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RESEARCH ARTICLE

UNDERSTANDING THE GEOMETRY OF LOW DIMENSIONAL MANIFOLDS

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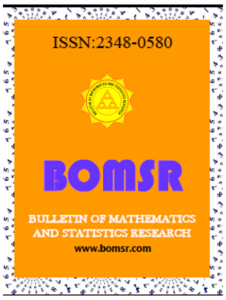

ABSTRACT

In this paper we have nice characterization of how differential equations arise from smooth manifolds and defining concepts of vector fields on integral curves and formulated proposition connecting to differential equations and have given brief account of dimension at most 5 and describe the smooth structure on them in particular we have examples coming from 2 and 3 dimensions.

Keywords: Smooth manifold ,topological manifold,diffeomorphism,vector fields.

1. INTRODUCTION

A well known concept in differential geometry is to associate a smooth manifold on M of a dimensional n of a smooth vector field, for the simple reason that vector fields gives rise to differential equations. Regarding vector fields as flows { Yamabe , Ricci} this becomes more imperative to see their evolution as a dynamical system problems relating to the PDE'S and associated geometry of underlying space are studied and investigated .

In section 1, in this exposition we have brought a nice account of how differential equation arises on smooth manifolds. We have seen that our manifolds are locally compact and hausdorff further it admits a countable base.

In section 2, we gather some basics on smooth manifolds and define concepts like vector fields and integral curves. In fact on such many manifolds integral curve generates a smooth vector field.

In the last section 3 we have formulated proposition and we can give a required connection of differential equation.

Section 1: Low dimensional manifolds of constant curvature

Low dimensional manifolds of constant curvature and flows on them have drawn considerable attention of researchers .This is because of the intrinsic relationship between PDE's and geometry associated with the flows. Most common among them are the Yamabe and Ricci flow introduced by Richard Hamilton in the early eighties .His idea of understanding the flow by controlling the metric (Ricci metric) lead to resolve a very interesting and vexed problem in topology namely Poincare Conjecture.

In another direction ,people were also attracted towards an interesting problem of finding a special class of solutions to PDE's, in particular the parabolic equations.It involved and directed towards the study of singularity associated with the flow developed over finite times .Some blow up techniques lead towards singularities enabled us to classify these solutions into two types, namely eternal and ancient. The solutions were termed as soliton solutions which keep to form a solution in time.

We concentrate on the manifold (which is low dimensional) and discussed them in some detail.

Section 2: Low dimensional manifolds and smooth structures

In this section we give brief account of dimension at most 5 and describe the smooth structure on them in particular we have examples coming from 2 and 3 dimensions.

Definition 2.1 : Topological Manifold:

An n-dimensional topological manifold M is an second countable Hausdorff topological space M, which is locally homeomorphic to an open subset of R^n. i.e M as a manifold possesses a countable basis for the topology. The most trivial example is R^n the n- dimensional real Euclidean space . Homeomorphism here that we have mentioned in the

definition implies for all m∈ M .There exists a homeomorphism $\Phi : U \rightarrow V$,
where U is a neighbourhood of m and V an open neighbourhood of R^n such a homeomorphism is called a chart.

Definition 2.2: An n dimensional smooth manifold is a topological manifold (differentiable of class Infinity) M is a topological manifold endowed with a collection of charts.

$\Phi_\alpha : U_\alpha \rightarrow V_\alpha$ called an atlas such that

(i) M = U $U_\propto$, the domain of the charts cover M and

(ii) For all $\propto, \beta$ with $U_\propto \cap V_\propto \neq \emptyset$ the transition map $\Phi_{\alpha,\beta} = \Phi_\beta \circ \Phi_\alpha^{-1}$

i.e $\Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \Phi_\beta(U_\alpha \cap U_\beta)$, is a smooth map on R^n.

This can be verified however we do not provide this information in this exposition .Nevertheless we give an example for the transition map to be smooth. The following example is a clear indication to such a transition map.

Let M = S^1 the unit circle which is a non trivial example of a manifold other than R^1.Clearly co-ordinates are smooth, continuous and differentiable.

Example 2.3: Consider the one dimensional manifold s^1 the unit circle .Unlike R^1 ,the real line which is also one dimensional manifold s^1 is compact and hence closed .Its description in cartesian co-ordinates is given by the set of all ordered pairs (x,y) in R^1such that $x^2 + y^2$ =1 .

Let p∈ S^1 and U_p be a coordinate ,neighbourhood of p in S^1, then p(x ,y) some ordered pair such that $x^2 + y^2$ =1 in other words y = $\pm\sqrt{1 - x^2}$.

Let ϕ_p be a map that maps U_p homeomorphically onto an open subset of R^1i.e ,$U_P \to \phi_p(U_P)$ in R^1,and $\phi_p(U_P)$ is open.

For q in S^1there is a corresponding neighbourhood in U_q and a homeomorphism $\phi_q : U_q \to \phi_q(U_q)$ open in R^1.

The following scenario is non trivial suppose $U_p \cap U_q \neq \phi$
Then we compute ϕ_p $(U_p \cap U_q)$ and ϕ_q $(U_p \cap U_q)$
Since $(U_p \cap U_q) \subseteq U_p$
$(U_p \cap U_q) \subseteq U_q$
ϕ_p $(U_p \cap U_q) \subseteq \mathbb{R}^1$ is open and
ϕ_q $(U_p \cap U_q) \subseteq \mathbb{R}^1$ is open in $\mathbb{R}^1$
Pointwise computation would clearly give us $\phi_q \circ \phi_p^{-1}$ as an transition map, maps the points of $\phi_p(U_p \cap U_q)$ to the point of $\phi_q(U_p \cap U_q)$ diffeomorphically i.e differentiable and homeomorphic.
Observe that the co-ordinate function is smooth under this map.
Extending this argument to a smooth manifold obtained by taking n copies of S^1 i.e M=$S^1XS^1 \ldots\ldots.S^1$ one can workout the details of smooth differentiable structure on it .
$S^1XS^1 \ldots\ldots.S^1 = T^n$=M
This manifold which is cartesian product of n copies of S^1 is also called an n-tori.
For two dimensional case M=S^1XS^1: T^2 is torus. A neighbourhood U_p, $p \to T^2$ is locally euc lidean.
In verbatic, the same argument would hold good in case of torus as well.

3:Differential equations and integral curves

We are familiar with the vector field defined on smooth manifolds given a smooth manifold M or dim n, n≥1 .We can associate to each point of M as its tangent space denoted T_xM of M at x
From linear algebra we know that T_xM is an isomorphic copy of $\mathbb{R}^n$
In the case of S^1, $T_x S^1$ is a one dimensional tangent space of S^1at p while T_pT^2 is a tangent space of a two dimensional smooth manifold T^2.
Associated with local description we can get a global entity called its tangent bundle of the manifold. Loosely speaking it is union for each x∈M and denoted TM for the tangent bundle of M i.e; TM=U T_xM .
On its own TM is not a manifold i. e; if some U∈TM means U is some T_xM for x ∈ M therefore ,we can regard U as a frame and members belonging to it can be regarded as tangent vectors coming from this frame or fibre.
Definition:3.1: Let $\gamma: I \to M$ be a smooth curve and γ passes through x in M such that $\gamma(0)$=x, where 0∈I,where I = $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$.
Then gamma determines an integral curve by inverse function theorem the resulting differential equation possesses unique solution .In fact we can obtain a system of ODEs .Since M is a n-dimensional.
If (U ,p) is a smooth chart in M ,then the local diffeomorphism given by ϕ on U such that $\phi(U)$ is an open in $\mathbb{R}^n$ combined with gamma,the composition map is given as follows,
Since γ, ϕ are differentiable then $\phi \circ \gamma$ is differentiable and $(\phi \circ \gamma)(0)$ can be computed , t=0.
Observe that $(\phi\circ \gamma)(t) = \phi(\gamma(t)) \in \mathbb{R}^n$ is a n tuple of real numbers.
To get all these components we make use of projection maps $\pi_i : \mathbb{R}^n \to \mathbb{R}$.Therefore the composition when projected onto $\mathbb{R}$ from $\mathbb{R}^n$ i. e; $(\phi \circ \gamma)(t) = \phi(\gamma(t))$.........$(\phi \circ (\gamma(t))_n)$
π_i $(\phi \circ \gamma)(t))$=$(\phi \circ \gamma)_i$(t) for each i,

There $\pi_i(\phi \circ \gamma)^{'}(t)_{t=0}$=$(\phi \circ \gamma)^{'}(t)_i$(t)/t=0

Which gives us a system of n ordinary differential equation. We discuss further issues relating to geometry of the manifold and PDE arising to similar fashion in our other context.

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