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FIXED POINT THEOREMS IN CONE METRIC SPACES FOR C-DISTANCE OF CONTRACTIVE MAPPINGS

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Abstract

In 1906, the French mathematician Maurice Frechet introduced the concept of metric spaces, although the name "metric" is due to Hausdorff . In 1934, Duro Kurepa, proposed metric spaces in which an ordered vector space is used as the codomain of a metric instead of the set of real numbers. In literature Metric Spaces with Vector Valued Metrics are known under various names such as Pseudo Metric Spaces, K-Metric Spaces, Generalized Metric Spaces, Cone-Valued Metric Spaces, Cone Metric Spaces , Abstract Metric Spaces and Vector Valued Metric Spaces. Fixed point theory in K-metric spaces was developed by Perov in 1964.

In 2007 , L. G. Huang, Z. Xian[14] introduced the notion of a cone metric space and established some fixed point theorems in cone metric spaces, an ambient space which is obtained by replacing the real axis in the definition of the distance, by an ordered real Banach space whose order is induced by a normal cone P. Recently, Wang et. al. [32] introduced the concept of c-distance on a cone metric space, which is a cone version of the *w*-distance of Kada et.al.[20] and proved a common fixed point theorem. In this paper, we prove some common fixed point results with generalized contractive condition under the concept of a c-distance in cone metric spaces. Our results extend and improve some well-known contractive conditions in literature. Also we extend the results of A.K.Dubey et al.[13].

Keywords: Coincidence Points, Fixed Points, Complete Cone Metric Spaces,
c – Distance, Contractive Mappings

AMS Subject Classification: 45H10, 54H25, 47H10, 47H09

1 Introduction

Huang and Zhang [14] introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. In 1996, Kada et al. [20] defined the concept of *w*-distance in complete metric space and proved some fixed point theorems in complete metric spaces (see [1, 22, 23, 25]). Also Saadati et al. [28] introduced a probabilistic version of the *w*-distance of Kada et al.[20] in a Menger probabilistic metric space. Cho et al. [8], and Wang and Guo [32] defined a concept of the c-distance in a cone metric space, which is a cone version of the *w*-distance of Kada et al.[20] and proved some fixed point theorems in ordered cone metric spaces. Sintunavarat et al. [31] generalized the Banach contraction theorem on c-distance of Cho et al. [8]. Also, Dordevi´c et al. in [12] proved some fixed point and common fixed point theorems under c-distance for contractive mappings in tvs-cone metric spaces. H. Rahimi, G. Soleimani Rad [24] extended the Banach contraction principle [6] and Chatterjea contraction theorem [7] on c-distance of Cho et al. [8], and proved some fixed point and common fixed point theorems in ordered cone metric spaces.

Also Fadail et al. [19] proved some fixed point theorems on c-distance in cone metric spaces, under the continuity condition for maps. In this paper we extend and improve the results of A.K.Dubey et al.[13].

2 Preliminaries

First let us start with some basic definitions

Definition 2.1: ([11, 14])

Let E be a real Banach space and 0 denote the zero element in E. A subset P of E is said to be a cone if

(a) P is closed, non-empty and $P \neq \{0\}$;

(b) $a,b \in \mathbb{R}$, $a,b \geq 0$, $x,y \in P$ imply that $ax + by \in P$; where R is the real number system

(c) if $x \in P$ and $-x \in P$, then $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y \Leftrightarrow y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in \text{int } P$ (where int P is interior of P). If int $P \neq \emptyset$, the cone P is called solid. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$. The least positive number satisfying the above is called the normal constant of P.

Definition 2.2: ([14])

Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.3: ([14])

Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$.

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} d(x_n, x) = 0$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x_m) \ll c$ for all $m,n > n_0$, and we write $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) If every Cauchy sequence in X is convergent, then X is called a complete cone metric space.

Lemma 2.4: ([14, 26])

Let (X,d) be a cone metric space and P be a normal cone with normal constant k. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y \in X$. Then the following hold:

(c1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(c2) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.

(c3) If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

(c4) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

(c5) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n,m \to \infty$.

Lemma 2.5: ([4, 15]) Let E be a real Banach space with a cone P in E. Then, for all $u,v,w,c \in E$, the following hold:

(p1) If $u \leq v$ and $v \ll w$, then $u \ll w$.

(p2) If $0 \leq u \ll c$ for each $c \in \text{int } P$, then $u = 0$.

(p3) If $u \leq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = 0$.

(p4) Let $x_n \to 0$ in E, $0 \leq x_n$ and $0 \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

(p5) If $0 \leq u \leq v$ and k is a nonnegative real number, then $0 \leq ku \leq kv$.

(p6) If $0 \leq u_n \leq v_n$ for all $n \in N$ and $u_n \to u$, $v_n \to v$ as $n \to \infty$, then $0 \leq u \leq v$.

Definition 2.6: ([8, 32])

Let (X,d) be a cone metric space. A function $q : X{\times}X \to E$ is called a c-distance on X if the following are satisfied:

(q1) $0 \leq q(x, y)$ for all $x, y \in X$;

(q2) $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \leq u$ for some u , then $q(x, y) \leq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.7: ([8, 32]) Each *w*-distance q in a metric space (X,d) is a c-distance (with $E = R^+$ and $P = [0,\infty)$). But the converse does not hold. Therefore, the c-distance is a generalization of *w*-distance.

Examples 2.8: ([8, 31, 32])

(1) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.

(2) Let $E = R$, $X = [0,\infty)$ and $P = \{x \in E : x \geq 0\}$. Define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space.

Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c distance.

(3) Let $E = C^1_R[0,1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the cone $P = \{x \in E : x(t) \geq 0$ on $[0,1]\}$. Also, let $X = [0,\infty)$ and define a mapping $d : X{\times}X \to E$ by $d(x, y) = |x-y|\psi$ for all

$x, y \in X$, where $\psi : [0,1] \to R$ such that $\psi(t) = 2t$. Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\psi$ for all $x, y \in X$. Then q is c-distance.

(4) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(w, y)$ for all $x, y \in X$, where $w \in X$ is a fixed point. Then q is a c-distance.

Remark 2.9: ([8, 31, 32]) From Examples 2.8 (1,2,4), we have three important results

(i) Each cone metric d on X with a normal cone is a c-distance q on X.

(ii) For c-distance q, $q(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

(iii) For c-distance q, $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.

Lemma 2.10: ([8, 31, 32])

Let (X,d) be a cone metric space and let q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to 0. Then the following hold:

(qp1) If $q(x_n, y) \le u_n$ and $q(x_n, z) \le v_n$ for $n \in N$, then $y = z$.

Specifically, if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$.

(qp2) If $q(x_n, y_n) \le u_n$ and $q(x_n, z) \le v_n$ for $n \in N$, then $\{y_n\}$ converges to z.

(qp3) If $q(x_n, x_m) \le u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(qp4) If $q(y, x_n) \le u_n$ for $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.

Definition 2.11 : ([19])

Let S and T be self-mappings of a set X. If $u = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T and u is called a point of coincidence of S and T.

Definition 2.12 : ([19])

Two self-mappings S and T of a set X are said to be weakly compatible if they coincidence point. i.e; if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

A.K.Dubey, Rohit Verma and Ravi Prakash Dubey proved following theorems

Theorem 2.13 : (Theorem 2.1, A.K.Dubey, Rohit Verma and Ravi Prakash Dubey)

Let (X,d) be a complete cone metric space and q is a c-distance on X . Suppose that the mapping $f : X \to X$ is continuous and satisfies the contractive condition:

$q(fx,fy) \le a_1q(x,y) + a_2q(x,fx) + a_3q(y,fy) + a_4[q(fx,y) + q(fy,x)]$ for all $x,y \in X$,

where a_1,a_2,a_3,a_4 are nonnegative real numbers such that $a_1 + a_2 + a_3 + 2a_4 < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$ then $q(v,v) = 0$. The fixed point is unique.

Theorem 2.14 : (Theorem 2.3, A.K.Dubey, Rohit Verma and Ravi Prakash Dubey)

Let (X,d) be a complete cone metric space and q is a c-distance on X . Suppose that the mapping $f : X \to X$ is continuous and satisfies the contractive condition: $q(fx,fy) \le a_1q(x,y) + a_2[q(x,fy) + q(y,fx)] + a_3[q(x,fx) + q(y,fy)]$ for all $x,y \in X$, where a_1,a_2,a_3 are nonnegative real numbers such that $a1 + 2a2 + 2a3 < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point.

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If $v = fv$ then $q(v,v) = 0$. The fixed point is unique.

3 Main results

In this section , we extend the results of A.K.Dubey, Rohit Verma and Ravi Prakash Dubey [13] and We establish some common fixed point theorems under c-distance in cone metric spaces, for a pair of self-maps.

Theorem 3.1 : Let (X,d) be a complete cone metric space and q is a c-distance on X . Suppose that the mappings $f, g : X \to X$ are two continuous functions and f and g commute and q satisfies the contractive condition:

$$q(fx, fy) \le a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4[q(fx, gy) + q(fy, gx)] \quad \ldots(3.1.1)$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are nonnegative real numbers such that $a_1 + a_2 + a_3 + 4a_4 < 1$.

Then f and g have a unique common fixed point in X.

Proof : Let $x_0 \in X$, $x_1 = f x_0 = g y_1$ for some y_1

Now $f y_1 = g y_2$, $f y_2 = g y_3$, …, $f y_n = g y_{n+1}$

Write $x_n = f y_n$

Consider $q(f y_n, f y_{n+1}) \le a_1 q(g y_n, g y_{n+1}) + a_2 q(g y_n, f y_n) + a_3 q(g y_{n+1}, f y_{n+1}) + a_4[q(f y_n, g y_{n+1}) + q(f y_{n+1}, g y_n)]$

$= a_1 q(f y_{n-1}, f y_n) + a_2 q(f y_{n-1}, f y_n) + a_3 q(f y_n, f y_{n+1}) + a_4[q(f y_n, f y_n) + q(f y_{n+1}, f y_{n-1})]$

$\le a_1 q(f y_{n-1}, f y_n) + a_2 q(f y_{n-1}, f y_n) + a_3 q(f y_n, f y_{n+1}) + a_4[q(f y_n, f y_{n+1}) + q(f y_{n+1}, f y_n)] + a_4[q(f y_{n+1}, f y_n) + q(f y_n, f y_{n-1})]$ …(3.1.2)

Similarly

$q(f y_{n+1}, f y_n) \le a_1 q(f y_n, f y_{n-1}) + a_2 q(f y_n, f y_{n-1}) + a_3 q(f y_{n+1}, f y_n) + a_4[q(f y_{n+1}, f y_n) + q(f y_n, f y_{n+1})] + a_4[q(f y_n, f y_{n+1}) + q(f y_{n-1}, f y_n)]$ …(3.1.3)

Adding (3.1.2) & (3.1.3) we get

$q(f y_n, f y_{n+1}) + q(f y_{n+1}, f y_n)$

$\le a_1[q(f y_{n-1}, f y_n) + q(f y_n, f y_{n-1})] + a_2[q(f y_{n-1}, f y_n) + q(f y_n, f y_{n-1})] + a_3[q(f y_n, f y_{n+1}) + q(f y_{n+1}, f y_n)] + a_4[q(f y_n, f y_{n+1}) + q(f y_{n+1}, f y_n) + q(f y_n, f y_{n-1}) + q(f y_{n-1}, f y_n)] +$
$2a_4[q(f y_n, f y_{n+1}) + q(f y_{n+1}, f y_n)]$

$\Longrightarrow q(f y_n, f y_{n+1}) + q(f y_{n+1}, f y_n)$

$\le \left(\frac{a_1 + a_2 + a_4}{1 - a_3 - 3a_4}\right)[q(f y_n, f y_{n-1}) + q(f y_{n-1}, f y_n)]$

$\Longrightarrow \alpha_n \le \lambda \alpha_{n-1}$, $\alpha_n = q(f y_n, f y_{n+1}) + q(f y_{n+1}, f y_n)$, $\lambda = \left(\frac{a_1 + a_2 + a_4}{1 - a_3 - 3a_4}\right)$

Proceeding like this, we get

$\alpha_n \le \lambda^n \alpha_0$, $\alpha_n \to 0$ as $n \to \infty$

$q(f y_n, f y_{n-1}) + q(f y_{n-1}, f y_n) \to 0$

$q(f y_n, f y_{n-1}) \to 0$ and $q(f y_{n-1}, f y_n) \to 0$

Let $m > n \ge 1$. Then it follows that

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$q(x_n, x_m) + q(x_m, x_n)$
$\leq \alpha_n + \alpha_{n+1} + \ldots + \alpha_{m-1}$
$\leq \lambda^n \alpha_0 + \lambda^{n+1} \alpha_0 + \ldots + \lambda^{m-1} \alpha_0$
$\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}) \alpha_0$
$\leq (\frac{\lambda^n}{1-\lambda}) \alpha_0$
$\Rightarrow q(x_n, x_m) + q(x_m, x_n)$
$\leq \mu [q(x_0, x_1) + q(x_1, x_0)]$ where $\mu = \frac{\lambda^n}{1-\lambda}$
Thus by Lemma 2.10, $\{x_n\}$ is a Cauchy Sequence in X.
Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$
$fy_n \to x^*$ as $n \to \infty$
$gy_{n+1} \to x^*$ as $n \to \infty$
$fg\, y_{n+1} \to fx^*$ as $n \to \infty$ since f is continuous
$gf\, y_{n+1} \to fx^*$ as $n \to \infty$ since f & g commute
But $gf\, y_{n+1} \to gx^*$ as $n \to \infty$ since g is continuous
$\Rightarrow fx^* = gx^*$
To show that $q(fx^*, fx^*)$
$\leq a_1 q(gx^*, gx^*) + a_2 q(gx^*, fx^*) + a_3 q(gx^*, fx^*) + a_4 [q(fx^*, gx^*) + q(fx^*, gx^*)]$
$\leq a_1 q(fx^*, fx^*) + a_2 q(fx^*, fx^*) + a_3 q(fx^*, fx^*) + a_4 [q(fx^*, fx^*) + q(fx^*, fx^*)]$
$= (a_1 + a_2 + a_3 + 2a_4) q(fx^*, fx^*)$
$\therefore q(fx^*, fx^*) \leq (a_1 + a_2 + a_3 + 2a_4) q(fx^*, fx^*)$
But $(a_1 + a_2 + a_3 + 2a_4) < (a_1 + a_2 + a_3 + 4a_4) < 1$
$\therefore q(fx^*, fx^*) = 0$
Now to show that $q(x^*, x^*) = 0$
Consider $q(fy_n, fy_n)$
$\leq a_1 q(gy_n, gy_n) + a_2 q(gy_n, fy_n) + a_3 q(gy_n, fy_n) + a_4 [q(fy_n, gy_n) + q(fy_n, gy_n)]$
on letting $n \to \infty$ we get
$q(x^*, x^*) \leq a_1 q(x^*, fx^*) + a_2 q(x^*, x^*) + a_3 q(x^*, x^*) + a_4 [q(x^*, x^*) + q(x^*, x^*)] = (a_1 + a_2 + a_3 + 2a_4) q(x^*, x^*)$
But $(a_1 + a_2 + a_3 + 2a_4) < 1$
$q(x^*, x^*) = 0$
To show that x^* is a common fixed point of f and g
Now $q(fy_n, fx^*)$
$\leq a_1 q(gy_n, gx^*) + a_2 q(gy_n, fy_n) + a_3 q(gx^*, fx^*) + a_4 [q(fy_n, gx^*) + q(fx^*, gy_n)]$
on letting $n \to \infty$ we get
$q(x^*, fx^*) \leq a_1 q(x^*, fx^*) + a_2 q(x^*, x^*) + a_3 q(fx^*, fx^*) + a_4 [q(x^*, fx^*) + q(fx^*, x^*)]$

$= a_1\, q(x^*, fx^*) + a_4 [\, q(x^*, fx^*) + q(fx^*, x^*)\,]$

similarly $q(fx^*, x^*) \le a_1\, q(fx^*, x^*) + a_4 [\, q(fx^*, x^*) + q(x^*, fx^*)\,]$

on adding we get,

$q(x^*, fx^*) + q(fx^*, x^*) \le a_1 [\, q(x^*, fx^*) + q(fx^*, x^*)] + 2a_4 [\, q(x^*, fx^*) + q(fx^*, x^*)\,]$

$\Longrightarrow (1- a_1 - 2\, a_4)[\, q(x^*, fx^*) + q(fx^*, x^*)] \le 0$

$\Longrightarrow q(x^*, fx^*) = 0$ and $q(fx^*, x^*) = 0$

$\Longrightarrow fx^* = x^* = gx^*$

Uniqueness :

Suppose that there is another fixed point y^* of f and g

$fy^* = y^*$ and $gy^* = y^*$

$q(x^*, y^*) = q(fx^*, fy^*)$

$\le a_1\, q(gx^*, gy^*) + a_2\, q(gx^*, fx^*) + a_3\, q(gy^*, fy^*) + a_4 [\, q(fx^*, gy^*) + q(fy^*, gx^*)\,]$

$= a_1 q(x^*, y^*) + a_2\, q(x^*, x^*) + a_3 q(y^*, y^*) + a_4 [\, q(x^*, y^*) + q(y^*, x^*)\,]$

similarly, $q(y^*, x^*) \le a_1 q(y^*, x^*) + a_2\, q(x^*, x^*) + a_3\, q(y^*, y^*) + a_4 [\, q(y^*, x^*) + q(x^*, y^*)\,]$

on adding, we get

$q(x^*, y^*) + q(y^*, x^*) \le a_1 [\, q(x^*, y^*) + q(y^*, x^*)\,] + 2a_4 [\, q(y^*, x^*) + q(x^*, y^*)\,]$

$\Longrightarrow (1- a_1 - 2a_4)\, [\, q(x^*, y^*) + q(y^*, x^*)\,] \le 0$

$\Longrightarrow [\, q(x^*, y^*) + q(y^*, x^*)\,] = 0$

$\Longrightarrow q(x^*, y^*) = 0$ and $q(y^*, x^*) = 0$

$\therefore\ x^* = y^*$

∴ Fixed Point is Unique

∴ f and g have a unique common fixed point in X.

Theorem 3.2 : Let (X,d) be a complete cone metric space and q is a c-distance on X . Suppose that the mappings $f, g : X \to X$ are two continuous functions and f and g commute and q satisfies the contractive condition:

$$q(fx, fy) \le a_1 q(gx, gy) + a_2\, [\, q(gx, fy) + q(gy, fx) + a_3[q(gx, fx) + q(gy, fy)] \quad \text{...(3.2.1)}$$

for all $x, y \in X$, where a_1, a_2, a_3 are nonnegative real numbers such that $a_1 + 4a_2 + 2a_3 < 1$.

Then f and g have a unique common fixed point in X.

Proof : Let $x_0 \in X$, $x_1 = f x_0 = g\, y_1$ for some y_1

Now $f\, y_1 = g\, y_2$, $f\, y_2 = g\, y_3, \ldots, f\, y_n = g\, y_{n+1}$

Write $x_n = f\, y_n$

Consider $q(f y_n, f y_{n+1})$

$\le a_1 q(g y_n, g y_{n+1}) + a_2 [\, q(g y_n, f y_{n+1}) + q(g y_{n+1}, f y_n)\,] + a_3[q(g y_n, f y_n) + q(g y_{n+1}, f y_{n+1})]$

$= a_1 q(f y_{n-1}, f y_n) + a_2 [\, q(f y_{n-1}, f y_{n+1}) + q(f y_n, f y_n)\,] + a_3 [q(f y_{n-1}, f y_n) + q(f y_n, f y_{n+1})]$

$\le a_1 q(f y_{n-1}, f y_n) + a_2 [\, q(f y_{n-1}, f y_n) + q(f y_n, f y_{n+1})\,] + a_2[\, q(f y_n, f y_{n-1}) + q(f y_{n-1}, f y_n)\,] + a_3[q(f y_{n-1}, f y_n) + q(f y_n, f y_{n+1})]$ ...(3.2.2)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
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Vol.4. S1.2016; ISSN: 2348-0580
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RADMAS– 2016

Similarly

$q(fy_{n+1}, fy_n) \le a_1 q(fy_n, fy_{n-1}) + a_2 [q(fy_n, fy_{n-1}) + q(fy_{n+1}, fy_n)] + a_2[q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1})] + a_3[q(fy_n, fy_{n-1}) + q(fy_{n+1}, fy_n)]$ ...(3.2.3)

Adding (3.2.2) & (3.2.3) we get

$q(fy_n, fy_{n+1}) + q(fy_{n+1}, fy_n)$

$\le a_1 q(fy_{n-1}, fy_n) + a_2 [q(fy_{n-1}, fy_n) + q(fy_n, fy_{n+1})] +$
$a_2[q(fy_n, fy_{n-1}) + q(fy_{n-1}, fy_n)] + a_3[q(fy_{n-1}, fy_n) + q(fy_n, fy_{n+1})] +$
$a_1 q(fy_n, fy_{n-1}) + a_2 [q(fy_n, fy_{n-1}) + q(fy_{n+1}, fy_n)] +$
$a_2[q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1})] + a_3[q(fy_n, fy_{n-1}) + q(fy_{n+1}, fy_n)]$

$\le a_1 [q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1})] + a_2 [q(fy_{n-1}, fy_n) + q(fy_n, fy_{n+1}) +$
$q(fy_n, fy_{n-1}) + q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1}) + q(fy_{n+1}, fy_n) +$
$q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1})] + a_3[q(fy_{n-1}, fy_n) + q(fy_n, fy_{n+1}) + q(fy_n, fy_{n-1}) + q(fy_{n+1}, fy_n)]$

$\le a_1 [q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1})] + 3 a_2 [q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1})] +$
$a_2 [q(fy_n, fy_{n+1}) + q(fy_{n+1}, fy_n)] + a_3[q(fy_{n-1}, fy_n) + q(fy_n, fy_{n-1})] +$
$a_3[q(fy_n, fy_{n+1}) + q(fy_{n+1}, fy_n)]$

$\Longrightarrow q(fy_n, fy_{n+1}) + q(fy_{n+1}, fy_n)$

$\le \left(\frac{a_1+3a_2+a_3}{1-a_2-a_3}\right) [q(fy_n, fy_{n-1}) + q(fy_{n-1}, fy_n)]$

$\Longrightarrow \alpha_n \le \lambda\, \alpha_{n-1}$, $\alpha_n = q(fy_n, fy_{n+1}) + q(fy_{n+1}, fy_n)$, $\lambda = \left(\frac{a_1+a_2+a_4}{1-a_3-3a_4}\right)$

Proceeding like this, we get

$\alpha_n \le \lambda^n \alpha_0$, $\alpha_n \to 0$ as $n \to \infty$

$q(fy_n, fy_{n-1}) + q(fy_{n-1}, fy_n) \to 0$

$q(fy_n, fy_{n-1}) \to 0$ and $q(fy_{n-1}, fy_n) \to 0$

Let $m > n \ge 1$. Then it follows that

$q(x_n, x_m) + q(x_m, x_n)$

$\le \alpha_n + \alpha_{n+1} + \ldots + \alpha_{m-1}$

$\le \lambda^n \alpha_0 + \lambda^{n+1} \alpha_0 + \ldots + \lambda^{m-1} \alpha_0$

$\le (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1})\, \alpha_0$

$\le \left(\frac{\lambda^n}{1-\lambda}\right) \alpha_0$

$\Longrightarrow q(x_n, x_m) + q(x_m, x_n)$

$\le \mu\, [q(x_0, x_1) + q(x_1, x_0)]$ where $\mu = \frac{\lambda^n}{1-\lambda}$

Thus by Lemma 2.10, $\{x_n\}$ is a Cauchy Sequence in X.

Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$

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RADMAS– 2016

$fy_n \to x^*$ as n $\to \infty$
$gy_{n+1} \to x^*$ as n $\to \infty$
$fg\, y_{n+1} \to fx^*$ as n $\to \infty$ since f is continuous
$gf\, y_{n+1} \to fx^*$ as n $\to \infty$ since f & g commute
But $gf\, y_{n+1} \to gx^*$ as n $\to \infty$ since g is continuous
$\Longrightarrow fx^* = gx^*$
To show that $q(fx^*, fx^*) = 0$
Now $q(fx^*, fx^*)$
$\leq a_1 q(gx^*, gx^*) + a_2 [q(gx^*, fx^*) + q(gx^*, fx^*)] + a_3 [q(gx^*, fx^*) + q(gx^*, fx^*)]$
$\leq a_1 q(fx^*, fx^*) + a_2 [q(fx^*, fx^*) + q(fx^*, fx^*)] + a_3 [q(fx^*, fx^*) + q(fx^*, fx^*)]$
$= (a_1 + 2a_2 + 2a_3)\, q(fx^*, fx^*)$
$\therefore q(fx^*, fx^*) \leq (a_1 + 2a_2 + 2a_3)\, q(fx^*, fx^*)$
But $(a_1 + 2a_2 + 2a_3) < (a_1 + 4a_2 + 2a_3) < 1$
$\therefore q(fx^*, fx^*) = 0$
Now to show that $q(x^*, x^*) = 0$
Consider $q(fy_n, fy_n)$
$\leq a_1 q(gy_n, gy_n) + a_2 [q(gy_n, fy_n) + q(gy_n, fy_n)] +$
$a_3 [q(gy_n, fy_n) + q(gy_n, fy_n)]$
on letting $n \to \infty$ we get
$q(x^*, x^*) \leq a_1 q(x^*, x^*) + a_2 [q(x^*, x^*) + q(x^*, x^*)] + a_3 [q(x^*, x^*) + q(x^*, x^*)]$
$= (a_1 + 2a_2 + 2a_3)\, q(x^*, x^*)$
But $(a_1 + 2a_2 + 2a_3) < 1$
$q(x^*, x^*) = 0$
To show that x^* is a common fixed point of f and g
Now $q(fy_n, fx^*)$
$\leq a_1 q(gy_n, gx^*) + a_2 [q(gy_n, fx^*) + q(gx^*, fy_n)] + a_3 [q(gy_n, fy_n) + q(fx^*, gx^*)]$
on letting $n \to \infty$ we get
$q(x^*, fx^*) \leq a_1 q(x^*, fx^*) + a_2 [q(x^*, fx^*) + q(fx^*, x^*)] + a_3 [q(x^*, x^*) + q(fx^*, fx^*)]$
$= a_1 q(x^*, fx^*) + a_2 [q(x^*, fx^*) + q(fx^*, x^*)]$
similarly $q(fx^*, x^*) \leq a_1 q(fx^*, x^*) + a_2 [q(fx^*, x^*) + q(x^*, fx^*)]$
on adding we get,
$q(x^*, fx^*) + q(fx^*, x^*) \leq a_1 [q(x^*, fx^*) + q(fx^*, x^*)] + 2a_2 [q(x^*, fx^*) + q(fx^*, x^*)]$
$(1 - a_1 - 2a_2)[q(x^*, fx^*) + q(fx^*, x^*)] \leq 0$
$q(x^*, fx^*) = 0$ and $q(fx^*, x^*) = 0$
$fx^* = x^* = gx^*$

Uniqueness

Suppose that there is another fixed point y^* of f and g

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RADMAS– 2016
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$f y^* = y^*$ and $g y^* = y^*$

$q(x^*, y^*) = q(fx^*, fy^*)$

$\leq a_1 q(gx^*, gy^*) + a_2[q(gx^*, fy^*) + q(gy^*, fx^*)] + a_3[q(gx^*, fx^*) + q(gy^*, fy^*)]$

$= a_1 q(x^*, y^*) + a_2[q(x^*, y^*) + q(y^*, x^*)] + a_3[q(x^*, x^*) + q(y^*, y^*)]$

similarly, $q(y^*, x^*) \leq a_1 q(y^*, x^*) + a_2[q(y^*, x^*) + a_3 q(x^*, y^*)] + a_3[q(x^*, x^*) + q(y^*, y^*)]$

on adding, we get

$q(x^*, y^*) + q(y^*, x^*) \leq a_1[q(x^*, y^*) + q(y^*, x^*)] + 2a_2[q(y^*, x^*) + q(x^*, y^*)]$

$\Longrightarrow (1 - a_1 - 2a_2)[q(x^*, y^*) + q(y^*, x^*)] \leq 0$

$\Longrightarrow [q(x^*, y^*) + q(y^*, x^*)] = 0$

$\Longrightarrow q(x^*, y^*) = 0$ and $q(y^*, x^*) = 0$

$\therefore\ x^* = y^*$

$\therefore$ Fixed Point is Unique

$\therefore$ f and g have a unique common fixed point in X.

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