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RESEARCH ARTICLE

On I_θ - Closedness and IH- Closedness in *I*-spaces

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ABSTRACT

This is the third in a series of papers on I-spaces. Here $\mathbf{I}_\theta$ - closedness and IH- closedness have been introduced for I- spaces, and many topological theorems related to θ -closedness and H-closedness have been generalized to I- spaces, as an extension of study of infratopological spaces.

Key Words: I-space,$\mathbf{I}_\theta$ - closedness, IH- closedness, $\mathbf{I}_\theta$ compactness and $\mathbf{I}_\theta$ connectedness, product I-structure, IH-continuum.

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1. Introduction

In a previous paper [1] we have introduced I-spaces and studied some of their properties. In this paper we use the terminology of [1]. Some study of these spaces was done previously in [8], [9], [10], [11] in less general form. These spaces were called infratopological spaces. Anti-Hausdorff I-spaces, Anti-Hausdorff U-spaces, Anti-Hausdorff topological-spaces, Hausdorff and compact U –spaces were introduced and studied in [4], [5], [6] and [7] respectively. θ- closedness and H-closedness topological spaces were defined and studied in [2] and [3] respectively. In this paper the concept of $\mathbf{I}_\theta$ - closedness, IH- closedness,$\mathbf{I}_\theta$ - compactness , $\mathbf{I}_\theta$ - connectedness, an IH-continuum in I-spaces have been introduced and a few important properties of such spaces have been studied.

Definition – 2.1. Let (X, **I**) be an I- space and let $A \subseteq X$. **The closure of A** written Icl A, is the subset of X consisting of the elements x such that for each I- open set G containing x, $G \cap A \neq \Phi$ .i.e., $IclA = \{x \in X \mid for\,each\ G \in$ **I**, with $x \in G, G \cap A \neq \Phi\}$

Definition – 2.2. **The θ - closure of A** in (X, **I**) , written I cl_θ (A), is defined as

I $cl_\theta(A) = \{x \in X \mid for\, each\ G \in$ **I**, with $x \in G \mid Icl\ G \cap A \neq \Phi\ \}$.

A is said to be I $_\theta$ **- closed** if $Icl_\theta(A) = A$, and A is called I $_\theta$ **- open** if X – A is I $_\theta$ - closed.

Thus,

A is I $_\theta$ - open if, $\forall x \in X, [\forall I - open\ set\ G\, in\ X\ with\, x \in G, IclG \cap (X - A) \neq \Phi]$

$\Leftrightarrow x \in X - A$**……..(** α **)**

Lemma-2.1 The intersection of a finite number of I $_\theta$ - open sets in X is I $_\theta$ -open.

Proof: Let G_1, G_2, G_3,………….,G_n be I $_\theta$ - open sets in X. Now let W∈ **I** and x∈W. Also, let

$$\overline{W} \cap (G_1 \cap G_2 \cap \cdots\cdots \cap G_n)^c \neq \Phi. Then\ \overline{W} \cap (G_1{}^c \cup G_2{}^c \cup \cdots\cdots \cup G_n{}^c) \neq \Phi,$$

$$and\ so\ (\overline{W} \cap G_1{}^c) \cup (\overline{W} \cap G_2{}^c) \cup \cdots\cdots \cup (\overline{W} \cap G_n{}^c) \neq \Phi.$$

$$Therfore\ (\overline{W} \cap G_i{}^c) \neq \Phi,\ for\, at\, least\ one\ i,\ 1 \le i \le n, say\, i_0\ , x \in G_{i_0}{}^c,\ i.e\ x \notin G_{i_0}.$$

By (α), since G_{i_0} is I $_\theta$ - open.

$$Hence\ \ x \notin G_1 \cap G_2 \cap \cdots\cdots \cap G_n\ \ i.e.,\ \ x \in (G_1 \cap G_2 \cap \cdots\cdots \cap G_n)^c$$

$$\therefore G_1 \cap G_2 \cap \cdots\cdots \cap G_n is\, I_\theta - open\, in\, X\ by\ (\alpha).$$

Obviously, X and Φ are both I $_\theta$ - closed and I $_\theta$ - open. [$\because Icl_\theta(X) = X,\ Icl_\theta(\Phi) = \Phi$].

We thus have the following Theorem

Theorem- 2.1 The I $_\theta$ - open sets in X form an I- structure on X.

If **I** is an I-structure on X, denote by **I** $_\theta$ the I- structure on X consisting of all I $_\theta$ - open sets in X,

We then have

Theorem- 2.2 I $_\theta \subseteq$ **I**

Proof: Let A be a subset of X. Then $A \subseteq \overline{A} \subseteq Icl_\theta A$. Hence if A is I $_\theta$ - closed, then $Icl_\theta A = A, and\, so, \overline{A} = A, i.e.,$ A is I- closed.

Therefore, for each G∈ I $_\theta$, X - G is I $_\theta$ - closed; and so, X – G is I- closed. Thus G∈ **I**

Lemma- 2.2: Any union of I $_\theta$ - open sets is I $_\theta$ - open.

Proof: Let $\{V_\alpha\}$ be a non-empty collection of I $_\theta$ - open sets. Let G be any I $_\theta$ - open set in X, and let x∈ G such that $\overline{G} \cap (\underset{\alpha}{\cup} V_\alpha)^c \neq \Phi. Then\ \overline{G} \cap (\underset{\alpha}{\cap} V_\alpha{}^c) \neq \Phi.$ Hence, $\overline{G} \cap V_\alpha{}^c) \neq \Phi$, for each α. Since each V_α is θ-open, $\forall \alpha, x \notin V_\alpha$, by (α). Hence $x \notin \underset{\alpha}{\cup} V_\alpha,\ i.e., x \in (\underset{\alpha}{\cup} V_\alpha)^c. Hence\ \underset{\alpha}{\cup} V_\alpha\ is\ I_\theta -$ open, again by (α).

Theorem- 2.3: **I** $_\theta$ is a topology on X.

Proof: The proof follows from Lemmas 2.1 and 2.2.

Definition-2.3 Let (X, **I** $_X$), (Y, **I** $_Y$) be I- spaces. Let G∈ **I** $_X$, H∈ **I** $_Y$. The product I-structure on $X \times Y$ is the I- structure generated by all sets of the terms $\pi_X{}^{-1}(G_X)$ and $\pi_Y{}^{-1}(G_Y)$, where $\pi_X\ and\ \pi_Y$

are the projection maps from $X \times Y$ onto X and Y respectively. Thus the product I- structure is the smallest I- structure on $X \times Y$ such that the projection maps π_X *and* π_Y are I- continuous.

Lemma- 2.3 Let (X, **I**$_X$), (Y, **I**$_Y$) be I- spaces and let $A \subseteq X, B \subseteq Y$. Then $Icl_\theta(A \times B) = Icl_\theta A \times Icl_\theta B.$

Proof: To see the truth of this statement, let (x, y) $\in Icl_\theta(A \times B)$. Then for each I- open set W in $X \times Y$ with (x, y) $\in$W, $\overline{W} \cap (A \times B) \neq \Phi$. Thus, by the definition of **I**$_{X\times Y}$, it follows that in particular, for each I- open G_x in X with x$\in G_x$ and for each I-open H_y in Y with y$\in H_y$, $\overline{G}_x \cap A \neq \Phi$, $\overline{H}_y \cap B \neq \Phi$, *i.e.*, $x \in Icl_\theta A$, $y \in Icl_\theta B$. Hence (x, y)$\in Icl_\theta A \times Icl_\theta B$.

Thus, $Icl_\theta(A \times B) \subseteq Icl_\theta A \times Icl_\theta B$. It is now obvious that the converse is also true.

Theorem- 2.4 Let (X, **I**$_X$), (Y, **I**$_Y$) be two I- spaces and let A and B be two I$_\theta$-closed subsets of X and Y respectively. Then $A \times B$ is I$_\theta$-closed in ($X \times Y$, **I**$_X$ × **I**$_Y$).

Proof: Since A and B are I$_\theta$- closed subsets of X and Y respectively, $A = Icl_\theta A$, and $B = Icl_\theta B$. Hence $A \times B = (Icl_\theta A) \times (Icl_\theta B) = Icl_\theta(A \times B)$, by lemma 2.2. Thus, $A \times B$ is I$_\theta$- closed.

Theorem- 2.5 The product of two I$_\theta$- open sets in I - spaces is I$_\theta$- open in their product spaces.

Proof: Let (X, **I**$_X$), (Y, **I**$_Y$) be two I- spaces and let A and B be two I$_\theta$-closed subsets of X and Y respectively. Then X – A and Y – B are I$_\theta$- closed in X and Y respectively. Now ($X \times Y$) – ($A \times B$) = $[(X - A) \times Y] \cup [X \times (Y - B)]$. X and Y are are I$_\theta$- closed in X and Y respectively.

Hence $(X - A) \times Y$ *and* $X \times (Y - B)$ are I$_\theta$-closed in $X \times Y$ .i.e.,

$$(X - A) \times Y = Icl_\theta[(X - A) \times Y] \text{ and } X \times (Y - B) = Icl_\theta[X \times (Y - B)]$$

$$\therefore [(X - A) \times Y] \cup [X \times (Y - B)] = Icl_\theta[(X - A) \times Y] \cup Icl_\theta[X \times (Y - B)]$$

$$\Rightarrow [(X - A) \times Y] \cup [X \times (Y - B)] = Icl_\theta[\{(X - A) \times Y\} \cup \{X \times (Y - B)\}].$$

Hence ($X \times Y$) – ($A \times B$) is I$_\theta$- closed, and so, ($A \times B$) is I-open.

Definition-2.4 I$_\theta$- compactness and I$_\theta$- connectedness: Since **I**$_\theta$ is an I- structure for every I- structure I on X, **I**$_\theta$- compactness and **I**$_\theta$- connectedness are defined in the usual manner. Since **I**$_\theta \subseteq$**I**, X is compact $\Rightarrow$X is **I**$_\theta$- compact and X is connected $\Rightarrow$X is **I**$_\theta$- connected.

A pair (P, Q) of non-empty subsets of X is called **I$_\theta$- separation** relative to X if $(P \cap Icl_\theta Q) \cup (Q \cap Icl_\theta P) = \Phi$.

A subset A of X is called **I**$_\theta$- connected if $A \neq P \cup Q$, where (P,Q) is a **I**$_\theta$ **-separation** relative to X.

Theorem- 2.6 X is connected $\Rightarrow$X is I$_\theta$- connected $\Rightarrow$X is **I**$_\theta$- connected.

Proof: Suppose X is connected. If possible, let X be I$_\theta$- disconnected.

Then X = P$\cup$Q, where P, Q are non-empty and $P \cap Icl_\theta Q = \Phi$ *and* $Q \cap Icl_\theta P = \Phi$ *Clearly* $P \cap Q = \Phi$.

Let x$\in$P. Then x$\notin$ Icl$_\theta$Q, and so, there exists an open set G in X such that x$\in$G and $Q\cap\overline{G}=\Phi$. Then $\overline{G}\subseteq X-Q=P$ and so G$\subseteq$P. Hence P is open.

Similarly, we can show that Q is open. Therefore X is disconnected. The contradiction proves that X is I$_\theta$- connected.

Next let X be I$_\theta$- connected. Suppose X is not I$_\theta$ connected. Then X = P$\cup$Q for disjoint non- empty I$_\theta$- open sets P and Q. Since X is I$_\theta$- connected,

either $P\cap Icl_\theta Q\neq\Phi$ $or,$ $Q\cap Icl_\theta P\neq\Phi$. i.e., either $Icl_\theta Q\not\subset X-P=Q\,or\;Icl_\theta\not\subset X-Q=P$. i.e., either Q is not I$_\theta$- closed or P is not I$_\theta$- closed. But this is a contradiction to the hypothesis. Hence X is **I**$_\theta$- connected.

Comment: 2.1 While connectedness implies **I**$_\theta$- connectedness, the converse is not true.

Theorem- 2.7 Sum of two **I**$_\theta$- connected spaces is also **I**$_\theta$- connected.

Proof: Let X and Y be two **I**$_\theta$- connected spaces. If possible suppose X + Y is not **I**$_\theta$-connected. Then there exists two non- empty subsets P, Q of X + Y such that

X+Y = P$\cup$Q with $P\cap Icl_\theta Q=\Phi\cdots\cdots(1)$ and $Q\cap Icl_\theta P=\Phi\cdots\cdots\cdots(2)$

Let $P_1=P\cap X,\ Q_1=Q\cap X\cdots\cdots\cdots(3)$ and $P_2=P\cap Y,\ Q_2=Q\cap Y\cdots\cdots\cdots(4)$

Then $X=P_1\cup Q_1,\ Y=P_2\cup Q_2$ .Since X and Y are **I**$_\theta$- connected,

$$P_1\cap Icl_\theta Q_1\neq\Phi\ or,\ Q_1\cap Icl_\theta P_1\neq\Phi\cdots\cdots\cdots(5)$$

$$P_2\cap Icl_\theta Q_2\neq\Phi\ or,\ Q_2\cap Icl_\theta P_2\neq\Phi\cdots\cdots\cdots(6)$$

Now $P\cap Icl_\theta Q=(P_1\cup P_2)\cap Icl_\theta(Q_1\cup Q_2)=(P_1\cup P_2)\cap(Icl_\theta Q_1\cup Icl_\theta Q_2)$

$=(P_1\cap Icl_\theta Q_1)\cup(P_1\cap Icl_\theta Q_2)\cup(P_2\cap Icl_\theta Q_1)\cup(P_2\cap Icl_\theta Q_2)\neq\Phi$ by (5) and (6)

[By (1) $P_1\cap Icl_\theta Q_1=\Phi$ and $P_2\cap Icl_\theta Q_2=\Phi$. So from (5) and (6) $Q_1\cap Icl_\theta P_1\neq\Phi$ *and* $Q_2\cap Icl_\theta P_2\neq\Phi$]

Similarly, $Q\cap Icl_\theta P=\ (Q_1\cap Icl_\theta P_1)\cup(Q_1\cap Icl_\theta P_2)\cup(Q_2\cap Icl_\theta P_1)\cup(Q_2\cap Icl_\theta P_2)\neq\Phi$

This is contradicts (1) and (2). Hence, X + Y is **I**$_\theta$- connected.

Definition-2.5 Velicko [12] defined a space X to be **H-closed** if every open cover $\{V_\alpha\}$ of X has a finite sub collection $\{V_{\alpha_1},V_{\alpha_2},\cdots\cdots,V_{\alpha_n}\}$ such that $X=\overline{V_{\alpha_1}}\cup\cdots\cdots\cup\overline{V_{\alpha_n}}$.

Ganguly and Bandyopadhyaya [3] defined and studied H-continua. An **H-continum** is a topological space which is both connected and H- closed.

Definition-2.6 IH-closed sets: An I-space X will be called an **IH-closed** if for every I-open cover U = $\{V_\alpha\}$ of X , there exists a finite subcollection $\{V_{\alpha_1},V_{\alpha_2},\cdots\cdots,V_{\alpha_n}\}$of U such that $X\subseteq\overline{V_{\alpha_1}}\cup\cdots\cdots\cup\overline{V_{\alpha_n}}$.

Definition-2.7 IH-continum: An I-space X will be called an **IH-continuum** if X is **I**$_\theta$-connected and H θ-closed.

Similar definitions for topological spaces were introduced by Velicko [12] and Ganguly and Bandyopadhyaya [3] respectively.

The following theorems for IH-closed and IH-continua hold:

Theorem- 2.8 The product of two IH-closed spaces is IH-closed.

Theorem- 2.9 The product of two IH-continua spaces is also IH-continuum.

Theorem- 2.10 If X and Y are IH-continua, then X + Y will be an IH- continuum iff $X \cap Y \neq \Phi$.

Theorem- 2.11 Let X be an IH-continuum and Y an I – spaces and let f: X $\rightarrow$Y be both continuous and open. Then f(X) is an IH- continuum.

Proof: The proofs of these theorems are exactly the same as those for Theorems 4.1, 4.2, 4.3, and 4.4 respectively of [2] M. Mitra and S. Majumdar.

Lemma- 2.4 If X and Y are compatible IH-closed spaces then X + Y is IH-closed.

Proof: Let $\{W_\alpha\}$ be an I-open cover of X + Y. Then each $W_\alpha = U_\alpha \cup V_\alpha$, for some U_α, V_α I-open in X and Y respectively. Then $\{U_\alpha\}$ and $\{V_\alpha\}$ are I- open covers of X and Y respectively. Since X and Y are IH-closed, $X = \overline{U}_{\alpha_1} \cup \overline{U}_{\alpha_2} \cup \cdots\cdots \cup \overline{U}_{\alpha_m}$ and $Y = \overline{V}_{\beta_1} \cup \overline{V}_{\beta_2} \cup \cdots\cdots \cup \overline{V}_{\beta_n}$ for some $\alpha_1, \alpha_2, \cdots\cdots \alpha_m, \beta_1, \beta_2, \cdots\cdots \beta_n$. Then X + Y = $\overline{W}_{\alpha_1} \cup \overline{W}_{\alpha_2} \cup \cdots\cdots \cup \overline{W}_{\alpha_m} \cup \overline{W}_{\beta_1} \cup \overline{W}_{\beta_2} \cup \cdots\cdots \cup \overline{W}_{\beta_n}$. Hence X + Y is IH- closed.

Comment: 2.2 A subspace of an IH-continuum space need not be so.

For [0,1] is an IH- continuum, but the subspace {0,1} is not IH- continuum as it is not **I** $_\theta$ - connected.

The property of being an IH- continuum does not hold for intersection.

If C $= \{(x,y) | x^2 + y^2 = 1\}, C_1 = \{(x,y) \in C | x \leq 0\}$ *and* $C_2 = \{(x,y) \in C | x \geq 0\}$, then $C \cap C_1 \cap C_2 = \{(0,1),(0,-1)\}$ is not IH- continuum as it is not **I** $_\theta$ - connected.

Comment: 2.3 If f is only I-open or only I- continuous, then f(X) need not be an IH-continuum.

For, if (X, **I**) is a IH- continuum and X has at least two elements and f:(X, **I**) $\rightarrow$(X,D) is the identity map on X where D is the discrete I- space, then f is I- open but f(X) is not an IH- continuum because it is **I** $_\theta$ - disconnected.

If for the above IH- continuum (X, I), f:(X, **I**) $\rightarrow$(X, $\mathbf{I}_0$) is the identity map on X where $\mathbf{I}_0$ denotes the indiscrete I- space, then f is I- continuous, but f(X) = X is not an IH-continuum since f(X) is not I-Hausdorff.

Comment: 2.4 If X is an IH- continuum and R an equivalence relation on X, then the identification space X/R is an IH- continuum as the projection map X $\rightarrow$ X/R is onto and both I- continuous and I-open.

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