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# THE BIFURCATION ANALYSIS OF AN ECO-EPIDEMIOLOGICAL MODEL WITH TWO INFECTIOUS DISEASES IN PREY

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**ABSTRACT**

In this paper, the conditions, which guarantee the occurring of local bifurcations such as (saddle node, transcritical and pitchfork) of all equilibrium points of eco-epidemiological model consisting of the prey-predator model with ( SI and SIS ) epidemic diseases in prey population only, are established, it's observed that there is transcritical bifurcation near vanishing equilibrium point and predator free equilibrium point while there is a saddle–node bifurcation near coexistence equilibrium point, on the other hand there is no pitchfork bifurcation near all of the equilibrium points. Further investigations for the Hopf bifurcation near coexistence equilibrium point are carried out. Finally, numerical simulations are used to illustration the occurrence of local bifurcation of this model.

Keywords: Eco-Epidemiological model, local bifurcation, Sotomayor's theorem, Hopf bifurcation



## 1. INTRODUCTION

Most dynamical system contains parameters in addition to variables. A general system of ordinary differential equation could therefore be written as: $\dot{x} = f(\ x\ ;\ \mathbf{a}\ )$, where $\mathbf{a}$ is a set of parameters on which the equation, and thus their solutions, depend. finding the solution of a set of differential equations at different parameter values, gives qualitatively. However, in some models, there is a set of parameter values which are close to each other but where the behavior of the model is in some way qualitatively different for one set or the other. For instance, a stable equilibrium point might have become unstable. We then say that system has a bifurcation.

In the other word, bifurcation occurs when a small smooth change made to the parameter value (the bifurcation parameter) of the system causes a sudden "qualitative" or topological change in its behavior. Generally, at a bifurcation, the local stability properties of equilibrium, periodic orbits or other invariant sets change [1]. The name "bifurcation" was first introduced by Henri Poincaré in

1885 in the first paper in mathematics showing such a behavior [2]. Moreover the bifurcation occurs in both continuous systems (described by ordinary differential equations) [3, 4, 5] and discrete systems (described by maps) [6, 7, 8, 9]. The bifurcation is divided into two principal classes: local bifurcations and global bifurcations. Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibrium, periodic orbit [10]. While global bifurcations occur when larger periodic orbits, collide with equilibrium. This causes changes in the topology of the trajectories in phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations [11].

The Hopf bifurcation is a local bifurcation in which equilibrium point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the equilibrium point cross the imaginary axis of complex plane, this type is also known as a Poincare Andronov Hopf bifurcation.

In this paper, an application of Sotomayor's theorem [10,12] for local bifurcation is used to study the occurrence of local bifurcation near the equilibria, furthermore Hopf bifurcation near positive equilibrium point conditions are established of a mathematical model proposed by M. A. Azhar and Sh. I. Inaam [13].

## 2. mathematical model [13]

An eco-epidemiological mathematical model consisting of prey-predator model involving SI, SIS infectious disease in prey population, is proposed and formulation in [13] as in the following:

$$
\begin{aligned}
\frac{dS}{dT} &= r\,S\left(1-\frac{S+I_2}{K}\right)-(\beta_1 I_1+\alpha_1)S-(\beta_2 I_2+\alpha_2)S+\sigma_1 I_2\\
\frac{dI_1}{dT} &= (\beta_1 I_1+\alpha_1)S-c_1 I_1 Y-d_1 I_1 \qquad (1)\\
\frac{dI_2}{dT} &= (\beta_2 I_2+\alpha_2)S-c_2 I_2 Y-d_2 I_2-\sigma_1 I_2\\
\frac{dY}{dT} &= e_1 c_1 I_1 Y+e_2 c_2 I_2 Y-d_3 Y
\end{aligned}
$$

Where $0 < e_1 < 1$ , $0 < e_2 < 1$ represent the conversion rate constants. This model consists of a prey, whose total population density at time T is denoted by N (T), interacting with predator whose total population density at time T is denoted by Y(T). Note that there are two different epidemic diseases (SI, SIS), divides the prey population in to three classes namely S(T) that represents the density of susceptible prey, $I_1$(T) which represents the density of infected prey by first disease and $I_2$(T) which represents the density of infected prey by second disease. Therefore, at any time T, we have N(T) = $S$(T) +$I_1$(T) + $I_2$(T).All the parameters of the model are moreover assumed to be positive and described as given in [13]

Now, for further simplification of the system (1), the following dimensionless variables are used in [13].

$$t=r\,T\,,\ s=\frac{S}{K}\ ,\ i_1=\frac{I_1}{K}\ ,\ i_2=\frac{I_2}{K}\,,\ y=\frac{c_1\,Y}{r}$$

As well as system (1) are written in the following dimensionless form:

$$\frac{ds}{dt} = s\big(1 - s - (1 + a_3)\, i_2 - a_1 i_1 - (a_2 + a_4)\big) + a_5 i_2 = f_1(\, s\,, i_1\,, i_2\,, y\,)$$

$$\frac{di_1}{dt} = i_1(a_1\, s - y - a_6) + a_2\, s = f_2(\, s\,, i_1\,, i_2\,, y\,) \qquad (2)$$

$$\frac{di_2}{dt} = i_2(a_3\, s - a_7\, y - (a_5 + a_8)\,) + a_4\, s = f_3(\, s\,, i_1\,, i_2\,, y\,)$$

$$\frac{dy}{dt} = y(a_9\, i_1 + a_{10}\, i_2 - a_{11}) = f_4(\, s\,, i_1\,, i_2\,, y\,)$$

Where:

$$a_1 = \frac{\beta_1 K}{r}\ ,\ a_2 = \frac{\alpha_1}{r}\ ,\ a_3 = \frac{\beta_2 K}{r}\ ,\ a_4 = \frac{\alpha_2}{r}\ ,\ a_5 = \frac{\sigma_1}{r},\ a_6 = \frac{d_1}{r},$$
$$a_7 = \frac{c_2}{c_1}\ ,\ a_8 = \frac{d_2}{r},\ a_9 = \frac{e_1 c_1 K}{r}\ ,\ a_{10} = \frac{e_2 c_2 K}{r}\ ,\ a_{11} = \frac{d_3}{r}$$

with $s(0) \geq 0$, $i_1(0) \geq 0$, $i_2(0) \geq 0$ and $y(0) \geq 0$ and it is observed that the number of parameters have been reduced from fourteen in the system (1) to eleven in the system (2). Obviously that all the interaction functions $f_1$, $f_2$, $f_3$ and $f_4$ on the right hand side of system (2) are continuous and have continuous partial derivatives on $R^4_+$ with respect to dependent variables $s$, $i_1$, $i_2$ and $y$. Therefore these functions are Lipschitzian and hence system (2) has a unique solution for each non-negative initial condition. Further the boundedness of the system is proved in [13] by theorem(1).

**3. Existence and stability analysis of system (2)**

It is observed that, system (2) has at most three biologically feasible equilibrium points, namely $E_0 = (0\,,0\,,0\,,0\,)$, $E_1 = (\,\bar{s}\,, \bar{i}_1\,, \bar{i}_2, 0\,)$ and $E_2 = \big(\,s^*\,, i_1^*\,, i_2^*\,, y^*\,\big)$ which are mentioned with their existence conditions in [13] as in the following:

1) The vanishing equilibrium point $E_0 = (0\,,0\,,0\,,0\,)$ always exist and it is a locally asymptotically stable if the following condition holds:

$$a_2 > 1 \qquad (3)$$

2) The predator free equilibrium point $E_1 = (\,\bar{s}\,, \bar{i}_1\,, \bar{i}_2, 0\,)$ exists uniquely in the $Int.R^3_+$ of $s i_1 i_2 -$ space if the following conditions hold:

$$a_2 < 1 - \frac{a_4 a_8}{a_5 + a_8} \qquad (4)$$

$$a_4 < 1 + \frac{a_5}{a_8} \qquad (5)$$

$$\bar{s} < \min\left\{\frac{a_6}{a_1}\,, \frac{a_5 + a_8}{a_3}\right\} \qquad (6)$$

And it is a locally asymptotically stable if the following conditions hold:

$$a_9\, \bar{i}_1 + a_{10} \bar{i}_2 < a_{11} \qquad (7)$$

$$2\bar{s} + (1 + a_3)\,\bar{\iota_2} + a_1\bar{\iota_1} + (a_2 + a_4) > 1 \tag{8}$$

$$\bar{s} > \frac{a_5}{1+a_3} \tag{9}$$

3) The coexistence equilibrium point $E_2 = (\, s^*\,, i_1^*\,, i_2^*\,, y^*\,)$ exists uniquely in $Int.\,R_+^4$, if the following conditions hold:

$$i_1^* < \frac{a_{11}}{a_9} \tag{10}$$

$$s^* > \frac{a_5+a_8}{a_3} \tag{11}$$

$$\left(\frac{\partial L_1}{\partial i_1}\right) > 0, \left(\frac{\partial L_1}{\partial s}\right) > 0 \quad \text{OR} \quad \left(\frac{\partial L_1}{\partial i_1}\right) < 0, \left(\frac{\partial L_1}{\partial s}\right) < 0. \tag{12}$$

$$\left(\frac{\partial L_2}{\partial i_1}\right) > 0, \left(\frac{\partial L_2}{\partial s}\right) < 0 \quad \text{OR} \quad \left(\frac{\partial L_2}{\partial i_1}\right) > 0, \left(\frac{\partial L_2}{\partial s}\right) < 0. \tag{13}$$

Accordingly, in addition to the conditions $(10-13)$ hold the isoclines $L_1(s, i_1) = 0$ intersect the s-axis at the positive value namely $s_1$ , for more details see [13].

And it is a locally asymptotically stable if the following conditions hold:

$$2s^* + (1 + a_3)i_2{}^* + a_1 i_1{}^* + (a_2 + a_4) > 1 \tag{14}$$

$$\frac{a_5}{1+a_3} < s^* < \frac{\left(a_{11}\text{-}a_9 i_1{}^*\right)(a_6 a_7\text{-}(a_5+a_8))}{\left(a_{11}\text{-}a_9 i_1{}^*\right)(a_1 a_7\text{-}a_3)+a_4 a_{10}} \tag{15}$$

$$-\mu_4\mu_7 > \mu_9 + \mu_3\mu_8 > 0 \tag{16}$$

$$\mu_7 < -\frac{\mu_4}{2} \tag{17}$$

## 4. The local bifurcation analysis of system (2)

In this section, the effect of varying the parameter values on the dynamical behavior of the system (2) around each equilibrium points is studied. Recall that the existence of non hyperbolic equilibrium point of system (2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems an application to the Sotomayor's theorem is appropriate.

Now, according to Jacobian matrix $J$ of system (2) given in [13] , it is clear to verify that for any non zero vector $\Lambda = (\Lambda_1,\ \Lambda_2,\ \Lambda_3,\ \Lambda_4)^T$ we have:

$$D^2F(X,\mu)(\Lambda,\Lambda)=\begin{bmatrix}-2\Lambda_1(\Lambda_1+a_1\Lambda_2+(1+a_3)\Lambda_3)\\ 2\Lambda_2(a_1\Lambda_1-\Lambda_4)\\ 2\Lambda_3(a_3\Lambda_1-a_7\Lambda_4)\\ 2\Lambda_4(a_9\Lambda_2+a_{10}\Lambda_3)\end{bmatrix} \tag{18}$$

and $D^3F(X,\mu)(\Lambda,\Lambda,\Lambda)=(0,0,0,0)^T$ .

Where $X=(s,i_1,i_2,y)^T$ and $\mu$ is any bifurcation parameter. So, according to Sotomayor's theorem the pitchfork bifurcation does not occur for each point $E_i$, $i=0,1,2$.

### *4.1 Local bifurcation analysis near* $E_0$

**Theorem 1:** Assume that condition (5) is satisfied. Then system (2) at the equilibrium point $E_0=(0,0,0,0)$ with the parameter $\tilde{a}_2=1-\frac{a_4a_8}{a_5+a_8}$ has :

- no saddle –node bifurcation.
- transcritical bifurcation.

**Proof:** According to the Jacobian matrix $J_0$ given in [13], system (2) at the equilibrium point $E_0$ has zero eigenvalue (say $\lambda_{0S}=0$) at $a_2=\tilde{a}_2$ , it is clear that $\tilde{a}_2>0$ provided that the condition (5) holds, and the Jacobian matrix $J_0$ with $a_2=\tilde{a}_2$ becomes:

$$\tilde{J}_0=J_0(\tilde{a}_2)=\begin{bmatrix}-\frac{a_4a_5}{a_5+a_8} & 0 & a_5 & 0\\ \tilde{a}_2 & -a_6 & 0 & 0\\ a_4 & 0 & -(a_5+a_8) & 0\\ 0 & 0 & 0 & -a_{11}\end{bmatrix}$$

Now, let $\Lambda^{[0]}=\left(\Lambda_1^{[0]},\Lambda_2^{[0]},\Lambda_3^{[0]},\Lambda_4^{[0]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{0s}=0$. Thus $(\tilde{J}_0-\lambda_{0s}I)\Lambda^{[0]}=0$, which gives:

$$\Lambda_2^{[0]}=\frac{\tilde{a}_2}{a_6}\Lambda_1^{[0]},\quad \Lambda_3^{[0]}=\frac{a_4}{a_5+a_8}\Lambda_1^{[0]},\quad \Lambda_4^{[0]}=0$$

and $\Lambda_1^{[0]}$ is any nonzero real number. Let $\mho^{[0]}=\left(\mho_1^{[0]},\mho_2^{[0]},\mho_3^{[0]},\mho_4^{[0]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{0s}=0$ of the matrix $\tilde{J}_0^T$ . Then we have, $(\tilde{J}_0^T-\lambda_{0s}I)\mho^{[0]}=0$ . By solving this equation for $\mho^{[0]}$ we obtain, $\mho^{[0]}=\left(\mho_1^{[0]},0,\frac{a_5}{a_5+a_8}\mho_1^{[0]},0\right)^T$, where $\mho_1^{[0]}$ is any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial a_2}=f_{a_2}(X,a_2)=\left(\frac{\partial f_1}{\partial a_2},\frac{\partial f_2}{\partial a_2},\frac{\partial f_3}{\partial a_2},\frac{\partial f_4}{\partial a_2}\right)^T=(-s,s,0,0)^T .$$

So, $f_{a_2}(E_0,\tilde{a}_2)=(0,0,0,0)^T$ and hence $\left(\mho^{[0]}\right)^T f_{a_2}(E_0,\tilde{a}_2)=0$ .

Thus, according to Sotomayor's theorem for local bifurcation, the saddle-node bifurcation condition can't occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_2}(X\,,a_2)=\begin{bmatrix}-1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

where $Df_{a_2}(X\,,a_2)$ represents the derivative of $f_{a_2}(X\,,a_2)$ with respect to $X=(s\,,i_1\,,i_2,y)^T$ .Further ,it is observed that

$$Df_{a_2}(\,E_0\,,\tilde{a}_2)\Lambda^{[0]}=\begin{bmatrix}-1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\Lambda_1^{[0]}\\ \frac{\tilde{a}_2}{a_6}\Lambda_1^{[0]}\\ \frac{a_4}{a_5+a_8}\Lambda_1^{[0]}\\0\end{bmatrix}=\begin{bmatrix}-\Lambda_1^{[0]}\\ \Lambda_1^{[0]}\\0\\0\end{bmatrix}$$

$$\left(\mho^{[0]}\right)^T\left[Df_{a_2}(E_0\,,\tilde{a}_2)\Lambda^{[0]}\right]=\left(\mho_1^{[0]}\,,0\,,\frac{a_5}{a_5+a_8}\,\mho_1^{[0]}\,,0\right)\left(-\Lambda_1^{[0]}\,,\Lambda_1^{[0]}\,,0\,,0\right)^T=-\Lambda_1^{[0]}\mho_1^{[0]}\neq 0$$

Moreover, by substituting $\Lambda^{[0]}$ in (18) we get:

$$D^2f(E_0\,,\tilde{a}_2)\left(\Lambda^{[0]}\,,\Lambda^{[0]}\right)=\begin{bmatrix}-2\left(\Lambda_1^{[0]}\right)^2\left(1+\frac{a_1\tilde{a}_2}{a_6}+\frac{(1+a_3)a_4}{a_5+a_8}\right)\\ 2\frac{a_1a_2^\star}{a_6}\left(\Lambda_1^{[0]}\right)^2\\ 2\frac{a_3a_4}{a_5+a_8}\left(\Lambda_1^{[0]}\right)^2\\0\end{bmatrix}$$

Hence, it is obtain that:

$$\left(\mho^{[0]}\right)^T\left[D^2f(E_0\,,\tilde{a}_2)\left(\Lambda^{[0]}\,,\Lambda^{[0]}\right)\right]=-2\left(\Lambda_1^{[0]}\right)^2\mho_1^{[0]}\left(1+\frac{a_1\tilde{a}_2}{a_6}+\frac{a_4(a_5+a_8(1+a_3))}{(a_5+a_8)^2}\right)\neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at $E_0$ with the parameter $a_2=\tilde{a}_2$ ■

### *4.2 Local bifurcation analysis near $E_1$*

**Theorem 2:** Assume that conditions (6), (8) and (9) with the following conditions are satisfied:

$$h_1< min\left\{\frac{1}{a_6}\,,\frac{a_7}{a_5+a_8}\,,\frac{1}{a_1\bar{s}}\,,\frac{a_7}{a_3\bar{s}}\right\} \tag{19}$$

$$(1+a_3)\bar{s}h_1\frac{\bar{u}_{33}}{\bar{u}_{13}}>(a_3\bar{s}h_1-a_7) \tag{20}$$

$$\frac{\overline{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]\left(-a_1\bar{s}h_1\frac{\bar{u}_{33}}{\bar{u}_{13}}+\frac{a_7\overline{\iota_2}}{\overline{\iota_1}}[a_1\bar{s}h_1-1]\right)$$

$$-\frac{\overline{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\left((1+a_3)\bar{s}h_1\frac{\bar{u}_{33}}{\bar{u}_{13}}+[a_3\bar{s}h_1-a_7]\right)>\bar{s}^2h_1^2\frac{\bar{u}_{33}}{\bar{u}_{13}} \tag{21}$$

Where: $h_1=\frac{g_1}{g_2}$

With: $g_1=a_2\bar{u}_{12}\bar{u}_{33}^2+a_4a_7\bar{u}_{13}\bar{u}_{33}^2$

$g_2=\bar{u}_{11}\bar{u}_{22}^2\bar{u}_{33}^2+a_2a_6\bar{u}_{12}\bar{u}_{33}^2+a_4(a_5+a_8)\bar{u}_{13}\bar{u}_{22}^2$

Then system (2) at the equilibrium point $E_1=(\bar{s},\overline{\iota_1},\overline{\iota_2},0)$ with the parameter $\bar{a}_{11}=a_9\,\overline{\iota_1}+a_{10}\overline{\iota_2}$ has:

- No saddle-node bifurcation.
- Transcritical bifurcation.

**Proof:** According to the Jacobian matrix $J_1=[\,u_{ij}\,]_{4\times4}$given in [13], system (2) at the equilibrium point $E_1$ has zero eigenvalue (say $\lambda_{1y}=0$) at $a_{11}=\bar{a}_{11}$ and the Jacobian matrix $J_1$ with $a_{11}=\bar{a}_{11}$ becomes:

$$\bar{J}_1=J_1(\bar{a}_{11})=[\bar{u}_{ij}]_{4\times4}$$

where $\bar{u}_{ij}=u_{ij}$ for all $i,j=1,2,3,4$ except $\bar{u}_{44}=0$.

Now, let $\Lambda^{[1]}=\left(\Lambda_1^{[1]},\Lambda_2^{[1]},\Lambda_3^{[1]},\Lambda_4^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1y}=0$. Thus $(\bar{J}_1-\lambda_{1y}I)\Lambda^{[1]}=0$, gives:

$$\Lambda_1^{[1]}=\bar{s}h_1\Lambda_4^{[1]},\quad \Lambda_2^{[1]}=-\frac{\overline{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]\Lambda_4^{[1]},\quad \Lambda_3^{[1]}=-\frac{\overline{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\Lambda_4^{[1]}$$

and $\Lambda_4^{[1]}$ is any nonzero real number. It is clear that $\Lambda_1^{[1]}$ not equal to zero provided that condition $(6),(8)$ and $(9)$ hold, while $\Lambda_2^{[1]}$ and $\Lambda_3^{[1]}$ not equal to zero provided that condition $(19)$ holds in addition to condition $(6),(8)$ and $(9)$.

Let $\mho^{[1]}=\left(\mho_1^{[1]},\mho_2^{[1]},\mho_3^{[1]},\mho_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1y}=0$ of the matrix $\bar{J}_1^T$.

Then we have, $(\bar{J}_1^T-\lambda_{1y}I)\mho^{[1]}=0$. By solving this equation for $\mho^{[1]}$ we obtain ,

$\mho^{[1]}=\left(-\frac{\bar{u}_{33}}{\bar{u}_{13}}\mho_3^{[1]},-\frac{a_7\overline{\iota_2}}{\overline{\iota_1}}\mho_3^{[1]},\mho_3^{[1]},\mho_4^{[1]}\right)^T$, where $\mho_3^{[1]}$, $\mho_4^{[1]}$ are any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial a_{11}}=f_{a_{11}}(X,a_{11})=\left(\frac{\partial f_1}{\partial a_{11}},\frac{\partial f_2}{\partial a_{11}},\frac{\partial f_3}{\partial a_{11}},\frac{\partial f_4}{\partial a_{11}}\right)^T=(0,0,0,-y)^T\ .$$

So, $f_{a_{11}}(E_1,\bar{a}_{11})=(0,0,0,0)^T$ and hence $\left(\mho^{[1]}\right)^Tf_{a_{11}}(E_1,\bar{a}_{11})=0$ .

Thus, according to the Sotomayor's theorem for local bifurcation the saddle-node bifurcation condition can't occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_{11}}(X\,,a_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

where $Df_{a_{11}}(X\,,a_{11})$ represents the derivative of $f_{a_{11}}(X\,,a_{11})$ with respect to $X = (s\,,i_1\,,i_2,y)^T$.Further ,it is observed that

$$Df_{a_{11}}(\,E_1\,,\bar{a}_{11})\Lambda^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{s}h_1\Lambda_4^{[1]} \\ -\dfrac{\bar{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]\Lambda_4^{[1]} \\ -\dfrac{\bar{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\Lambda_4^{[1]} \\ \Lambda_4^{[1]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\Lambda_4^{[1]} \end{bmatrix}$$

$$\left(\mho^{[1]}\right)^T\left[Df_{a_{11}}(E_1\,,\bar{a}_{11})\Lambda^{[1]}\right] = -\Lambda_4^{[1]}\mho_4^{[1]} \neq 0$$

Moreover, by substituting $\Lambda^{[1]}$ in (18) we get:

$$D^2f(E_1\,,\bar{a}_{11})\left(\Lambda^{[1]}\,,\Lambda^{[1]}\right) = \begin{bmatrix} -2\bar{s}h_1\left(\Lambda_4^{[1]}\right)^2\left(\bar{s}h_1-\dfrac{a_1\bar{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]-\dfrac{(1+a_3)\bar{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\right) \\ -2\dfrac{\bar{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]\left(\Lambda_4^{[1]}\right)^2(a_1\bar{s}h_1-1) \\ -2\dfrac{\bar{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\left(\Lambda_4^{[1]}\right)^2(a_3\bar{s}h_1-a_7) \\ -2\left(\Lambda_4^{[1]}\right)^2\left(\dfrac{a_9\bar{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]+\dfrac{a_{10}\bar{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\right) \end{bmatrix}$$

Hence, it is obtain that:

$$\begin{aligned}\left(\mho^{[1]}\right)^T&\left[D^2f(E_1\,,\bar{a}_{11})\left(\Lambda^{[1]}\,,\Lambda^{[1]}\right)\right] \\ &= 2\left(\Lambda_4^{[1]}\right)^2\left(\mho_3^{[1]}\left[\frac{\bar{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]\left(-a_1\bar{s}h_1\frac{\bar{u}_{33}}{\bar{u}_{13}}+\frac{a_7\bar{\iota_2}}{\bar{\iota_1}}[a_1\bar{s}h_1-1]\right)\right.\right. \\ &\quad\left.-\frac{\bar{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\left((1+a_3)\bar{s}h_1\frac{\bar{u}_{33}}{\bar{u}_{13}}+[a_3\bar{s}h_1-a_7]\right)+\bar{s}^2h_1^2\frac{\bar{u}_{33}}{\bar{u}_{13}}\right] \\ &\quad\left.-\,\mho_4^{[1]}\left[\frac{a_9\bar{\iota_1}}{\bar{u}_{22}}[a_6h_1-1]+\frac{a_{10}\bar{\iota_2}}{\bar{u}_{33}}[(a_5+a_8)h_1-a_7]\right]\right)\end{aligned}$$

So, according to conditions (19), (20) and (21) we obtain that:

$$\left(\mho^{[1]}\right)^T\left[D^2f(E_1\,,\bar{a}_{11})\left(\Lambda^{[1]}\,,\Lambda^{[1]}\right)\right] \neq 0 \quad .$$

Thus, according to Sotomayor's theorem of local bifurcation system (2) has transcritical bifurcation at $E_1$ with the parameter $a_{11} = \bar{a}_{11}$

### 4.3 *Local bifurcation analysis near* $\mathrm{E}_2$

**Theorem 3:** Assume that conditions (1) and (15) with the following conditions are satisfied:

$$s^* > \max\left\{\frac{a_5 a_9}{a_9(1+a_3)-a_1 a_{10}}, \frac{a_{10} y^*}{a_1 a_7}\right\} \quad (22)$$

$$\frac{a_3}{a_7} < a_1 < \frac{a_9(1+a_3)}{a_{10}} \quad (23)$$

$$a_4 < \frac{(a_1 a_7 - a_3) i_1^* i_2^* + a_2 a_7 i_2^*}{i_1^*} \quad (24)$$

$$k_1\left((1+a_3)k_2 + \frac{a_1 a_7 a_{10} i_2^*}{a_9 i_1^*} + a_3\right) \neq k_1 k_2\left(k_1 - \frac{a_1 a_{10}}{a_9}\right) + \left[z_{31}^* k_1 - \frac{a_4 s^*}{i_2^*}\right]\left(\frac{a_{10}}{a_9 i_1^*} + \frac{1}{i_2^*}\right) \quad (25)$$

Where: $k_1 = \frac{\varepsilon_1}{\varepsilon_2}$ and $k_2 = \frac{\varepsilon_3}{\varepsilon_4}$

with $\varepsilon_1 = a_{10} z_{12}^* - a_9 z_{13}^*$ , $\varepsilon_2 = a_9 z_{11}^*$

$\varepsilon_3 = a_7 i_2^* z_{21}^* - i_1^* z_{31}^*$ , $\varepsilon_4 = i_1^* z_{11}^*$

Then system (2) at the equilibrium point $E_2 = (s^*, i_1^*, i_2^*, y^*)$ with the parameter $a_6^\star = \frac{z_{24}(a_{10} z_{12} z_{13} - a_9 \mu_8) + z_{34}(z_{21}(a_9 z_{13} - a_{10} z_{12}) - z_{11}(a_7 z_{12} + a_{10} y^*))}{a_{10} z_{34} z_{11}}$ has:

- No transcritical bifurcation.
- Saddle –node bifurcation.

**Proof:** The characteristic equation of Jacobian matrix $J_2$ given by (4.6) in [13] having zero eigenvalue (say $\lambda_2 = 0$ ) if and only if $D_4 = 0$ and then $E_2$ becomes a non-hyperbolic equilibrium point. Clearly the Jacobian matrix of system (2) at the equilibrium point $E_2$ with parameter $a_6 = a_6^*$ becomes:

$$J_2^* = J_2(a_6^*) = [\, z_{ij}^* \,]_{4\times 4}$$

where $z_{ij}^* = z_{ij}$ for all $i,j = 1,2,3,4$ except $z_{22}^* = a_1 s^* - y^* - a_6^*$ . Note that, $a_6^* > 0$ provided that the conditions (22) and (23) hold in addition to conditions (14) and (15) .

Now, let $\Lambda^{[2]} = \left(\Lambda_1^{[2]}, \Lambda_2^{[2]}, \Lambda_3^{[2]}, \Lambda_4^{[2]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_2 = 0$. Thus $(J_2^* - \lambda_2 I)\Lambda^{[2]} = 0$, gives:

$$\Lambda_1^{[2]} = k_1 \Lambda_3^{[2]}, \quad \Lambda_2^{[2]} = -\frac{a_{10}}{a_9}\Lambda_3^{[2]}, \Lambda_4^{[2]} = \frac{1}{a_7 i_2^*}\left(z_{31}^* k_1 - \frac{a_4 s^*}{i_2^*}\right)\Lambda_3^{[2]}$$

and $\Lambda_3^{[2]}$ is any nonzero real number. It is clear that $\Lambda_1^{[2]}$ and $\Lambda_4^{[2]}$ not equal to zero provided that conditions and (22) and (23) hold in addition to condition (14).

Let $\mho^{[2]} = \left(\mho_1^{[2]}, \mho_2^{[2]}, \mho_3^{[2]}, \mho_4^{[2]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_2 = 0$ of the matrix $J_2^{*T}$ . Then we have, $(J_2^{*T} - \lambda_2 I)\mho^{[2]} = 0$ . By solving this equation for $\Psi^{[2]}$ we obtain:

$$\mho^{[2]} = \left(k_2 \mho_3^{[2]}, -\frac{a_7 i_2^*}{i_1^*}\mho_3^{[2]}, \mho_3^{[2]}, \frac{1}{a_{10 y^*}}\left(\frac{a_4 s^*}{i_2^*} - z_{13}^* k_2\right)\mho_3^{[2]}\right)^T$$

and $\mho_3^{[1]}$ is any nonzero real number. It is clear that $\mho_1^{[2]}$ not equal to zero provided that conditions (24) and (25) hold in addition to condition (14), Now, consider:

$$\frac{\partial f}{\partial a_6} = f_{a_6}(X\,,a_6) = \left(\frac{\partial f_1}{\partial a_6},\frac{\partial f_2}{\partial a_6},\frac{\partial f_3}{\partial a_6},\frac{\partial f_4}{\partial a_6}\right)^T = (0\,,-i_1\,,0,0\,)^T\ .$$

So, $f_{a_6}(E_2\,,a_6^*) = (0\,,-i_1^*\,,0\,,0)^T$

and hence $\left(\mho^{[2]}\right)^T f_{a_6}(E_2,a_6^*) = a_7 i_2^* \mho_3^{[2]} \neq 0$.

Thus, according to the Sotomayor's theorem for local bifurcation the transcritical bifurcation condition can't occur. While the first condition of saddle-node bifurcation is satisfied.

Moreover, by substituting $\Lambda^{[2]}$ in (18) we get:

$$D^2 f(E_2\,,a_6^*)\left(\Lambda^{[2]}\,,\Lambda^{[2]}\right) = \begin{bmatrix} -2k_1\left(\Lambda_3^{[2]}\right)^2\left(k_1 - \dfrac{a_1a_{10}}{a_9} + (1+a_3)\right) \\ -2\dfrac{a_{10}}{a_9}\left(\Lambda_3^{[2]}\right)^2\left(a_1k_1 - \dfrac{1}{a_7 i_2^*}\left(z_{31}^*k_1 - \dfrac{a_4 s^*}{i_2^*}\right)\right) \\ 2\left(\Lambda_3^{[2]}\right)^2\left(a_3k_1 - \dfrac{1}{i_2^*}\left(z_{31}^*k_1 - \dfrac{a_4 s^*}{i_2^*}\right)\right) \\ 0 \end{bmatrix}$$

Hence, it is obtain that:

$$\left(\mho^{[2]}\right)^T\left[D^2 f(E_2\,,a_6^*)\left(\Lambda^{[2]}\,,\Lambda^{[2]}\right)\right] = 2\,\mho_3^{[2]}\left(\Lambda_3^{[2]}\right)^2\left(k_1\left((1+a_3)k_2 + \frac{a_1a_7a_{10}i_2^*}{a_9 i_1^*} + a_3\right.\right.$$
$$\left.\left. -k_1k_2\left(k_1 - \frac{a_1a_{10}}{a_9}\right) - \left[z_{31}^*k_1 - \frac{a_4 s^*}{i_2^*}\right]\left(\frac{a_{10}}{a_9 i_1^*} + \frac{1}{i_2^*}\right)\right)\right)$$

So, according to conditions (22), (23)(24) and (25) we obtain that:

$$\left(\mho^{[2]}\right)^T\left[D^2 f(E_2\,,a_6^*)\left(\Lambda^{[2]}\,,\Lambda^{[2]}\right)\right] \neq 0\quad .$$

Thus, according to Sotomayor's theorem system (2) has saddle-node bifurcation at $E_2$ with the parameter $a_6 = a_6^*$

## 5 The Hopf bifurcation analysis of system (2)

In this section, the occurrence of a Hopf bifurcation near the positive equilibrium point $E_2$ of system (2) is investigated, therefore an application to the hopf bifurcation theorem [14] for local bifurcation is appropriate as shown in the following theorem.

### *5.1 Hopf bifurcation analysis near $\mathrm{E}_2$:*

To discuss the occurrence of Hopf bifurcation , first we need to know that the Hopf bifurcation for $n = 4$ are constructed according to the Haque and Venturino methods [14]. Consider the characteristic equation given by:

$$P_4(\tau) = \tau^4 + D_1\tau^3 + D_2\tau^2 + D_3\tau + D_4 = 0$$

here $D_1 = -tr\left(J(x^*)\right)$, $D_2 = M_1\left(J(x^*)\right)$, $D_3 = -M_2\left(J(x^*)\right)$ and $D_4 = det\left(J(x^*)\right)$ with $M_1\left(J(x^*)\right)$ and $M_2\left(J(x^*)\right)$ represent the sum of the principal minors of order two and three of $J(x^*)$ respectively.

Clearly, the first condition of Hopf bifurcation holds if and only if:

$D_{\mathrm{i}} > 0$ ; $\mathrm{i} = 1,3$ , $\Delta_1 = D_1D_2 - D_3 > 0$ , $D_1^3 - 4\Delta_1 > 0$ , $\Delta_2 = D_3(D_1D_2 - D_3) - D_1^2 D_4 = 0$

Consequently, $D_4 = \frac{D_3(D_1D_2 - D_3)}{D_1^2}$ So, the characteristic equation becomes:

$$P_4(\tau) = \left(\tau^2 + \frac{D_3}{D_1}\right)\left(\tau^2 + D_1\tau + \frac{\Delta_1}{D_1}\right) = 0 \qquad (26)$$

Clearly, the roots of equation (26) are:

$$\tau_{1,2} = \mp i\sqrt{\frac{D_3}{D_1}}\ , \tau_{3,4} = \frac{1}{2}\left(-D_1 \mp \sqrt{D_1^2 - 4\frac{\Delta_1}{D_1}}\right)$$

Now, to verify the transversality condition of Hopf bifurcation, we substitute $\tau(\mu) = \varsigma_1(\mu) \mp i\varsigma_2(\mu)$ into equation (26), and then calculating its derivative with respect to the bifurcation parameter $\mu$ , $P_4'(\tau(\mu)) = 0$ comparing the two sides of this equation and then equating their real and imaginary parts, we have:

$$\left.\begin{aligned} \bar{\Psi}(\mu)\,\varsigma_1'(\mu) - \bar{\Phi}(\mu)\,\varsigma_2'(\mu) + \bar{\Theta}(\mu) = 0 \\ \bar{\Phi}(\mu)\,\varsigma_1'(\mu) + \bar{\Psi}(\mu)\,\varsigma_2'(\mu) + \bar{\Gamma}(\mu) = 0 \end{aligned}\right\} \qquad (27)$$

Where

$$\bar{\Psi}(\mu) = 4\,(\varsigma_1(\mu))^3 + 3D_1(\mu)(\varsigma_1(\mu))^2 + D_3(\mu) + 2D_2(\mu)\varsigma_1(\mu) \\ -12\,\varsigma_1(\mu)\,\varsigma_2^2(\mu) - 3D_1(\mu)(\varsigma_2(\mu))^2$$

$$\bar{\Phi}(\mu) = 12\,(\varsigma_1(\mu))^2\varsigma_2(\mu) + 6D_1(\mu)\,\varsigma_1(\mu)\,\varsigma_2(\mu) + 2\,D_2(\mu)\varsigma_2(\mu) - 4\,\left(\varsigma_2(\mu)\right)^3 \qquad (28)$$

$$\bar{\Theta}(\mu) = (\varsigma_1(\mu))^3 D_1'(\mu) + D_3'(\mu)\,\varsigma_1(\mu) + D_2'(\mu)(\varsigma_1(\mu))^2 + D_4'(\mu) \\ -3\,D_1'(\mu)\,\varsigma_1(\mu)(\varsigma_2(\mu))^2 - D_2'(\mu)(\varsigma_2(\mu))^2$$

$$\bar{\Gamma}(\mu) = 3\,(\varsigma_1(\mu))^2\varsigma_2(\mu)D_1'(\mu) + D_3'(\mu)\varsigma_2(\mu) + 2D_2'(\mu)\,\varsigma_1(\mu)\,\varsigma_2(\mu) \\ - D_1'(\mu)(\varsigma_2(\mu))^3$$

Solving the linear system (27) by using Cramer's rule for the unknowns $\varsigma_1'(\mu)$ and $\varsigma_2'(\mu)$ , gives that:

$$\varsigma_1'(\mu) = -\frac{\bar{\Theta}(\mu)\bar{\Psi}(\mu) + \bar{\Gamma}(\mu)\bar{\Phi}(\mu)}{(\bar{\Psi}(\mu))^2 + (\bar{\Phi}(\mu))^2} \text{ and } \varsigma_2'(\mu) = \frac{-\bar{\Gamma}(\mu)\bar{\Psi}(\mu) + \bar{\Theta}(\mu)\bar{\Phi}(\mu)}{(\bar{\Psi}(\mu))^2 + (\bar{\Phi}(\mu))^2}$$

Hence the second necessary and sufficient condition which is called (transversality condition) of Hopf bifurcation

$\frac{d}{d\mu}Re(\tau)\,\Big|_{\mu=\tilde{\mu}} = \varsigma_1'(\mu)\,\Big|_{\mu=\tilde{\mu}}$ is not equal to zero if and only if:

$$\bar{\Theta}(\mu)\bar{\Psi}(\mu) + \bar{\Gamma}(\mu)\bar{\Phi}(\mu) \neq 0 \qquad (29)$$

Moreover, according to the above results, the occurrence of Hopf bifurcation near the positive equilibrium point $E_2$ is established as it shows in the following theorem.

**Theorem (4):** Assume that conditions $(10), (14), (15), (17), (23)$ and $(24)$ with the following conditions are satisfied:

$$a_4 < -\frac{a_1(a_{11} - a_9 i_1^*)z_{31}}{a_9 z_{11}} \qquad (30)$$

$$D_3 < \Delta_1 < \frac{D_1^3}{4} \qquad (31)$$

**Proof:** Consider the characteristic equation of system (2) at $E_2$ which is given by $(4.6)$ in [13], now to verify the necessary and sufficient conditions for Hopf bifurcation to occur we need to find a parameter say $(a_5^\star)$ satisfy that:

$D_{\mathrm{i}}(a_5^\star) > 0$ ; $\mathrm{i} = 1; 3$, $\Delta_1(a_5^\star) > 0$, $D_1^3(a_5^\star) - 4\Delta_1(a_5^\star) > 0$ and $\Delta_2(a_5^\star) = 0$.

Where $D_{\mathrm{i}}$; $\mathrm{i} = 1,3$ represent the coefficients of characteristic Straightforward computation gives that: $D_{\mathrm{i}}(a_5^\star) > 0$ ; $\mathrm{i} = 1,3$ and $\Delta_1(a_5^\star) > 0$ provided that conditions $(14)$ and $(15)$ hold, While $D_1^3(a_5^\star) - 4\Delta_1(a_5^\star) > 0$ provided that condition $(31)$ holds.

On the other hand, it is observed that $\Delta_2 = 0$ gives:

$$a_5^2 R_1 + a_5 R_2 + R_3 = 0 \tag{32}$$

Where

$$R_1 = z_{22} z_{31}^2 \mu_2$$
$$R_2 = -(1 + a_3) s^* R_1 - \eta_1$$
$$R_3 = (1 + a_3) s^* \eta_1 + \eta_2$$

with

$$\begin{aligned}\eta_1 &= z_{31}(\, z_{22}[(1 + a_3)s^* z_{31}\mu_2 - z_{33}\mu_1 D_1 - z_{22}\mu_3 + \mu_1\mu_7 - z_{33}\mu_4] - \mu_2[\mu_1\mu_4 + \mu_2\mu_3 - z_{33}\mu_7]\,) \\ &\quad + z_{42} D_1^2 (z_{31} z_{24} - z_{21} z_{34}) \\ \eta_2 &= (z_{33}\mu_1 D_1 + z_{22}\mu_3)(\mu_1\mu_4 + \mu_2\mu_3 - z_{33}\mu_7) - \mu_1\mu_7(\mu_2\mu_3 - z_{33}\mu_7) \\ &\quad + z_{33}\mu_4(\mu_1[\mu_4 + 2\mu_7] + \mu_2\mu_3) + z_{24} D_1^2 (z_{12} z_{31} z_{43} + z_{11} z_{33} z_{42})\end{aligned}$$

it is easy to verify that, the equation (32) has a unique positive root

$$a_5^\star = \frac{1}{2R_1}\left(-R_2 + \sqrt{R_2^2 - 4\, R_1 R_3}\right)$$

provided that conditions $(23), (24)$ and $(30)$ are hold, in addition to conditions $(10), (14), (15)$ and $(17)$.

Now, at $a_5 = a_5^\star$ the characteristic equation given by equation (4.6) in [13] can be written as:

$$\left(\lambda_2^2 + \frac{D_3}{D_1}\right)\left(\lambda_2^2 + D_1\lambda_2 + \frac{\Delta_1}{D_1}\right) = 0 \quad ,$$

which has four roots $\lambda_{2\;1,2} = \pm \mathrm{i}\sqrt{\frac{D_3}{D_1}}$ and $\lambda_{2\;3,4} = \frac{1}{2}\left(-D_1 \pm \sqrt{D_1^2 - 4\frac{\Delta_1}{D_1}}\right)$.

Clearly, at $a_5 = a_5^\star$ there are two pure imaginary eigenvalues ( $\lambda_{2\;1}$ and $\lambda_{2\;2}$ ) and two eigenvalues which are real and negative. Now for all values of $a_5$ in the neighborhood of $a_5^\star$, the roots in general of the following form:

$$\lambda_{2\;1} = \varsigma_1 + i\varsigma_2 \;, \lambda_{2\;2} = \varsigma_1 - i\varsigma_2 \;, \lambda_{2\;3,4} = \frac{1}{2}\left(-D_1 \pm \sqrt{D_1^2 - 4\frac{\Delta_1}{D_1}}\right).$$

Clearly, $Re\left(\lambda_{2\;1,2}(a_5)\right)\big|_{a_5 = a_5^\star} = \varsigma_1(a_5^\star) = 0$ that means the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at $a_5 = a_5^\star$. Now, according to verify the transversality condition we must prove that:

$$\overline{\Theta}(a_5^\star)\, \overline{\Psi}(a_5^\star) + \overline{\Gamma}(a_5^\star)\, \overline{\Phi}(a_5^\star) \neq 0 \quad ,$$

where $\overline{\Theta}, \overline{\Psi}$, $\overline{\Gamma}$ and $\overline{\Phi}$ are given in $(28)$. Note that for $a_5 = a_5^\star$ we have $\varsigma_1 = 0$ and $\varsigma_2 = \sqrt{\frac{D_3}{D_1}}$, substituting into $(28)$ gives the following simplifications:

$\overline{\Psi}(a_5^\star) = -2\, D_3(a_5^\star)$ ,

$\overline{\Phi}\,(a_5^\star) = 2\,\dfrac{\varsigma_2(a_5^\star)}{D_1}(D_1 D_2 - 2\, D_3)$ ,

$\overline{\Theta}(a_5^\star) = D_4'(a_5^\star) - \dfrac{D_3}{D_1} D_2'(a_5^\star)$ ,

$\overline{\Gamma}(a_5^\star) = \varsigma_2(a_5^\star)\left(D_3'(a_5^\star) - \dfrac{D_3}{D_1} D_1'(a_5^\star)\right)$ ,

where

$D_1' = \dfrac{dD_1}{da_5}\,\Big|_{a_5=a_5^\star} = 0$ ,

$D_2' = \dfrac{dD_2}{da_5}\,\Big|_{a_5=a_5^\star} = -\,z_{31}$ ,

$D_3' = \dfrac{dD_3}{da_5}\,\Big|_{a_5=a_5^\star} = z_{22}z_{31}$ ,

$D_4' = \dfrac{dD_4}{da_5}\,\Big|_{a_5=a_5^\star} = z_{31}\,\mu_3 - z_{21}z_{34}\,z_{42}$ .

Then substituting into $(29)$ we get that:

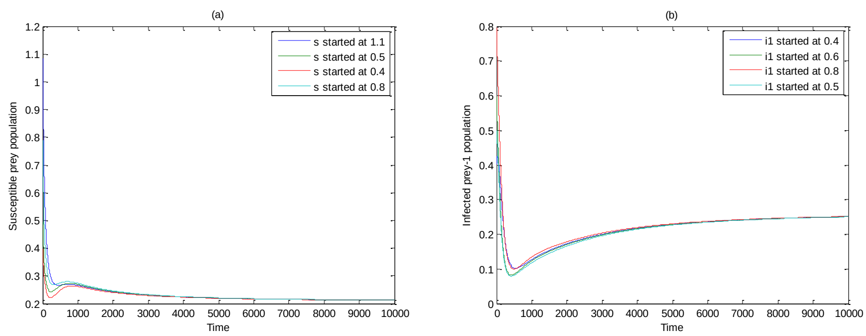
$$\overline{\Theta}(a_5^\star)\,\overline{\Psi}(a_5^\star) + \overline{\Gamma}(a_5^\star)\,\overline{\Phi}(a_5^\star) = 2\left(z_{42}\,D_1[-z_{31}z_{24} + z_{21}z_{34}] - z_{31}\frac{D_3}{D_1}\left[D_3 - \frac{z_{22}}{D_1}(\Delta_1 - D_3)\right]\right) \neq 0$$

provided that conditions $(23), (24)$ and $(31)$ are hold. So, we obtain that the Hopf bifurcation occurs around the equilibrium point $E_2$ at the parameter $a_5 = a_5^\star$ and the proof is complete.

## 6 Numerical Simulation of system (2) [13]

In this section, the dynamical behavior of system $(2)$ is studied numerically for different sets of parameters and different sets of initial points. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system $(2)$ has a globally asymptotically stable positive equilibrium point as shown in Fig. $(1)$.

$$\left.\begin{array}{c} a_1 = 0.5\;, a_2 = 0.3\;, a_3 = 0.3\;, a_4 = 0.3\,, a_5 = 0.2\;, a_6 = 0.3\;, \\ a_7 = 0.5\;, a_8 = 0.2\,, a_9 = 0.5\,, a_{10} = 0.4\,, a_{11} = 0.2\,. \end{array}\right\} (33)$$



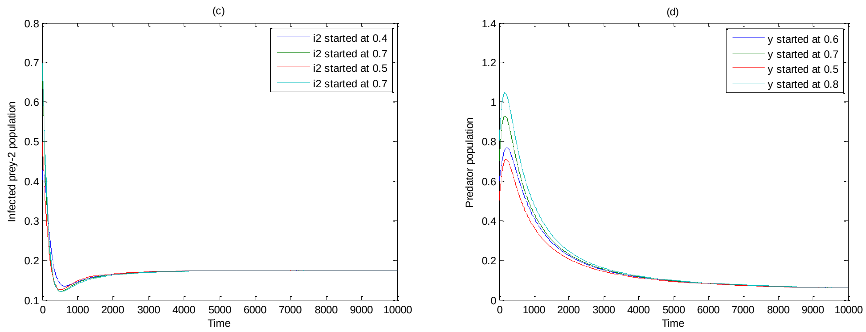



Fig.1: Time series of the solution of system (2) that started from four different initial point $(1.1\,,0.4\,,0.4\,,0.6)$, $(0.5\,,0.6\,,0.7\,,0.7)$, $(0.4\,,0.8\,,0.5,0.5)$ and $(0.8\,,0.5\,,0.7\,,0.8)$ for the data given in (33). (a) trajectories of $s$ as a function of time, (b) trajectories of $i_1$ as a function of time, (c) trajectories of $i_2$ as a function of time and (d) trajectories of $y$ as a function of time .

Clearly, figure (1) shows that the solution of system $(2)$ approaches asymptotically to the positive equilibrium point $E_2 = (\,0.212\,,0.251\,,0.174\,,0.059\,)$ starting from four different initial points and this is confirming our obtained analytical results, see [13].

Moreover system (2) is solved numerically for the data given in $(33\,)$ with varying one parameter at each time and the obtained results are given in table (1), for more details see [13].

TABLE **1:** NUMERICAL BEHAVIORS OF SYSTEM $(2)$ FOR THE DATA GIVEN IN $(33\,)$ WITH VARYING ONE PARAMETER AT EACH TIME

| **Range of parameter** | **Numerical behavior of system (2)** |
|---|---|
| $0 < a_1 < 1.47$<br>$1.47 \le a_1$ | Approaches to the positive equilibrium point $E_2$<br>Approaches to the predator free equilibrium point $E_1$ |
| $0.001 \le a_2 < 0.51$<br>$0.51 \le a_2 \le 0.84$<br>$0.89 \le a_2$ | Approaches to the positive equilibrium point $E_2$<br>Approaches to the predator free equilibrium point $E_1$<br>Approaches to the vanishing equilibrium point $E_0$ |
| $0 < a_3 < 1.57$<br>$1.57 \le a_3$ | Approaches to the positive equilibrium point $E_2$<br>Approaches to the predator free equilibrium point $E_1$ |
| $0.004 \le a_4 < 0.54$<br>$0.54 \le a_4 \le 1.4$<br>$1.5 \le a_4$ | Approaches to the positive equilibrium point $E_2$<br>Approaches to the predator free equilibrium point $E_1$<br>Approaches to the vanishing equilibrium point $E_0$ |
| $0 < a_5 < 0.04$<br>$0.04 \le a_5 \le 1$ | Approaches to the predator free equilibrium point $E_1$<br>Approaches to the positive equilibrium point $E_2$ |
| $0 < a_6 < 0.64$<br>$0.64 \le a_6 \le 1$ | Approaches to the positive equilibrium point $E_2$<br>Approaches to the predator free equilibrium point $E_1$ |
| $0 < a_9 < 0.3$<br>$0.3 \le a_9 < 1$ | Approaches to the predator free equilibrium point $E_1$<br>Approaches to the positive equilibrium point $E_2$ |
| $0 < a_{10} < 0.09$<br>$0.09 \le a_{10} < 0.5$ | Approaches to the predator free equilibrium point $E_1$<br>Approaches to the positive equilibrium point $E_2$ |
| $0.017 \le a_{11} < 0.256$<br>$0.256 \le a_{11} \le 1$ | Approaches to the positive equilibrium point $E_2$<br>Approaches to the predator free equilibrium point $E_1$ |

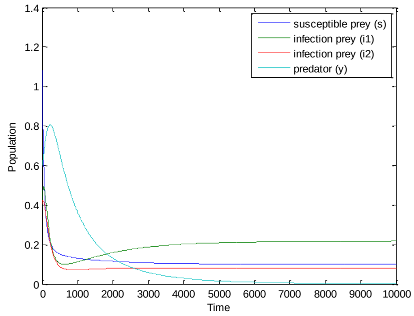



Fig.2: Time series of the solution of system (2) approaches asymptotically to the predator free equilibrium point $E_1 = (0.099\ , 0.217\ ,0.08\ ,0)$ for the data given in (33) with $a_2 = 0.55$.

Clearly, figure (2) shows that system (2) has a bifurcation since varying the first external infection rate in the range $0.51 \leq\ a_2 \leq 0.84$ keeping other parameters as data given in $(\ 33)$ observed that system (2) approach the predator free equilibrium point $E_1$.

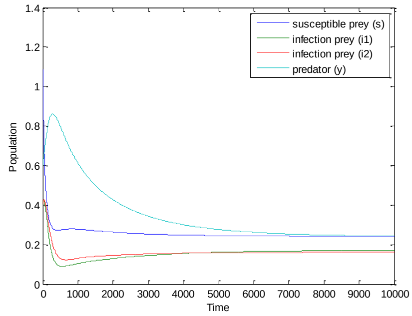



Fig.3: Time series of the solution of system (2) approaches asymptotically to the positive equilibrium point $E_2 = (\ 0.24\ , 0.17\ , 0.16, 0.242)$ for the data given in (33) with $a_{11} = 0.15$.

Also, figure (3) shows that system (2) has a bifurcation since varying the death rate of the predator in the range $0.017 \leq\ a_{11} < 0.256$ keeping other parameters as data given in $(\ 33)$ observed that system (2) approach asymptotically to the positive equilibrium point $E_2$.

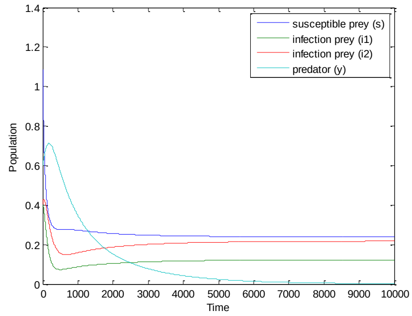



Fig.4: Time series of the solution of system (2) approaches asymptotically the predator free equilibrium point $E_1 = (0.24\ , 0.12\ ,0.22\ ,0)$ for the data given in (33) with $a_6 = 0.7$ .

Finally, figure (4) shows that system (2) has a bifurcation since varying the death rate of the infected prey by first disease in the range $0.64 \leq\ a_6 \leq 1$ keeping other parameters as data given in $(\ 33)$ observed that system (2) approach the predator free equilibrium point $E_1$.

## 7 Conclusion

In this paper , we studied the effect of varying the parameter values on the dynamical behavior of the system (2) around each equilibrium points , as well as some sufficient conditions of the occurrence of local bifurcation such as (saddle- node, transcritical and pitchfork) are presented, moreover the Hopf bifurcation near positive equilibrium point conditions are also derived of eco-epidemiological mathematical model with ( SI and SIS ) infectious diseases in prey which is transmitted within the same species by contact and external source. Further, it is observed that:

1) For the set of hypothetical parameters value given in (33), system (2) do not have a periodic dynamics, while still has possibility to have a periodic dynamics for other set of parameters, especially Hopf bifurcation existing analytically.
2) For the parameters value ($a_7$ and $a_8$ ) given in (33) there is no any kind of bifurcation, since they do not have any effect on the dynamical behavior of system (2).
3) For the parameters value ( $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$, $a_9$, $a_{10}$, $a_{11}$ ) given in (33), system (2) has one bifurcation .
4) The parameters value $a_2$ given in (33) has no bifurcation occur near $E_0$ , While still has has possibility to have a bifurcation for other set of parameters, especially transcritical bifurcation existing analytically near $E_0$.

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