BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
RADMAS– 2016
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com

A FIXED POINT THEOREM IN METRIC SPACES AND APPLICATION TO CONE METRIC SPACES

Prof. K. P. R. SASTRY[1], Capt. K. SUJATHA[2]
[1]8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam -530 017, India
[2]Selection Grade Lecturer in Mathematics, St. Joseph's College for Women (Autonomous), Visakhapatnam-530 004, India
kambhampati.sujatha@yahoo.com

ABSTRACT

G.V.R. Babu, G.N. Alameyehu and K.N.V.V. Varaprasad [11] proved common fixed point theorems for generalized contraction pair of self maps on a complete cone metric space. In this paper we introduce the notion of quasi generalized contraction pair of self maps and observe that every quasi generalized contraction pair of self maps is a generalized contraction pair. We show that a quasi generalized contraction pair need not be a generalized contraction pair and prove some theorems and provide examples in support of our results

1 INTRODUCTION

Guang and Xian [35] generalized the notion of metric spaces replacing the set of all real numbers by an ordered Banach space defining in this way a cone metric space. These authors also described the convergence in this cone metric spaces and the corresponding notation of complete cone metric spaces. These authors, also proved some fixed point theorems of contractive mappings on complete cone metric spaces. Later Rezapore and Hamlbarani [68] proved some of the results of Guang and Xiang [35] by omitting the assumption of normality on the cone.

G.V.R. Babu, G.N. Alameyehu and K.N.V.V. Varaprasad [11] proved common fixed point theorems for generalized contraction pair of self maps on a complete cone metric space. In this paper we introduce the notion of quasi generalized contraction pair of self maps and observe that every quasi generalized contraction pair of self maps is a generalized contraction pair. We show that a quasi generalized contraction pair need not be a generalized contraction pair and prove some theorems and provide examples in support of our results.

1.1 Definition : (G.V.R.Babu, G.N.Alameyehu and K.N.V.V.Varaprasad [11])

Let (X, d) be a cone metric space and P be a cone with non empty interior. Let f, g be self maps on X. Suppose that there exists a constant $k \in (0,1)$ and

$p(x, y) \in \{\, d(x, y)\,, d(x, fx)\,, d(y, gy)\,, \frac{d(x,gy)+ d(y,fx)}{2}\}$ such that

$d(fx, gy) \leq k\, p(x, y)$

for all x, y in X. Then the pair *(f, g)* is called a generalized contraction pair on X.

G.V.R. Babu, G.N. Alameyehu and K.N.V.V. Varaprasad [11] proved the following fixed point theorem in a complete cone metric space, for a generalized contraction pair on X.

1.2 Theorem : ([1], Theorem 2.1) Let *(X, d)* be a complete cone metric space. Suppose that (*f, g*) is a generalized contraction pair on *X* .Then *f* and g have a unique common fixed point in *X.*

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com
RADMAS– 2016

2 Main results

In this section we introduce the notion of a quasi generalized contraction pair of self maps on a cone metric space (X, *d)*. We observe that every generalized contraction pair is a quasi generalized contraction pair and the other way is not true.

2.1 Definition : Let *(X, d)* be a cone metric space and (*f, g)* be a pair of self maps on *X.* The pair (*f , g)* is said to be a quasi generalized contraction pair on X if there exists $\delta \in (0,1)$ such that the following conditions hold.

$(i)\ d(fx, gfx) \le \delta\ d(x, fx)\ and$

$(ii)\ d(gx, fgx) \le \delta d(x, gx)\ for\ all\ x \in X.$

We observe that every generalized contraction pair is a quasi generalized contraction pair.

We have the following lemma.

2.2 Lemma : For any quasi generalized contraction pair (*f, g)* of self maps on a cone metric space X ,the fixed point sets of *f* and *g* are the same .

Proof : Follows from the definition of a quasi generalized contraction pair.

2.3 Remark : In Lemma 5.4 , if (f, g) *is a* generalized contraction pair, then the fixed point set of f (and hence of *g)* is at most singleton.

The following example shows that a quasi generalized contraction pair need not be a generalized contraction pair.

2.4 Example : Let $X = \left\{\frac{1}{(n+1)^2} : n = 1, 2, ..\right\} \cup \{0\}$ and *d* be the usual metric on X. Define f: X → X by f$(x) = x^2\ if\ x \in X\ and\ x \neq 0, f(0) = \frac{1}{4}\ and\ put\ g = f.$

Then the pair *(f , g) is a* quasi generalized contraction pair but not a generalized contraction pair, because this pair has no common fixed points in view of Theorem 5.2.

2.5 Notation: Suppose *f* and *g* are self maps on a non empty set X and $x_0 \in$ X. Then we define sequence { x_n} in *X* iteratively as follows: $x_1 = f(x_0),\ x_2 = g(x_1)$ and

In general $x_{2n+1} = f(x_{2n})\ and\ x_{2n+2} = g(x_{2n+1}),$ for $n = 0, 1, 2 ...$ ... (A)

2.6 Theorem : Let X be a complete cone metric space and (f, g) be a quasi generalized contraction pair on X. Then we have the following

(i) The sequence defined in (A) is a Cauchy sequence in X and hence is convergent in X

(ii) If one of the two functions *f* and is continuous then the fixed point sets of *f* and *g* are non empty.

Proof : Let us observe that for any $n \in N$,

$(a)\ d(x_{2n}, x_{2n+1}) \le \delta^{2n}\ d(x_0, x_1)\ and$

$(b) d(x_{2n+1}, x_{2n+2}) \le \delta^{2n+1}\ d(x_0, x_1)\ for\ the\ sequence\ \{x_n\}$ denoted by (A) in X and $\delta \in (0,1)$ satifying (i) and (ii) of Definition 5.3. To see this,

we have that $d(x_{2n}, x_{2n+1}) = d(gx_{2n-1}, fx_{2n})$

(by the definition of the sequence $\{x_n\}$)

$= d(gx_{2n-1}, fgx_{2n-1})$

$\le \delta\ d(x_{2n-1}, gx_{2n-1})$

$= \delta\ d(fx_{2n-2}, gx_{2n-2}) \le \delta^2\ d(x_{2n-2}, x_{2n-1})$

Continuing this way we get (a) after a finite number steps.

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com
RADMAS– 2016

In a similar way (b) can be established.

Now, from (a) and (b) it follows that

$$d(x_n, x_{n+1}) \leq \delta^n\, d(x_0, x_1) \text{ for any } n \geq 1 \qquad \ldots \quad \text{(B)}$$

Let n, m be positive integers such that $n < m$.

Then $d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$

$\leq \delta^n\, d(x_0, x_1) + \delta^{n+1}\, d(x_0, x_1) + \cdots + \delta^{m-1}\, d(x_0, x_1)$

(by B)

$= (\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})\, d(x_0, x_1)$

$\leq \{\frac{\delta^n}{1-\delta}\}\, d(x_0, x_1) \to 0 \text{ as } n \to \infty$, since $0 < \delta < 1$.

Hence the sequence $\{x_n\}$ is Cauchy. Since the space X is complete, this sequence $\{x_n\}$ converges to some point *x* in X.

Now suppose f is continuous.

Then, we have that $f(x) = f(lim x_{2n}) = lim f(x_{2n}) = lim\, x_{2n+1} = x$.

Thus *x* is a fixed point of *f* and hence the fixed point set of *f* is non-empty. Now by Lemma 5.4 the fixed point set of g is also non empty. Similarly we can prove that if g is continuous then *x* is fixed point of g. Consequently the fixed point sets of f and g are non empty.

2.7 Remark : If $f = g$ = identity function on X then the pair (f, g) *is* quasi generalized contraction pair. Further the fixed point sets of f and g are the same and may contain more than one fixed point.

2.8 Remark : If both *f* and , g **a**re not continuous in theorem 5.8 , then the fixed point sets of f and , g may be empty.

This is justified in the following example:

2.9 Example : Let $X = [0, \frac{1}{2}]$ and *d* usual metric on X.

Define f: X $\rightarrow$ X by $f(x) = x^2$ *if* $x \in (0, \frac{1}{2}]$ and $f(0) = \frac{1}{2}$ and put $g = f$.

Then the pair (f, , g) is a quasi generalized contraction pair. Also both *f* and g are not continuous and *f* and g have no fixed points in X.

References

[1]. Aage C.T and Salunke J.N : On common fixed points for contractive type mappings in Cone metric spaces , Bulletin of Math-Analy and Appl. Vol 1, Issue 3 (2009), 10 – 15.

[2]. Babu G.V.R , Alameyehu G.N and Varaprasad K.N.V.V : Common fixed point theorems of generalized contraction, Zamfirescu pair of maps in cone metric spaces, Albanian Journal of Mathematics. 4 (1) (2010), 19-29.

[3]. Huang Long – Guang, Zhan Xian : “Cone metric spaces and fixed point theorems of contractive mapping.” J.Math.Anal.Appl. 332 (2007), 1468 - 1476.

[4]. Rezapour Sh and Hamlbarani R : Some notes on the paper “cone metric spaces and fixed point theorems of contractive mappings”, J.Math, Anal. Appl., 345 (2008), 719 – 724.