooth manifolds.

3.1. Algebra of smooth functions on smooth manifolds. If M is a smooth n-manifold, a function $f : M\ \to \mathbb{R}$ is said to be smooth if for every $p \in M$, there exists a smooth chart $(U, \varphi\ )$ for M whose domain contains p and such that the composite function $f\ \mathrm{o}\ \varphi^{-1}$ is smooth on the open subset $\varphi\ (U) \subset \mathbb{R}^{\boldsymbol{n}}$. The set of all such functions is denoted by $C^\infty(M,\mathbb{R})$. Because sums and constant multiples of smooth functions are smooth, the space $C^\infty(M,\mathbb{R})$ is a vector space. It is easy to show that [2, 5] pointwise multiplication turns $C^\infty(M,\mathbb{R})$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$.

Let denote by $Al_{C^\infty(M_i)}$ the set of all associated algebras of smooth functions on the smooth manifolds M_i for all $i \in I$ with I the set of indices. We have the following result.

Proposition 3.1. *The collection* $Al_{C^\infty(M_i)}$ *is the category of the algebras* $C^\infty(M_i,\mathbb{R})$.

Proof. We consider the collection of the $\mathbb{R}$ -algebras $C^\infty(M_i,\mathbb{R})$ associated to the smooth manifolds M_i for all $i \in I$. We define the morphism between them as follow: for all differentiable application $f_{ij} : M_i \to M_j$ between two smooth manifolds, we define the morphism f_{ij}^* between the $\mathbb{R}$ -algebras $C^\infty(M_i,\mathbb{R})$ and $C^\infty(M_j,\mathbb{R})$ by

$$f_{ij}^* : C^\infty(M_j,\mathbb{R}) \to C^\infty(M_i,\mathbb{R}) : g \mapsto f_{ij}^*(g) := g \circ f_{ij}.$$

The collection $Hom\,(C^\infty(M_j,\mathbb{R}),(C^\infty(M_i,\mathbb{R}))$ of all morphisms f_{ij}^* admits the identity morphism $\mathrm{id}_{\boldsymbol{Mi}}: C^\infty(M_i,\mathbb{R}) \to C^\infty(M_i,\mathbb{R})$ defined naturally and it also admits the composite $f_{ik}^*: C^\infty(M_k,\mathbb{R}) \to C^\infty(M_i,\mathbb{R})$ of the morphisms $f_{ij}^*: C^\infty(M_j,\mathbb{R}) \to C^\infty(M_i,\mathbb{R})$ and $f_{jk}^*: C^\infty(M_k,\mathbb{R}) \to C^\infty(M_j,\mathbb{R})$ such that $f_{ik}^* = f_{ij}^* \text{ o } f_{jk}^*$.

It is easy to see that this composition is associative and the identities morphisms are exactly the identities for the composition of morphisms. So, the collection $Al_{C^\infty(M_i)}$ is a category and every $\mathbb{R}$ -algebra $C^\infty(M_i,\mathbb{R})$ is an object of the category $Al_{C^\infty(M_i)}$.

3.2. Equivalence of the categories $\mathcal{SM}$ and $Al_{C^\infty(M_i)}$. When we consider the category $\mathcal{SM}$ of smooth manifolds, we can define the functor which defines the equivalence between the category $Al_{C^\infty(M_i)}$ and the category $\mathcal{SM}$.

Definition 3.2. We define the application $C^\infty: \mathcal{SM} \to Al_{C^\infty(M_i)}$ such that for all M_i and M_j, we have

$$\left(M_i \overset{f_{ij}}{\to} M_j\right) \mapsto \left(C^\infty(M_j,\mathbb{R}) \overset{f_{ij}^*}{\to} C^\infty(M_i,\mathbb{R}) : g \mapsto f_{ij}^*(g) := g \circ f_{ij}\right)$$

The following result is obvious.

Proposition 3.3. The application C^∞ is a contravariant functor between the categories $\mathcal{SM}$ and $Al_{C^\infty(M_i)}$.

Proof. It suffices to see that the application C^∞ maps on every smooth manifold M_j to the $\mathbb{R}$ -algebra $C^\infty(M_i) := C^\infty(M_i,\mathbb{R})$. It is also easy to see that for every morphism $f_{ij} : M_i \to M_j$ in the category $\mathcal{SM}$ we associate the morphism

$$C^\infty(f_{ij}) = f_{ij}^* : C^\infty(M_j,\mathbb{R}) \to C^\infty(M_i,\mathbb{R})$$

such that for all function $g \in C^\infty(M_j,\mathbb{R})$ one has

$$C^\infty(f_{ij})(g) = f_{ij}^*(g) = g \circ f_{ij}.$$

It is obvious to see that the application C^∞ respects the identities and the composition, i.e.,

I. for all smooth manifold M_i, $C^\infty(\mathrm{id}_{M_i}) = \mathrm{id}_{C^\infty(M_i)}$;

II. for all objects M_i, M_j, M_k in the category $\mathcal{SM}$ and for all morphisms f_{ij} and f_{jk} in the category $\mathcal{SM}$ we obtain $C^\infty(f_{jk} \text{ o } f_{ij}) = C^\infty(f_{ij})\ o\ C^\infty(f_{jk})$ in the category $Al_{C^\infty(M_i)}$.

The following proposition concerns the equivalence between the categories $\mathcal{SM}$ and $Al_{C^\infty(M_i)}$.

Proposition 3.4. The contravariant functor C^∞ defined above establishes the equivalence of the categories $\mathcal{SM}$ and $Al_{C^\infty(M_i)}$.

Proof. The proof is facilitated by the one given on the proposition 2.5. Here, it suffices to verify the following facts.

Firstly, it is obvious that the functor C^∞ is faithful, i.e. for all smooth manifolds M_i, M_j it induces a bijection

$$Hom(M_i, M_j) \to Hom(C^\infty(M_j), C^\infty(M_i)).$$

Secondly, we can see that the functor C^∞ is essentially surjective, i.e. for all object $C^\infty(M_i,\mathbb{R}) \in \mathrm{O}b(Al_{C^\infty(M_i)})$, it exists an object $M_i \in \mathrm{O}b(\mathcal{SM})$ such that $C^\infty(M_i) = C^\infty(M_i,\mathbb{R})$.

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