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ON NON-HOMOGENEOUS SEXTIC EQUATION WITH FIVE UNKNOWNS

$$(x+y)(x^3-y^3)=26(z^2-w^2)T^4$$

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ABSTRACT

The non-homogeneous sextic equation with five unknowns given by $(x+y)(x^3-y^3)=26(z^2-w^2)T^4$ is considered and analysed for its non-zero distinct integer solutions. Employing the linear transformation $x=u+v, y=u-v, z=2u+v, w=2u-v, (u\neq v\neq 0)$ and applying the method of factorization, three different patterns of non-zero distinct integer solutions are obtained. A few interesting relations between the solutions and special numbers namely Four dimensional numbers, Polygonal numbers, Octahedral numbers, Pyramidal numbers, Centered Pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, Kynea numbers and Star numbers are presented.

Keywords: Non-homogeneous sextic equation, sextic equation with five unknowns, Integer solutions.

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1. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4] particularly, in [5, 6] Sextic equations with three unknowns are studied for their integral solutions. [7-12] analyze Sextic equations with four unknowns for their non-zero integer solutions. [13, 14] analyze Sextic equations with five unknowns for their non-zero integer solutions. This communication analyzes a Sextic equation with five unknowns given by $(x+y)(x^3-y^3)=26(z^2-w^2)T^4$.

Infinitely many Quintuples (x, y, z, w, T) satisfying the above equation are obtained. Various interesting properties among the values of x, y, z, w and T are presented.

Notations:

- Polygonal number of rank n with size m

$$T_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

- Pyramidal number of rank n with size m

$$P_n^m=\frac{1}{6}[n(n+1)][(m-2)n+(5-m)]$$

- Star number of rank n

$$S_n=6n(n-1)+1$$

- Octahedral number of rank n

$$OH_n=\frac{1}{3}\left(n\left(2n^2+1\right)\right)$$

- Centered Pyramidal number of rank n with size m

$$CP_{m,n}=\frac{m(n-1)n(n+1)+6n}{6}$$

- Four dimensional Figurate number of rank n whose generating polygon is a square

$$F_{4,n,4}=\frac{n^4+5n^3+8n^2+4n}{12}$$

- Four dimensional Figurate number of rank n whose generating polygon is a pentagon

$$F_{4,n,5}=\frac{3n^4+10n^3+9n^2+2n}{4!}$$

- Jacobsthal number of rank n

$$J_n=\frac{1}{3}\left(2^n-(-1)^n\right)$$

- Jacobsthal-Lucas number of rank n

$$j_n=2^n+(-1)^n$$

- Kynea number of rank n

$$Ky_n=(2^n+1)^2-2$$

2. METHOD OF ANALYSIS

The non-homogeneous sextic equation with five unknowns to be solved is given by

$$(x+y)(x^3-y^3)=26(z^2-w^2)T^4 \tag{1}$$

The substitution of the linear transformations

$$x=u+v,\ y=u-v,\ z=2u+v,\ w=2u-v,\ u\neq v\neq 0 \tag{2}$$

in (1) leads to

$$3u^2+v^2=52T^4 \tag{3}$$

Assume $T=T(a,b)=a^2+3b^2;\quad a,b>0$ (4)

(3) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

2.1 Pattern: 1

Write 52 as

$$52=\left(7+i\sqrt{3}\right)\left(7-i\sqrt{3}\right) \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization and equating positive factors we get

$$\left(v+i\sqrt{3}u\right)=\left(7+i\sqrt{3}\right)\left(a+i\sqrt{3}b\right)^4$$

Equating real and imaginary parts,

$$u=u(a,b)=a^4+28a^3b-18a^2b^2-84ab^3+9b^4$$

$$v=v(a,b)=7a^4-12a^3b-126a^2b^2+36ab^3+63b^4$$

Employing (2), the values of x, y, z, w and T are given by

$$x=x(a,b)=u+v=8a^4+16a^3b-144a^2b^2-48ab^3+72b^4$$

$$y=y(a,b)=u-v=-6a^4+40a^3b+108a^2b^2-120ab^3-54b^4$$

$$z=z(a,b)=2u+v=9a^4+44a^3b-162a^2b^2-132ab^3+81b^4$$

$$w=w(a,b)=2u-v=-5a^4+68a^3b+90a^2b^2-204ab^3-45b^4$$

$$T=T(a,b)=a^2+3b^2$$

which represent non-zero distinct integer solutions of (1) in two parameters.

Properties:

- $104SO_a-3x(a,1)-4y(a,1)\equiv 0\pmod{5}$
- $T(1,2^n)+9J_n+3j_n-4=3Ky_n$
- $72(T_{4,b})^2-x(1,b)-48CP_{6,b}-24S_b\equiv 0\pmod{2}$
- $w(a,a)-y(a,a)-z(a,a)$ is a nasty number
- $2\{T(1,b)-1\}$ is a nasty number

2.2 Pattern: 2

One may write (3) as

$$v^2+3u^2=52T^4*1 \tag{6}$$

Also, write 1 as

$$1=\frac{\left(1+i4\sqrt{3}\right)\left(1-i4\sqrt{3}\right)}{49} \tag{7}$$

Substituting (4), (5) and (7) in (6) and employing the method of factorization and equating positive factors we get

$$\left(v+i\sqrt{3}u\right)=\frac{\left(7+i\sqrt{3}\right)\left(1+i4\sqrt{3}\right)}{7}\left(a+i\sqrt{3}b\right)^4$$

Equating real and imaginary parts, we have

$$u=u(a,b)=\frac{1}{7}\left(29a^4-20a^3b-522a^2b^2+60ab^3+261b^4\right) \tag{8}$$

$$v=v(a,b)=\frac{1}{7}\left(-5a^4-348a^3b+90a^2b^2+1044ab^3-45b^4\right) \tag{9}$$

The choices a=7A and b=7B in (8), (9) lead to

$$u = u(A,B) = 9947A^4 - 6860A^3B - 179046A^2B^2 + 20580AB^3 + 89523B^4$$

$$v = v(A,B) = -1715A^4 - 119364A^3B + 30870A^2B^2 + 358092AB^3 - 15435B^4$$

In view of (2), the integer values of x, y, z, w and T are given by

$$x = x(A.B) = 8232A^4 - 12622A^3B - 148176A^2B^2 + 378672AB^3 + 74088B^4$$

$$y = y(A,B) = 11662A^4 + 112504A^3B - 209916A^2B^2 - 337512AB^3 + 104958B^4$$

$$z = z(A,B) = 18179A^4 - 133084A^3B - 327222A^2B^2 + 399252AB^3 + 163611B^4$$

$$w = w(A,B) = 21609A^4 + 105644A^3B - 388962A^2B^2 - 316932AB^3 + 194481B^4$$

$$T = T(A,B) = 49A^2 + 147B^2$$

which represent non-zero distinct integer solutions of (1) in two parameters.

Properties:

- $24F_{4,A,5} + 76829(T_{4,A})^2 - x(A,-A) - 3y(A,-A) - 15(OH_A) - 3T_{8,A} \equiv 0 \pmod 3$
- $218148F_{4,A,4} - z(A,1) - 223979CP_{6,A} - 135044T_{9,A} \equiv 0 \pmod 7$
- $w(1,B) - 194481T_{4,B^2} + 633864P_B^5 + 12005S_B \equiv 0 \pmod 2$
- $T(1,2^n) + 147(3J_n + j_n) - 196 = 147Ky_n$
- $2\{T(1,B) - 49\}$ is a nasty number

Remark:

It is worth to note that 52 in (5) and 1in (7) are also represented in the following ways

$$52 = (5 + i3\sqrt{3})(5 - i3\sqrt{3}) = (2 + i4\sqrt{3})(2 - i4\sqrt{3})$$

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} = \frac{(1 + i15\sqrt{3})(1 - i15\sqrt{3})}{676}$$

By introducing the above representations in (5) and (7), one may obtain different patterns of solutions to (1).

2.3 Pattern: 3

Write (3) as

$$3(u^2 - T^4) = 49T^4 - v^2 \tag{10}$$

Factorizing (10) we have

$$3(u + T^2)(u - T^2) = (7T^2 + v)(7T^2 - v) \tag{11}$$

This equation is written in the form of ratio as

$$\frac{3(u - T^2)}{7T^2 - v} = \frac{(7T^2 + v)}{u + T^2} = \frac{a}{b}, \quad b \neq 0 \tag{12}$$

which is equivalent to the system of double equations

$$3bu + av - (3b + 7a)T^2 = 0 \tag{13}$$

$$-au + bv + (7b - a)T^2 = 0 \tag{14}$$

Applying the method of cross multiplication, we get

$$u = -a^2 + 3b^2 + 14ab \tag{15}$$

$$v = 7a^2 - 21b^2 + 6ab \tag{16}$$

$$T^2 = a^2 + 3b^2 \quad (17)$$

Here $T^2(a,b)$ is of the form $z^2 = Dx^2 + y^2$ (D>0 and square free). Then the solution for (17) is

$$a = 3p^2 - q^2, \quad b = 2pq, \quad T = 3p^2 + q^2 \quad (18)$$

Using (18) in (15) and (16), we get

$$u = u(p,q) = -9p^4 + 84p^3q + 18p^2q^2 - 28pq^3 - q^4$$

$$v = v(p,q) = 63p^4 + 36p^3q - 126p^2q^2 - 12pq^3 + 7q^4$$

In view of (2), the integer values of x, y, z, w, T are given by

$$x = x(p,q) = u + v = 54p^4 + 120p^3q - 108p^2q^2 - 40pq^3 + 6q^4$$

$$y = y(p,q) = u - v = -72p^4 + 48p^3q + 144p^2q^2 - 16pq^3 - 8q^4$$

$$z = z(p,q) = 2u + v = 45p^4 + 204p^3q - 90p^2q^2 - 68pq^3 + 5q^4$$

$$w = w(p,q) = 2u - v = -81p^4 + 132p^3q + 162p^2q^2 - 44pq^3 - 9q^4$$

$$T = T(p,q) = 3p^2 + q^2$$

which represent non-zero distinct integer solutions of (1) in two parameters.

Properties:

- $w(1,q) - y(1,q) + (T_{4,q})^2 + 28(CP_{6,q}) - 3S_q \equiv 0 \pmod 2$
- $T(2^n,1) - 3ky_n + 6j_n = \begin{cases} -2, & \text{if } n \text{ is odd} \\ 10, & \text{if } n \text{ is even} \end{cases}$
- $y(p,1) + 72(T_{4,p})^2 - 72CP_{8,p} - 48T_{8,p} \equiv 0 \pmod 2$
- $2\{T(p,1) - 1\}$ is a nasty number

3. CONCLUSION

First of all, it is worth to mention here that in (2), the values of z and w may also be represented by $z = 2uv + 1, w = 2uv - 1$ and $z = uv + 2, w = uv - 2$ and thus will obtain other choices of solutions to (1). In conclusion, one may consider other forms of Sextic equation with five unknowns and search for their integer solutions.

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