BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH
A Peer Reviewed International Journal,
Contents available on www.bomsr.com
RADMAS– 2016
Vol.4. S1.2016; ISSN: 2348-0580
Email:editorbomsr@gmail.com

FIXED POINT THEOREM FOR TWO SELF-MAPS IN A *G*-METRIC SPACE

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Abstract. An extension of a result of Vats et al [3] is obtained to a pair of compatible self-maps on a *G*-metric space.

2010 Mathematics Subject Classification. 54H25
Key words. *G*-metric space, Compatible self-maps, *G*-Cauchy sequence, Common fixed point

1. Introduction

Let X be a nonempty set and $G: X \times X \times X \to [0,\infty)$ be such that

(G1) $G(x,y,z)=0$ $x,y,z \in X$ are such that $x=y=z$

(G2) $G(x,x,y)>0$ for all $x,y\in X$ with $x \neq y$

(G3) $G(x,x,y)<G(x,y,z)$ for all $x,y,z\in X$ with $z\neq y$,

(G4) $G(x,y,z)=G(\pi(x,y,z))$, where π is a permutation on the set $\{x,y,z\}$

(G5) $G(x,y,z)\le G(x,w,w)+G(w,y,z)$ for all $x,y,z,w\in X$.

Then *G* is called a *G*-metric [2] on *X*, and the pair (X,G) denotes a *G*-metric space. It was shown that $\rho_G(x,y)=G(x,y,y)+G(x,x,y)$ is a metric on *X*, and the family of *G*-balls of the form $B_G(x,r)=\{y\in X: G(x,y,y)<r\}$ is a base topology, called the *G*-metric topology $\tau(G)$ on *X*.

A sequence $\langle x_n\rangle_{n=1}^{\infty}\subset X$ is said to be *G*-convergent with limit $p\in X$, if it converges to *p* in $\tau(G)$. According to [2], $\langle x_n\rangle_{n=1}^{\infty}\subset X$ converges to $p\in X$, if and only if, $\lim_{n\to\infty} G(x_n,x_n,p)=0$ or $\lim_{n\to\infty} G(x_n,p,p)=0$. And, $\langle x_n\rangle_{n=1}^{\infty}\subset X$ is said to be *G*-Cauchy, if $G(fx_n, fx_m, fx_m)\to 0$ $m, n\to\infty$. The space *X* is said to be *G*-complete, if every *G*-Cauchy sequence in *X* converges in it.

From the definition of *G*-metric space, it follows that

$$G(x,y,y)\le 2G(x,x,y) \text{ for all } x, y\in X \quad \dots \quad (1.1)$$

Self-maps *f* and *g* on a *G*-metric space (X,G) are said to be compatible [1] if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n)=0$ whenever $\langle x_n\rangle_{n=1}^{\infty}\subset X$ is such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$.

2. Main Result

We prove the following common fixed point theorem:

Theorem 2.1. *Suppose that f and g are self-maps on a complete G-metric space* (X,G) *such that*

(a) $f(X)\subset g(X)$

(b) *f or g is continuous, and*

$$G(fx, fy, fz)\le k\max\{G(gx, fx, fx)+G(gy, fy, fy)+G(gz, fz, fz),$$

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$$G(gx, fy, fy)+G(gy, fx, fx)+G(gz, fy, fy),$$
$$G(gx, fz, fz)+G(gy, fz, fz)+G(gz, fx, fx)\}$$
$$\textit{for all } x, y, z, w \in X, \quad \ldots (2.1)$$

where $0<k<1/3$. *If f and g are compatible, then f and g have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. In view of the inclusion (a), we can choose points $x_1, x_2, \ldots, x_n, \ldots$ such that

$$fx_{n-1} = gx_n \text{ for } n = 1, 2, \ldots \quad \ldots \quad (2.2)$$

Writing $x = x_n$ and $y = z = x_{n+1}$ in (2.1), and then using (1.1), (2.2) and the rectangle inequality of *G*, we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le k\max\{G(fx_{n-1}, fx_n, fx_n)+G(fx_n, fx_{n+1}, fx_{n+1})+G(fx_n, fx_{n+1}, fx_{n+1}),$$
$$G(fx_{n-1}, fx_{n+1}, fx_{n+1})+G(fx_n, fx_n, fx_n)+G(fx_n, fx_{n+1}, fx_{n+1}),$$
$$G(fx_{n-1}, fx_{n+1}, fx_{n+1})+G(fx_n, fx_{n+1}, fx_{n+1})+G(fx_n, fx_n, fx_n)\}$$
$$\le k\max\{G(fx_{n-1}, fx_n, fx_n)+2G(fx_n, fx_{n+1}, fx_{n+1}),$$
$$[G(fx_{n-1}, fx_n, fx_n)+G(fx_n, fx_{n+1}, fx_{n+1})]+G(fx_n, fx_{n+1}, fx_{n+1})$$
$$= k\,[G(fx_{n-1}, fx_n, fx_n)+2G(fx_n, fx_{n+1}, fx_{n+1})]$$

or $$G(fx_n, fx_{n+1}, fx_{n+1}) \le \tfrac{k}{1-2k} G(fx_{n-1}, fx_n, fx_n).$$

By induction, we get

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le q^n G(fx_0, fx_1, fx_1), \text{ for all } n \ge 1. \quad \ldots \quad (2.3)$$

where $q = \frac{k}{1-2k} < 1$. By repeated use of rectangle inequality and (2.1), we have

$$G(fx_n, fx_m, fx_m) \le \sum_{j=n}^{m-1} G(fx_j, fx_{j+1}, fx_{j+1}) \le \sum_{j=n}^{m-1} q^j G(fx_0, fx_1, fx_1) \le \tfrac{q^n}{1-q} G(fx_0, fx_1, fx_1)$$

for $m > n$. As $m,n \to \infty$, this implies that $G(fx_n, fx_m, fx_m) \to 0$, proving that $\langle gx_n \rangle_{n=1}^{\infty}$ is *G*-Cauchy in *X*.

Since *X* is *G*-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} fx_{n-1} = \lim_{n\to\infty} gx_n = p. \quad \ldots (2.4)$$

Suppose that *g* is continuous. Then

$$\lim_{n\to\infty} gfx_{n-1} = \lim_{n\to\infty} ggx_n = gp. \quad \ldots (2.5)$$

Since *f* and *g* are compatible, $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, which implies that

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = gp. \quad \ldots \quad (2.6)$$

But from (2.1), we see that

$$G(ffx_n, fx_n, fx_n) \le k\max\{G(gfx_n, ffx_n, ffx_n)+G(gx_n, fx_n, fx_n)+G(gx_n, fx_n, fx_n),$$
$$G(gfx_n, fx_n, fx_n)+G(gx_n, ffx_n, ffx_n)+G(gx_n, fx_n, fx_n),$$

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$$G(gfx_n, fx_n, fx_n)+G(gx_n, fx_n, fx_n)+G(gx_n, ffx_n, ffx_n)\}.$$

In the limit as $n \to \infty$, from this, in view of (2.4), (2.5), (2.6), it follows that $G(gp, p, p) \le k[G(gp, p, p)+2G(gp, p, p)]$ or $(1-3k)G(gp, p, p) \le 0$ so that $gp = p$, since $k < 1/3$.

Again, from (2.1), we have

$$G(fx_n, fp, fp) \le k \max\{G(gx_n, fx_n, fx_n)+G(gp, fp, fp)+G(gp, fp, fp),$$
$$G(gx_n, fp, fp)+G(gp, fx_n, fx_n)+G(gp, fp, fp)$$
$$G(gx_n, fp, fp)+G(gp, fp, fp)+G(gp, fx_n, fx_n)\}.$$

Proceeding the limit as $n \to \infty$, in this and using $gp = p$ and (1.1), we get

$$G(p, fp, fp) \le 2k\, G(p, fp, fp). \qquad \text{... (2.7)}$$

If $G(p, fp, fp) > 0$, (2.7) would give $G(p, fp, fp) \le 2k\, G(p, fp, fp) < G(p, fp, fp)$, which would be a contradiction. Therefore, $G(p, fp, fp) = 0$ so that $fp = p$. In other words, p is a common fixed point of f and g.

The case of f being continuous can similarly be handled. The uniqueness of the common fixed point follows from (1.2), (2.1) and the choice of k.

Remark. Let g be the identity map i on X. we first observe that (a) holds good, and the pair (f, i) is compatible. Further, (2.1) reduces to (9) of Theorem 2 of [3]. Thus, with $g = i$, Theorem 2.1 reduces to Theorem 2 of Vats et al [3].

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