ON NON-HOMOGENEOUS SEXTIC EQUATION WITH FIVE UNKNOWNS

\[(x + y)(x^3 - y^3) = 26(z^2 - w^2)\Gamma^4\]

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ABSTRACT

The non-homogeneous sextic equation with five unknowns given by 
\[(x + y)(x^3 - y^3) = 26(z^2 - w^2)\Gamma^4\] is considered and analysed for its non-zero distinct integer solutions. Employing the linear transformation 
x = u + v, y = u - v, z = 2u + v, w = 2u - v, (u \neq v \neq 0) and applying the method of factorization, three different patterns of non-zero distinct integer solutions are obtained. A few interesting relations between the solutions and special numbers namely Four dimensional numbers, Polygonal numbers, Octahedral numbers, Pyramidal numbers, Centered Pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, Kynea numbers and Star numbers are presented.

Keywords: Non-homogeneous sextic equation, sextic equation with five unknowns, Integer solutions.

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1. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4] particularly, in [5, 6] Sextic equations with three unknowns are studied for their integral solutions. [7-12] analyze Sextic equations with four unknowns for their non-zero integer solutions. [13, 14] analyze Sextic equations with five unknowns for their non-zero integer solutions. This
communication analyzes a Sextic equation with five unknowns given by
\[(x + y)(x^3 - y^3) = 26(z^2 - w^2)T^4.\]

Infinitely many Quintuples \((x, y, z, w, T)\) satisfying the above equation are obtained. Various interesting properties among the values of \(x, y, z, w\) and \(T\) are presented.

**Notations:**
- Polygonal number of rank \(n\) with size \(m\)
  \[T_{m,n} = n \left[1 + \frac{(n-1)(m-2)}{2}\right]\]
- Pyramidal number of rank \(n\) with size \(m\)
  \[P^m_n = \frac{1}{6} [n(n+1)][(m-2)n + (5-m)]\]
- Star number of rank \(n\)
  \[S_n = 6n(n-1) + 1\]
- Octahedral number of rank \(n\)
  \[OH_n = \frac{1}{3} (n(2n^2 + 1))\]
- Centered Pyramidal number of rank \(n\) with size \(m\)
  \[CP_{m,n} = \frac{m(n-1)n(n+1)+6n}{6}\]
- Four dimensional Figurate number of rank \(n\) whose generating polygon is a square
  \[F_{4,n,4} = \frac{n^4 + 5n^3 + 8n^2 + 4n}{12}\]
- Four dimensional Figurate number of rank \(n\) whose generating polygon is a pentagon
  \[F_{4,n,5} = \frac{3n^4 + 10n^3 + 9n^2 + 2n}{4!}\]
- Jacobsthal number of rank \(n\)
  \[J_n = \frac{1}{3} \left(2^n - (-1)^n\right)\]
- Jacobsthal-Lucas number of rank \(n\)
  \[j_n = 2^n + (-1)^n\]
- Kynea number of rank \(n\)
  \[Ky_n = \left(2^n + 1\right)^2 - 2\]

2. **METHOD OF ANALYSIS**

The non-homogeneous sextic equation with five unknowns to be solved is given by
\[(x + y)(x^3 - y^3) = 26(z^2 - w^2)T^4.\]  
(1)

The substitution of the linear transformations
\[x = u + v, \ y = u - v, \ z = 2u + v, \ w = 2u - v, \ u \neq v \neq 0\]
in (1) leads to
\[3u^2 + v^2 = 52T^4\]  
(3)
Assume \(T = T(a, b) = a^2 + 3b^2; \ a, b > 0\)
(3) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

### 2.1 Pattern: 1

Write 52 as

\[
52 = (7 + i\sqrt{3})(7 - i\sqrt{3})
\]

(5)

Using (4) and (5) in (3) and employing the method of factorization and equating positive factors we get

\[
(v + i\sqrt{3}u) = (7 + i\sqrt{3})(a + i\sqrt{3}b)
\]

Equating real and imaginary parts,

\[
u = u(a,b) = a^4 + 28a^3b - 18a^2b^2 - 84ab^3 + 9b^4
\]

\[
v = v(a,b) = 7a^4 - 12a^3b - 126a^2b^2 + 36ab^3 + 63b^4
\]

Employing (2), the values of x, y, z, w and T are given by

\[
x = x(a,b) = u + v = 8a^4 + 16a^3b - 144a^2b^2 - 48ab^3 + 72b^4
\]

\[
y = y(a,b) = u - v = -6a^4 + 40a^3b + 108a^2b^2 - 120ab^3 - 54b^4
\]

\[
z = z(a,b) = 2u + v = 9a^4 + 44a^3b - 162a^2b^2 - 132ab^3 + 8b^4
\]

\[
w = w(a,b) = 2u - v = -5a^4 + 68a^3b + 90a^2b^2 - 204ab^3 - 45b^4
\]

\[
T = T(a,b) = a^2 + 3b^2
\]

which represent non-zero distinct integer solutions of (1) in two parameters.

**Properties:**

- \(104SO_n - 3x(a,1) - 4y(a,1) \equiv 0 \pmod{5}\)
- \(T(1,2^n) + 9J_n + 3j_n - 4 = 3Ky_n\)
- \(72(T_{4,b})^2 - x(1,b) - 48CP_{6,b} - 24S_b \equiv 0 \pmod{2}\)
- \(w(a,a) - y(a,a) - z(a,a)\) is a nasty number
- \(2[T(1,b) - 1]\) is a nasty number

### 2.2 Pattern: 2

One may write (3) as

\[
v^2 + 3u^2 = 52T^4 \times 1
\]

(6)

Also, write 1 as

\[
1 = \frac{(1 + 4\sqrt{3}) (1 - 4\sqrt{3})}{49}
\]

(7)

Substituting (4), (5) and (7) in (6) and employing the method of factorization and equating positive factors we get

\[
(v + i\sqrt{3}u) = \frac{(7 + i\sqrt{3})(1 + 4\sqrt{3}) (a + i\sqrt{3}b)}{7}
\]

Equating real and imaginary parts, we have

\[
u = u(a,b) = \frac{1}{7} (29a^4 - 20a^3b - 522a^2b^2 + 60ab^3 + 26b^4)
\]

(8)

\[
v = v(a,b) = \frac{1}{7} (-5a^4 - 348a^3b + 90a^2b^2 + 1044ab^3 - 45b^4)
\]

(9)
The choices \(a=7A\) and \(b=7B\) in (8), (9) lead to

\[
\begin{align*}
    u &= u(A, B) = 9947A^4 - 6860A^3B - 179046A^2B^2 + 20580AB^3 + 89523B^4 \\
    v &= v(A, B) = -1715A^4 - 119364A^3B - 30870A^2B^2 + 358092AB^3 - 15435B^4
\end{align*}
\]

In view of (2), the integer values of \(x, y, z, w\) and \(T\) are given by

\[
\begin{align*}
    x &= x(A, B) = 8232A^4 - 12622A^3B - 148176A^2B^2 + 378672AB^3 + 74088B^4 \\
    y &= y(A, B) = 11662A^4 + 112504A^3B - 209916A^2B^2 - 337512AB^3 + 104958B^4 \\
    z &= z(A, B) = 18179A^4 - 133084A^3B - 327222A^2B^2 + 399252AB^3 + 16361B^4 \\
    w &= w(A, B) = 21609A^4 + 105644A^3B - 388962A^2B^2 - 316932AB^3 + 19448B^4 \\
    T &= T(A, B) = 49A^2 + 147B^2
\end{align*}
\]

which represent non-zero distinct integer solutions of (1) in two parameters.

Properties:

- \(24F_{4,4,5} + 76829(T_{4,4})^2 - x(A, -A) - 3y(A, -A) - 15(OH_A) - 3T_{8,A} \equiv 0 \pmod{3}\)
- \(218148F_{4,4,4} - z(A, 1) - 223979CP_{6,6} - 135044T_{9,6} \equiv 0 \pmod{7}\)
- \(w(1, B) - 19448 \Pi_{4,6} + 633864P_B^5 + 12005S_B \equiv 0 \pmod{2}\)
- \(T(1, 2^n) + 147(3J_n + j_n) - 196 = 147K_n\)
- \(2[T(1, B) - 49] \) is a nasty number

Remark:

It is worth to note that 52 in (5) and 1 in (7) are also represented in the following ways

\[
\begin{align*}
    52 &= (5 + i3\sqrt{3})(5 - i3\sqrt{3}) \\
    &= (2 + i4\sqrt{3})(2 - i4\sqrt{3}) \\
    &= \frac{1 + i\sqrt{3}}{4} \frac{1 - i\sqrt{3}}{} \\
    &= \frac{1 + i15\sqrt{3}}{676} \frac{1 - i15\sqrt{3}}{} \\
    1 &= \frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}
\end{align*}
\]

By introducing the above representations in (5) and (7), one may obtain different patterns of solutions to (1).

2.3 Pattern: 3

Write (3) as

\[
3(u^2 - T^2) = 49T^4 - v^2
\]

Factorizing (10) we have

\[
3(u + T^2)(u - T^2) = (7T^2 + v)(7T^2 - v)
\]

This equation is written in the form of ratio as

\[
\frac{3(u - T^2)}{7T^2 - v} = \frac{7T^2 + v}{u + T^2} = \frac{a}{b}, \quad b \neq 0
\]

which is equivalent to the system of double equations

\[
\begin{align*}
    3bu + av - (3b + 7a)T^2 &= 0 \\
    - au + bv + (7b - a)T^2 &= 0
\end{align*}
\]

Applying the method of cross multiplication, we get

\[
\begin{align*}
    u &= -a^2 + 3b^2 + 14ab \\
    v &= 7a^2 - 2ab^2 + 6ab
\end{align*}
\]
Here $T^2(a,b)$ is of the form $z^2 = Dx^2 + y^2$ (D>0 and square free). Then the solution for (17) is

$$a = 3p^2 - q^2, \quad b = 2pq, \quad T = 3p^2 + q^2$$

Using (18) in (15) and (16), we get

$$u = u(p,q) = -9p^4 + 84p^3q + 18p^2q^2 - 28pq^3 - q^4$$
$$v = v(p,q) = 63p^4 + 36p^3q - 126p^2q^2 - 12pq^3 + 7q^4$$

In view of (2), the integer values of $x, y, z, w, T$ are given by

$$x = x(p,q) = u + v = 54p^4 + 120p^3q - 108p^2q^2 - 40pq^3 + 6q^4$$
$$y = y(p,q) = u - v = -72p^4 + 48p^3q + 144p^2q^2 - 16pq^3 - 8q^4$$
$$z = z(p,q) = 2u + v = 45p^4 + 204p^3q - 90p^2q^2 - 68pq^3 + 5q^4$$
$$w = w(p,q) = 2u - v = -81p^4 + 132p^3q + 162p^2q^2 - 44pq^3 - 9q^4$$

$$T = T(p,q) = 3p^2 + q^2$$

which represent non-zero distinct integer solutions of (1) in two parameters.

**Properties:**

- $w(1,q) - y(1,q) + (T_{4,q})^2 + 28(CP_{6,q}) - 3S_q = 0 (mod 2)$
- $T(2^n,1) - 3ky_n + 6j_n = \begin{cases} -2, & \text{if } n \text{ is odd} \\ 10, & \text{if } n \text{ is even} \end{cases}$
- $y(p,1) + 72(T_{s,p})^2 - 72CP_{s,p} - 48T_{s,p} = 0 (mod 2)$
- $2[T(p,1) - 1]$ is a nasty number

**3. CONCLUSION**

First of all, it is worth to mention here that in (2), the values of $z$ and $w$ may also be represented by $z = 2uv + 1, w = 2uv - 1$ and $z = uv + 2, w = uv - 2$ and thus will obtain other choices of solutions to (1). In conclusion, one may consider other forms of Sextic equation with five unknowns and search for their integer solutions.

**4. REFERENCES**


[12] Gaussian Integer Solutions of Sextic Equations with four unknowns \( x^6 - y^6 = 4z(x^4 + y^4 + w^4) \), Archimedes, J.Math, 3(3), 263-266, (2013)
