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RESEARCH ARTICLE

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OBSERVATIONS ON THE CONE $15x^2 - 32y^2 = 7z^2$

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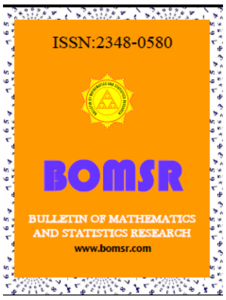

ABSTRACT

The ternary quadratic equation representing the cone given by $15x^2 - 32y^2 = 7z^2$ is analyzed for determining its infinitely many non-zero distinct integer solutions. A few interesting properties among its solutions are given. Also, formulas for generating sequences of integer solutions based on its given initial solution are presented

Keywords : Ternary quadratic, Homogeneous cone, integer solutions

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1. INTRODUCTION

The quadratic Diophantine equations with three unknowns offers an unlimited field for research because of their variety [2,24]. For an extensive review of various problems on ternary quadratic Diophantine equations representing specific 3 dimensional surfaces, one may refer [1, 3 – 23, 25].

In this communication, we present a problem of the ternary quadratic equation representing the cone given by $15x^2 - 32y^2 = 7z^2$ is analyzed for determining its infinitely many non-zero distinct integer solutions. A few interesting properties among its solutions are given. Also, formulas for generating sequences of integer solutions based on its given initial solution are presented

2. METHOD OF ANALYSIS

Consider the cone represented by the ternary quadratic equation given by

$$15x^2 - 32y^2 = 7z^2 \tag{1}$$

To start with (1) is satisfied by the following triples of integers (x, y, z): $(18,118,76)$, $(108,54,108)$ and$(168k^2 + 12, 112k^2 + 14k - 8, -56k^2 + 128k + 4)$. However, we have other different sets of integer solutions satisfying (1) which are presented below:

Introducing the linear transformations

$$x = 12U, \ y = \alpha + 7\beta, \ z = 4\alpha - 8\beta \tag{2}$$

in (1), it is written as

$$15U^2 = \alpha^2 + 14\beta^2 \qquad (3)$$

Again, introducing the linear transformations

$$U = \gamma + 14\delta,\ \ \beta = \gamma + 15\delta \qquad (4)$$

(3) is written as

$$\gamma^2 = 210\delta^2 + \alpha^2 \qquad (5)$$

which is equivalent to the following systems of double equations:

	System 1	System 2	System 3	System 4	System 5	System 6	System 7
γ + α	δ^2	$105\delta^2$	$15\delta^2$	15δ	105δ	210δ	30δ
γ - α	210	2	14	14δ	2δ	δ	7δ

Solving each of the above systems inturn, the corresponding values of γ, α *and* δ are obtained. In view of (4) and (2), one obtains the corresponding integer solutions to (1) which are presented below:

Set 1	**Solutions obtained from system 1**		
	$x = 24k^2 + 336k + 1260$	$y = 16k^2 + 210k + 630$	$z = -8k^2 - 240k - 1260$
Set 2	**Solutions obtained from system 2**		
	$x = 2520k^2 + 336k + 12$	$y = 1680k^2 + 210k + 6$	$z = -840k^2 - 240k - 12$
Set 3	**Solutions obtained from system 3**		
	$x = 360k^2 + 336k + 84$	$y = 240k^2 + 210k + 42$	$z = -120k^2 - 240k - 84$
Set 4	**Solutions obtained from system 4**		
	$x = 684k$	$y = 414k$	$z = -468k$
Set 5	**Solutions obtained from system 5**		
	$x = 1620k$	$y = 1062k$	$z = -684k$
Set 6	**Solutions obtained from system 6**		
	$x = 2868k$	$y = 1896k$	$z = -1092k$
Set 7	**Solutions obtained from system 7**		
	$x = 780k$	$y = 492k$	$z = -444k$

In addition to the above sets of integer solutions, we have some more sets of different solutions satisfying (1) and they are illustrated below:

Set 8

Assume $U = a^2 + 14b^2$

Write 15 as $15 = (1 + i\sqrt{14})(1 - i\sqrt{14})$ (6)

Using (6) in (3) and employing the method of factorization, define

$$\alpha + i\sqrt{14}\,\beta = (1 + i\sqrt{14})(a + i\sqrt{14}\ b)^2$$

Equating the real and imaginary parts, we have

$$\alpha = a^2 - 14b^2 - 28ab \text{ and } \beta = a^2 - 14b^2 + 2ab$$

Thus, the corresponding integer solutions of (1) are given by

$$x(a,b) = 12a^2 + 168b^2$$
$$y(a,b) = 8a^2 - 112b^2 - 14ab$$
$$z(a,b) = -4a^2 + 56b^2 - 128ab$$

Properties observed from the above :

1. $\frac{1}{6}[x(a,b)-3z(a,b)]$ is written as the difference of two squares
2. Each of the following expressions is a nasty number:
   (i). $x(a,1)-3z(a,1)+1536$
   (ii). $y(2b-1,b)-x(2b-1,b)-z(2b-1,b)+228T_{3,b}$
   (iii). $y(3b,b)-x(3b,b)-z(3b,b)$
   (iv). $30\left[x(1,b)-2y(1,b)-z(1,b)-312T_{3,b}\right]$

Set 9

(3) is expressed in the form of ratio as

$$\frac{\alpha+U}{7(U+\beta)}=\frac{2(U-\beta)}{\alpha-U}=\frac{P}{Q}, Q\neq 0 \tag{7}$$

which is equivalent to the system of equations

$$\alpha Q+U(Q-7P)-7P\beta=0$$
$$\alpha P+U(-P-2Q)+2Q\beta=0$$

Applying the method of cross-multiplication, we have

$$\alpha=2Q^2-28PQ-7P^2$$
$$U=-7P^2-2Q^2$$
$$\beta=7P^2-2PQ-2Q^2$$

Hence, in this case, the corresponding integer solutions to (1) is given by

$$x=-84P^2-24Q^2$$
$$y=42P^2-42PQ-12Q^2$$
$$z=24Q^2-96PQ-84P^2$$

2.1 Remark

In addition to (7), one may also have the following representations from (3):

(i) $\frac{\alpha+U}{2(U+\beta)}=\frac{7(U-\beta)}{\alpha-U}=\frac{P}{Q}$

(ii) $\frac{\alpha+U}{14(U+\beta)}=\frac{(U-\beta)}{\alpha-U}=\frac{P}{Q}$

(iii) $\frac{\alpha+U}{(U+\beta)}=\frac{14(U-\beta)}{\alpha-U}=\frac{P}{Q}$

Following the procedure presented above , other sets of distinct integer solutions to (1) are obtained.

3. GENERATION OF SOLUTIONS

In this section, we obtain general formulas for generating sequences of integer solutions to (1) based on its initial solution

3.1 Formula 1

Let (x_0, y_0, z_0) be the given integer solution to (1).

Let $x_1=2h+x_0,\ y_1=y_0,\ z_1=3h-z_0$ (8)

be the first solution to (1) where h is the non-zero integer to be determined.

Substituting (8) in (1) and simplifying we get

$$h=20x_0+14z_0$$

Therefore, $x_1=41x_0+28z_0,\ z_1=60x_0+41z_0$

Expressing the above equations in the matrix form , we have

$$\begin{pmatrix}x_1\\z_1\end{pmatrix}=M\begin{pmatrix}x_0\\z_0\end{pmatrix}$$

where $M=\begin{pmatrix}41 & 28\\60 & 41\end{pmatrix}$

Repeating the above process, the general values of x and z are given by

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha-\beta)}(M - \beta I) + \frac{\beta^n}{(\beta-\alpha)}(M - \alpha I)$$

where α, β are the eigen values of M and I is the unit matrix of order 2 .
For our problem,

$$\alpha = 41 + 4\sqrt{105}, \quad \beta = 41 - 4\sqrt{105}$$

Therefore, $$M^n = \begin{pmatrix} Y_n & 7X_n \\ 15X_n & Y_n \end{pmatrix}$$

where $Y_n = \frac{1}{2}(\alpha^n + \beta^n)$ and $X_n = \frac{1}{2\sqrt{105}}(\alpha^n - \beta^n)$

Thus the general solution to (1) based on its initial solution is

$$x_n = Y_n x_0 + 7X_n z_0$$
$$y_n = y_0$$
$$z_n = 15X_n x_0 + Y_n z_0$$

3.2 Formula 2

$$x_n = \frac{1}{2}(\alpha^n + \beta^n)x_0 + \frac{4}{\sqrt{30}}(\alpha^n - \beta^n)y_0$$
$$y_n = \frac{15}{8\sqrt{30}}(\alpha^n - \beta^n)x_0 + \frac{1}{2}(\alpha^n + \beta^n)y_0$$
$$z_n = 7^n z_0$$

where $\alpha = 263 + 48\sqrt{30}$, $\beta = 263 - 48\sqrt{30}$

4. CONCLUSION

In this paper, we have made an attempt to obtain infinitely many non- zero distinct integer solutions to the homogeneous cone given by $15x^2 - 32y^2 = 7z^2$. As ternary quadratic equations are rich in variety, one may search for integer solutions to other choices of homogeneous cones and determine their corresponding properties.

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6. REFERENCES

[1]. Devibala, S., and Gopalan. M.A.,On The Ternary Quadratic Diophantine Equation $z^2 = 7x^2 + y^2$, International Journal of Emerging Technologies in Engineering Research, 4(9), 2016, 6 – 7.

[2]. Dickson. L.E. "History of Theory of Numbers and Diophantine Analysis", Vol.2, Dove Publications, New York 2005.

[3]. Carmicheal,R.D., "The Theory of Numbers and Diophantine Analysis", Dover Publications, New York 1959.

[4]. Gopalan.M.A.,Geetha.D, Lattice points on the Hyperboloid of two sheets $x^2 - 6xy + y^2 + 6x - 2y + 5 = z^2 + 4$ Impct J.Sci.Tech,2010,4,23-32.

[5]. Gopalan.M.A.,Vidhyalakshmi.S,Kavitha..A, Integral points on the Homogeneous cone $z^2 = 2x^2 - 7y^2$,The Diophantus J.Math,2012,1(2),127-136.

[6]. Gopalan.M.A.,Vidhyalakshmi.S,Sumathi.G:Lattice points on the Hyperboloid of one sheet $4z^2 = 2x^2 + 3y^2 - 4$,Diophantus J.Math .,2012,1(2) 109-115.

[7]. Gopalan.M.A.,Vidhyalakshmi.S,Lakshmi.K Integral points on the Hyperboloid of two sheets $3y^2 = 7x^2 - z^2 + 21$ Diophantus J. of Math.2012,1(2),99-107.

[8]. Gopalan.M.A., Vidhyalakshmi.S, Mallika.S, Observation on Hyperboloid of one sheet $x^2+2y^2-z^2=2$, Bessel J. of Math.,2012,2(3),221-226.

[9]. Gopalan.M.A.,Vidhyalakshmi.S,Usha Rani.T.R. Mallika.S,Integral points on the Homogeneous cone $6z^2+3y^2-2x^2=0$ Impact J.Sci.Tech.,2012,6(1),7-13.

[10]. Gopalan.M.A.,Vidhyalakshmi.S,Sumathi.G: Lattice points on the Elliptic paraboloid $9x^2+4y^2=z$, Advances in Theoretical and Applied Mathematics,2012,7(4), 379-385.

[11]. Gopalan..M.A,Vidhyalakshmi.S, Usha Rani.T.R Integral points on the non-homogeneous cone $2z^2+4xy+8x-4z=0$,Global Journal of Mathematics and Mathematical Sciences, 2012, 2(1), 61-67.

[12]. Gopalan.M.A, Vidhyalakshmi.S, Lakshmi.K, Lattice points on the elliptic paraboloid $16y^2+9z^2=4x$ Bessel J. of Math, 2013, 3(2),137-145.

[13]. Gopalan.M.A,Vidhyalakshmi.S,UmaRani.J,Integral points on the homogeneous cone $x^2+4y^2=37z^2$ Cayley J of Math,2013,2(2),101-107.

[14]. Gopalan.M.A,Vidhyalakshmi.S,Kavitha.A,Observations on the Hyperboloid of two Sheets $7x^2-3y^2=z^2+z(y-x)+4$ International Journal of Latest Research in Science and technology 2013, 2(2), 84 - 86.

[15]. Gopalan.M.A,Sivagami.B,Integral points on the homogeneous cone $z^2=3x^2+6y^2$ IOSR Journal of Mathematics, 2013, 8(4), 24 - 29.

[16]. Gopalan.M.A,Geetha.V,Lattice points on the Homogeneous cone $z^2=2x^2+8y^2-6xy$ Indian journal of Science, 2013, 2, 93 - 96.

[17]. Gopalan.M.A,Vidhyalakshmi.S,Maheswari.D,Integral points on the homogeneous cone $2x^2+3y^2=35z^2$ Indian Journal of Science, 2014, 7, 6-10.

[18]. Gopalan.M.A,Vidhyalakshmi.S,Umarani.J,On the Ternary Quadratic Diophantine equation $6(x^2+y^2)-8xy=21z^2$,Sch.J.Eng Tech., 2014, 2(2A), 108-112.

[19]. Gopalan.M.A, Vidhyalakshmi.S, Nivetha.S On Ternary Quadratics equation $4(x^2+y^2)-7xy=31z^2$, Diophantus J.Math., 2014, 3(1), 1-7.

[20]. Gopalan. M. A., Vidhyalakshmi. S and Mallika. S , On ternary quadratic Diophantine equation $8(x^2+y^2)-15xy=80z^2$, Bulletin of Mathematics and statistics Research, Vol.2, No.4, 2014, 429-433

[21]. Gopalan. M. A., Vidhyalakshmi. S, Thiruniraiselvi. N, Observations on the ternary quadratic diophantine equation $x^2+9y^2=50z^2$, International Journal of Applied Research, 1(2), 51 -53, 2015.

[22]. Meena.K,Vidhyalakshmi.S,Gopalan.M.A,Priya.IK,Integral points on the cone $3(x^2+y^2)-5xy=47z^2$,Bulletin of Mathematics and Statistic Research, 2014, 2(1), 65-70.

[23]. Meena.K,Vidhtalakshmi.S,Gopalan.M.A,Thangam.SA Integer Solutions on the Homogeneous cone $4x^2+3y^2=28z^2$ Bulletin of Mathematics and statistics Research,2 014,2(1), 47-53.

[24]. Mordell L.J., "Diophantine Equations" Academic Press, New York,1970

[25]. Shanthi.J,Gopalan.M.A,Vidhyalakshmi,Lattice points on the Homogeneous cone $8(x^2+y^2)-15xy=56z^2$, Sch.Journal of Phy.Math.Stat.,2014,Vol-1,issue-1, 29-32.