BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com
RADMAS– 2016

EQUIVALENCE RELATIONS AND CONGRUENCES IN PARTIALLY ORDERED SEMIGROUPS

Dr.P.M.PADMALATHA
Department of Mathematics, J.M.J. College, Tenali.
padmalatha323@gmail.com

ABSTRACT
In this paper the terms; equivalence relation, left congruence, right congruence, congruence generated by ρ, band, semilattice, semilattice congruence and complete are introduced. It is proved that an equivalence relation ρ on a po semigroup S is a congruence if and only if for all *a*, *b*, *c*, *d* ∈ S, *a* ρ *b* and *c* ρ *d* implies (*ac*)ρ(*bd*). It is proved that if ρ_1 and ρ_2 are two left congruences (resp. right congruences, congruences) of a po semigroup S, then (ρ_1 o ρ_2) is a left congruence (resp. right congruence, congruence) of S. Also it is proved that if $\rho_1, \rho_2, \ldots, \rho_n$ are left congruences (resp. right congruences, congruences) of a po semigroup S, then $\rho_1 o \rho_2 o \ldots o \rho_n$ is a left congruence (resp. right congruence, congruence) of S. It is proved that the intersection of family of congruences on a po semigroup S is again a congruence on S. Futher it is proved that the union of a non-empty family of congruences on a po semigroup S is a congruence on S.
KEY WORDS : Equivalence relation, left congruence, right congruence, congruence generated by ρ, band, semilattice, semilattice congruence and complete.

1. CONGRUENCES:

DEFINITION 1.1 : A relation ρ on a po semigroup S is said to be ***reflexive*** on S if *x*ρ*x* for all *x* ∈ S.

DEFINITION 1.2 : A relation ρ on a po semigroup S is said to be ***symmetric*** on S if *x, y* ∈ S and *x*ρ*y* implies *y*ρ*x*.

DEFINITION 1.3 : A relation ρ on a po semigroup S is said to be ***transitive*** on S if *x, y, z* ∈ S, *x*ρ*y*, *y*ρ*z* implies *x*ρ*z*.

DEFINITION 1.4 : A relation ρ on a po semigroup S is said to be an ***equivalence relation*** on S if (i) *x*ρ*x* for all *x* ∈ S, (ii) *x, y* ∈ S, *x*ρ*y* implies *y*ρ*x* (iii) *x, y, z* ∈ S, *x*ρ*y*, *y*ρ*z* implies *x*ρ*z*.

NOTE 1.5 : Let S be a po semigroup. A relation ρ on S is an equivalence relation on S iff ρ is (i) reflexive (ii) symmetric and (iii) transitive.

DEFINITION 1.6 : Let S be a po semigroup. An equivalence relation ρ on S is said to be a ***left congruence***, if *a, b, c* ∈ S, *a* ρ *b* implies (*ca*)ρ(*cb*).

DEFINITION 1.8 : Let S be a po semigroup. An equivalence relation ρ on S is said to be a ***right congruence***, if *a, b, c* ∈ S, *a* ρ *b* implies (*ac*)ρ(*bc*).

DEFINITION 1.10 : Let S be a po semigroup. An equivalence relation ρ on S is said to be a ***congruence***, if *a, b, c* ∈ S, *a* ρ *b* implies (*ca*)ρ(*cb*) and (*ac*)ρ(*bc*).

NOTE 1.12 : An equivalence relation ρ on a po semigroup S is a congruence iff it is both a left congruence and a right congruence on S.

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com
RADMAS– 2016

THEOREM 1.13 : An equivalence relation ρ on a po semigroup S is a congruence if and only if *a*, *b*, *c*, *d* ∈ S, *a* ρ *b* and *c* ρ *d* implies *ac* ρ bd.

***Proof* :** Let ρ be an equivalence relation on a po semigroup S.
Suppose that ρ is a congruence on S. Let *a*, *b*, *c*, *d* ∈ S, *a* ρ *b* and *c* ρ *d*
a, *b*, *c* ∈ S, *a* ρ *b* and ρ is right congruence ⇒ (*ac*)ρ(*bc*).
b, *c*, *d* ∈ S, *c* ρ *d* and ρ is left congruence ⇒ (*bc*)ρ(*bd*).
Now (*ac*) ρ (*bc*), (*bc*) ρ (*bd*), ρ is transitive ⇒ (*ac*)ρ(*bd*).
Conversely suppose that ρ is an equivalence relation on a po semigroup S such that
a, *b*, *c*, *d* ∈ S, *a* ρ *b* and *c* ρ *d* ⇒ (*ac*)ρ(*bd*).
Now *c* ρ *c*, *a* ρ *b* ⇒ (*ca*)ρ(*cb*) ⇒ ρ is a left congruence.
a ρ *b*, *c* ρ *c* ⇒ (*ac*)ρ(*bc*) ⇒ ρ is a right congruence and hence ρ is a congruence.

NOTATION 1.14 : Let ρ be a congruence relation on a po semigroup S. We denote the set {*b* ∈ S / *a*ρ*b*} by a_ρ and is called ρ-class containing *a*. The set of all ρ-classes is denoted by S/ρ.

THEOREM 1.15 : If S is a po semigroup and ρ is a congruence on S then S/ρ is a semigroup with respect to the operation defined by $a_\rho b_\rho = (ab)_\rho$ for all $a_\rho, b_\rho \in S/\rho$.

***Proof* :** If $a_\rho, b_\rho \in S/\rho$, then we define the multiplication on S/ρ,
given by $(a_\rho)(b_\rho) = (ab)_\rho$ for all *a*, *b* ∈ S.
This is well defined, since for all *a*, *b*, *c*, *d* ∈ S, $a_\rho = b_\rho$ and $c_\rho = d_\rho$ ⇒ *a*ρ*b*, *c*ρ*d*
⇒ (*ac*)ρ(*bc*), (*bc*)ρ(*bd*)
⇒ (*ac*)ρ(*bd*) ⇒ $(ac)_\rho = (bd)_\rho$.
Let $(a)_\rho, (b)_\rho, (c)_\rho \in S/\rho$.
Then $[(a)_\rho(b)_\rho](c)_\rho = (ab)_\rho(c)_\rho = [(ab)c]_\rho = [a(bc)]_\rho = (a)_\rho(bc)_\rho = (a)_\rho[(b)_\rho(c)_\rho]$. Therefore S/ρ is a semigroup.

DEFINITION 1.16 : Let ρ be a congruence relation on a po semigroup S. Then the semigroup S/ρ of all ρ-classes with respect to the operation defined as $(a_\rho)(b_\rho) = (ab)_\rho$ for all *a*, *b* ∈ S is called the ***quotient semigroup*** of S relative to the congruence ρ.

NOTE 1.17 : If S is a po semigroup and ρ is a congruence on S, then the quotient semigroup S/ρ is not a po semigroup w. r. t the relation ≼ on S/ρ defined by means of the order ≤ on S, that is, $(a)_\rho \preccurlyeq (b)_\rho$ ⇔ there exist $x \in (a)_\rho$ and $y \in (b)_\rho$ such that *x* ≤ *y*. But the relation is not a partial order, in general. We show it in the following example.

EXAMPLE 1.18 : We consider the po semigroup S = {*a*, *b*, *c*, *d*, *e*} defined by the multiplication and the order ≤ below:

α	a	b	c	d	e
a	a	e	c	d	e
b	a	e	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

β	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

and ≤ = { (*a*, *a*), (*a*, *d*), (*b*, *b*), (*c*, *c*), (*c*, *e*), (*d*, *d*), (*e*, *e*) }.
For *x*, *y*, *z* ∈ S, we have

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
RADMAS– 2016
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com

$(xy)a = a = x(ya)$, $(xy)c = c = x(yc)$
$(xy)d = d = x(yd)$, $(xy)e = e = x(ye)$
$(xy)b = e = x(yb)$,
$(xy)b = e = x(yb)$ if $y \neq b$
$(xb)b = e = x(bb)$ if $x \neq b$
$(bb)b = e = b(bb)$, $(bb)b = b = b(bb)$.

Then S is a semigroup. Since $xa \leq xd$, $ax = dx$, $xc \leq xe$, $cx = ex$ for all $x \in S$, S is a po semigroup.

Let ρ be the congruence on S defined as follows:

$$\rho = \{ (a, a), (b, b), (c, c), (d, d), (e, e), (a, e), (e, a), (c, d), (d, c)\}.$$

Let $\preccurlyeq$ be an order on S/ρ defined by means of the order $\leq$ on S, that is,

$(a)_\rho \preccurlyeq (b)_\rho \Leftrightarrow$ there exist $x \in (a)_\rho$ and $y \in (b)_\rho$ such that $x \leq y$.

We have $a_\rho = \{a, e\}$, $b_\rho = \{b\}$ and $c_\rho = \{c, d\}$. Also we have $a_\rho \preccurlyeq c_\rho$ and $c_\rho \preccurlyeq a_\rho$ but $a_\rho \neq c_\rho$. Thus $\preccurlyeq$ is not an order relation on S/ρ .

THEOREM 1.19 : Let S be a po semigroup. If ρ_1 and ρ_2 are two left congruences of S, then $(\rho_1 \circ \rho_2)$ is a left congruence on S.

Proof : Let ρ_1 and ρ_2 be two left congruences on S.

Clearly $(\rho_1 \circ \rho_2)$ is an equivalence relation on S.

Let $a, b \in S$ and $s \in S$, $a(\rho_1 \circ \rho_2)b$.

Since $a(\rho_1 \circ \rho_2)b$ there exists $c \in S$ such that $a\, \rho_1 c$ and $c\, \rho_2\, b$.

Since ρ_1, ρ_2 are left congruences on S, it follows that $(sa)\rho_1(sc)$ and $(sc)\rho_2(sb)$

This implies that $(sa)(\rho_1 o \rho_2)(sb)$ and hence $(\rho_1 o \rho_2)$ is a left congruence on S.

From theorem 1.19, it can be easily prove the following result by induction:

COROLLARY 1.20 : Let S be a po semigroup. If $\rho_1, \rho_2, \ldots, \rho_n$ are left congruences on S, then $\rho_1 o \rho_2 o \ldots . \rho_n$ is a left congruence on S.

THEOREM 1.21 : Let S be a po semigroup. If ρ_1 and ρ_2 are two right congruences on S, then $(\rho_1 o \rho_2)$ is a right congruence on S.

Proof : Let ρ_1 and ρ_2 be two right congruences on S.

Suppose $a(\rho_1 \circ \rho_2)b$ holds for $a, b \in S$.

Then there exists $c \in S$ such that $a\, \rho_1 c$ and $c\, \rho_2\, b$ hold.

Since ρ_1, ρ_2 are right congruences on S, it follows that $(as)\rho_1(cs)$ and $(cs)\rho_2(bs)$ for all $s \in S$. This implies that $(as)(\rho_1 o \rho_2)(bs)$ hold for all $s \in S$ and hence $(\rho_1 o \rho_2)$ is a right congruence on S.

From theorem 1.21, the following corollary can be proved easily by induction:

COROLLARY 1.22 : Let S be a po semigroup. If $\rho_1, \rho_2, \ldots, \rho_n$ are right congruences on S, then $\rho_1 o \rho_2 o \ldots . \rho_n$ is a right congruence on S.

THEOREM 1.23 :Let S be a po semigroup. If ρ_1 and ρ_2 are two congruences on S, then $(\rho_1 o \rho_2)$ is a congruence of S.

Proof : By theorem 1.19, $(\rho_1 o \rho_2)$ is a left congruence on S.

By theorem 1.21, $(\rho_1 o \rho_2)$ is a right congruence on S and

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com
RADMAS– 2016

hence ($\rho_1 o \rho_2$) is a congruence on S.

From theorem 1.23, the following result can be proved easily by induction:

COROLLARY 1.24: Let S be a po semigroup. If $\rho_1, \rho_2, \ldots, \rho_n$ **are congruences on S, then** $\rho_1 o \rho_2 o \ldots . \rho_n$ **is a congruence of S.**

THEOREM 1.25 : The intersection of any family of congruences on a po semigroup S is again a congruence on S.

Proof : Let $\{\rho_i / i \in \Delta\}$ be a family of congruences on S.

Let ψ = $\bigcap_{i\in\Delta} \rho_i$. Clearly Ψ is an equivalence relation on S.

Let *a*, *b*, *c*, *d* ∈ S.

Suppose that *a* ψ *b*, *c* ψ *d*.

a ψ *b*, *c* ψ *d* ⇒ *a* $\bigcap_{i\in\Delta} \rho_i$ *b*, *c* $\bigcap_{i\in\Delta} \rho_i$ *d* ⇒ *a* ρ_i *b*, *c* ρ_i *d* for all ρ_i

⇒ *ac* ρ_i *bd* for all ρ_i ⇒ *ac* $\bigcap_{i\in\Delta} \rho_i$ *bd* ⇒ *ac* ψ *bd*.

Hence the intersection of any family of congruences on a po semigroup S is again a congruence on S.

THEOREM 1.26 : The union of any family of congruences on a po semigroup S is a congruence on S.

Proof : Let $\{\rho_i / i \in \Delta\}$ be a family of congruences on S.

Let ψ = $\bigcup_{i\in\Delta} \rho_i$ where ρ_i is a congruence on po semigroup S.

Let *a*, *b*, *c* ∈ S. Suppose that *a* ψ *b*.

a ψ *b* ⇒ *a* $\bigcup_{i\in\Delta} \rho_i$ *b* ⇒ *a* ρ_i *b* for some ρ_i on S

⇒ *a* ρ_i *b* for some ρ_i on S and ρ_i is a congruence on S⇒ *ca* ρ_i *cb* ⇒ *ca* $\bigcup_{i\in\Delta} \rho_i$ *cb*

⇒ *ca* ψ *cb* ⇒ ψ is a left congruence on S

Now *a* ρ_i *b* for some ρ_i on S, ρ_i is a congruence on S⇒ *ac* ρ_i *bc* ⇒ *ac* $\bigcup_{i\in\Delta} \rho_i$ *bc*

⇒ *ac* ψ *bc* ⇒ ψ is right congruence on S.

So, Ψ is a congruence on the po semigroup S. Therefore the union of a non-empty family of congruences on a po semigroup S is a congruence on S.

NOTE 1.27 : The set of all congruences on a po semigroup *S* is denoted by *C*(*S*).

DEFINITION 1.28 : The intersection of all congruences on a po semigroup S containing a binary relation ρ on S is called the ***congruence generated by* ρ .**

DEFINITION 1.29 : A po semigroup S is said to be a ***band*** if every element of S is a idempotent.

DEFINITION 1.30 : A po semigroup S is said to be a ***semilattice*** if S is a commutative band.

DEFINITION 1.31 : A congruence ρ on a po semigroup S is said to be ***semilattice congruence*** if *a*, *b* ∈

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
RADMAS– 2016
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com

S ⇒ *aa* ρ *a* and *ab* ρ *ba*.

DEFINITION 1.32 : A semilattice congruence ρ on a po semigroup S is said to be ***complete*** if for any *a*, *b* ∈ S, *a* ≤ *b* implies *a* ρ *ab*.

NOTE 1.33 : A semilattice congruence ρ on a po semigroup S is ***complete*** iff *a*, *b* ∈ S, *a* ≤ *b* implies *a*ρ(*ab*).

REFERENCES

[1]. ANJANEYULU A., *Primary ideals in semigroups* – Semigroup Forum, Vol.20 (1980) 129-144.

[2]. ANJANEYULU A., *On primary semigroups* – Czechoslovak Mathematical Journal, 30(105) 1980, Praha.

[3]. BOURNE S.G., *Ideal theory in a commutative semigroup* - Dissertation, John Hopkins University (1949).

[4]. BRAJA ISLAM., *Characterizations of Regular Gamma Semigroups Using Quasi-Ideals* – International Journal of Mathematical Analysis, Vol. 3, 2009, No. 36, 1789-1794.

[5]. CHINRAM. R and SIAMMAI. P., *On Green's Relations for* **Γ**-*semigroups and Reductive* **Γ**-*semigroups* – International Journal of Algebra, Vol. 2, 2008, No.4, 187-195.

[6]. CLIFFORD A. H. and PRESTON G. B., *The algebraic theory of semigroups* – Vol-I, American Mathematical Society, Providence (1961).

[7]. CLIFFORD A. H. and PRESTON G. B., *The algebraic theory of semigroups* – Vol-II, American Mathematical Society, Providence (1967).

[8]. DUTTA. T. K and CHATTERJEE. T. K., *Green's equivalence on* **Γ**-*semigroup,* Bull.Col.Soc., 80, (1987), 30-35.

[9]. JUN. Y. B., **Γ**-*filters and semilattice* **Γ**-*congruences in po*-**Γ**-*semigroups,* PU. M. A. No. 2-3-4 (1977), 275-280.

[10]. PETRICH. M., *Introduction to semigroups* - Merril Publishing Company, Columbus, Ohio, (1973).

[11]. SCWARTZ S., *Prime ideals and maximal ideals in semigroups* - Czechoslovak Mathematical Journal.,19(94), (1969), 72-79.

[12]. SEN. M.K., *On* **Γ**-*semigroups,* Proceeding of International Conference on Algebra and it's Applications, Decker Publication, New York 301.

[13]. SEN. M.K. and SAHA. N.K., *On* Γ - *Semigroups-II*, Bull. Calcutta Math. Soc. 79 (1987), No.6, 331-335.

[14]. SIRIPITUKDET. M and IAMPAN. A., *On the least* (*ordered*) *Semilattice Congruence in Ordered* **Γ**-*semigroups* – The Journal of Mathematics Volume 4(2006) Number 2 : 403 – 415.