Vol.5.Issue.3.2017 (July-Sept.)
©KY PUBLICATIONS

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

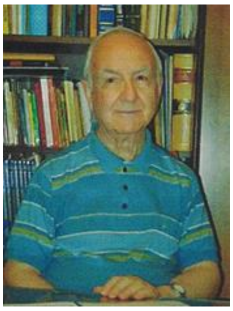
http://www.bomsr.com
Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

THE RIEMANN HYPOTHESIS CONCERNING THE ZETA FUNCTION

Prof. Aldo P.Peretti
Professor (Ex), Universidad Kennedy
Buenos Aires, Argentina
Email: aldopperetti@gmail.com



ABSTRACT

We present here six variants of proofs of the Riemann hypothesis that all the imaginary zeros of the zeta function lie on the line $\sigma = \frac{1}{2}$

2010 Mathematics subject classification: 11MO6, 11M26 11-02
Key Words : Riemann hypothsis, Argument of the zeta function, Dirichiet series

1. Introduction

In 1859, G.F.B.Riemann published a most famous paper concerning the distribution of prime numbers, with the title: “On the quantity of prime numbers below a given quantity”, where, for the first time were used the methods of complex variable functions in order to determine π (x) : the quantity of prime numbers ≤ x.

His starting formula was the product decomposition that Euler had found for the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

i.e. the formula

$$\zeta(s) = \prod_p \frac{1}{1-1/p^s}$$

where p stands for the prime numbers. (Riemann used the letter s to denote the variable, s = σ + it ; and this way of notation was unanimously used after him)

In the first part of the memoir, he proves the functional equation of the zeta function, and after this he deduces the formula

$$\frac{\log \zeta(s)}{s} = \int_0^\infty \frac{f(x)}{x^{s+1}} dx$$

valid for $\sigma > 1$, and where

f(x) = $\pi(x) + 1/2\,\pi\left(\sqrt{x}\right) + 1/3\,\pi\left(\sqrt[3]{x}\right) + \ldots$

This formula had been obtained formerly in 1848 by Chebysev (whose work on the subject very likely was known to Riemann).

But he was unable to make the inversion of this formula, that Riemann succeeded to do, obtaining thus:

$$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\log \zeta(s)}{s} x^s \, ds \qquad (a > 1)$$

The remaining part of Riemann's paper is very obscure and confusing because of its excessive brevity. Fortunately some years ago E.C. Edwards published a wonderful book (ref. (1)) explaining and justifying step by step Riemann's reasoning.

This required about 200 pages, and put in evidence that he performed six hypotheses before arriving at his final formula:

1). There are infinitely many zeros of $\zeta(s)$ in the "critical strip" $0 \le \sigma \le 1$.

2) The quantity N(T) of these zeros in the rectangle $0 \le \sigma \le 1 \quad 0 < t < T$ is

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + O\left(\frac{1}{T}\right)$$

3) The series $\sum |\rho|^{-2}$ is convergent, but $\sum |\rho|^{-1}$ diverges

4) The entire function

$\xi(s) = ½\, \pi^{-s/2} s(s-1)\, \zeta(s)\, \Gamma(s/2)$

admits the product decomposition

$$\xi(s) = a\, e^{bs} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

5) All the imaginary zeros ρ have real part 1/2

6) Let

$$\pi^*(x) = \sum_{2 \le n \le x} \frac{\Lambda(n)}{\log n} \qquad \pi_0(x) = \frac{1}{2}\left\{\pi^*(x+0) + \pi^*(x-0)\right\}$$

Then holds that

$\pi_0(x) = \pi(x) + 1/2\,\pi\left(\sqrt{x}\right) + 1/3\,\pi\left(\sqrt[3]{x}\right) + \ldots$

$$= li(x) - \sum_\rho li\left(x^\rho\right) + \int_x^\infty \frac{du}{(u^2-1)u.\log u} - \log 2$$

where li(x) denotes the logarithmic integral function; $\pi(x)$ is the quantity of primes $\le x$, and ρ denotes the imaginary zeros of the zeta function.

The hypothesis 1), 3) and 4) were proved in 1892 and 1893 by Hadamard. The hypothesis 2), with error term 0(log T) was proved in 1894 by von Mangoldt, who also proved hypothesis 6) (but he used an alternative way).

There is besides a numerical and irrelevant mistake in the formula for $\pi_0(x)$, where Riemann writes $\xi(½)$ instead of -log 2.

Hence at present remains unproved only hypothesis 5).

Remark that Riemann's formula for N(T) indeed gives the quantity of Gram points for $t \leq T$, up to a difference of $\pi/8$.

2. How the hypothesis was not proved.

Variant A)

It is mentioned in § 10.1, p. 213-214 of Titchmarsh's textbook. (ref[2])

We have:

$$\xi\,(1/2 + it) = 2 \int_0^\infty \Phi(u)\ \cos ut\ dt$$

where

$$\Phi\ (u) = 2 \sum_{n=1}^{\infty} (2n^4 \pi^2 e^{\frac{9}{2}u} - 3n^2 \pi e^{\frac{5}{2}u})\ e^{-n^2 \pi e^{2u}}$$

This series converges very rapidly and one might suppose that an approximation to the truth could be obtained by replacing it by their first terms.

The author has performed the exact calculation, and he proved that we are thus led to the formula

$$\xi(s) = \frac{1}{2} + \frac{s(s-1)}{2}\left\{\pi^{-s/2}\left(\sum_{n=1}^{N}\frac{\Gamma_{n^2\pi}(s/2)}{n^s} + O\left(\frac{e^{-\frac{\pi}{4}t}}{t^{3/2}}\right)\right) + \pi^{\frac{1-s}{2}}\left(\sum_{n=1}^{N}\frac{\Gamma_{n^2\pi}((1-s)/2)}{n^s} + O\left(\frac{e^{-\frac{\pi}{4}t}}{t^{3/2}}\right)\right)\right\}$$

where $N = \left[\frac{\sqrt{t}}{2}\right]$ and $\Gamma_{n^2\pi}(x)$ denotes the incomplete gamma function. See ref. (3).

It is evident now the slowness of convergence of both series at right, because are necessary $0(\sqrt{t})$ terms in order to obtain a satisfactory accuracy. Hence, it has not any special advantage over the use of the Riemann – Siegel formula.

Variant B)

In other place of his book ref[2] (Chapter III § 3,1 p.38) Titchmarsh states that "The problem of the zero-free region" (of the zeta function) "appears to be a question of extending the sphere of influence of the Euler product beyond its actual region of convergence , ...In fact, the deepest theorems on the distribution of the zeros of ζ(s) are obtained in the way suggested.

But the problem of extending the sphere of influence of (the Euler product) ... to the left of σ = 1 in any effective way appears to be of extreme difficulty"

The "extremely difficulty problem" was solved in 1991 by the author (ref[4]), who proved the product formula

$$\zeta(s) = \prod_{p \le x} \frac{1}{1-\frac{1}{p^s}} \frac{e^{\sum Li_1(x^{\theta-s})}}{e^{\sum Li_1(x^{1-s})}} e^{\sum Li_1(x^{-2-s})}$$

for integral positive $x \geq 2$ and every s. Unfortunately enough, this does not give us any information concerning the zeros. In p.50 of the same reference, the author proved that there are infinitely many natural numbers x such that if

$$\theta = 1 - \frac{2\log\ \log x}{\log x} + 0\left(\frac{1}{\log x}\right)$$

there are not zeros for $\sigma > \theta$

3. How the hypothesis was proved

In the Nº 10, January 2001 issue of the “Italian Journal of Pure and Applied Mathematics” was published a paper of the author entitled “The functions N(T) and N_0 (T) of the Riemann zeta function”, (ref.(5)).

The contents of the first two pages of the paper can be summarized in a few lines as follows:

The function

$\xi(s) = 1/2\, s\,(s-1)\, \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$

is real for $s = 1/2 + it$ and real t, as proved by Riemann. Hence, the same thing is true for

(3.1) $F(s) = \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$

Equating arguments of both sides in (3..1.) we obtain:

(.3..2) $\arg F(1/2 + it) = -t/2 \log \pi + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it) \pm m\pi$

As $F(1/2 + it)$ is a real quantity, must hold that

(3.3) $\arg F(1/2 + it) = \pm \pi \pm 2k\pi$

Replacing in (3.2.) we obtain:

(3.4) $\pm \pi \pm 2k\pi = -t/2 \log + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it) \pm m\pi$

which is clearly equivalent to:

(3.5) $\boxed{\pm n\pi = -t/2 \log \pi + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it)}$

This equation must hold identically for every real t.

The preceding formula admits a double interpretation:

A) It gives the quantity of zeros $N_0(T)$ in the critical line $\sigma = ½$ between coordinates t=0 and $t = \pm T$

B) It enables us to evaluate $\arg \zeta(1/2 + it)$

The validity of (3.5) can be checked in a variety of ways.

1) By use of the Haselgrove-Miller tables (ref. (6)). This can be accomplished in two different ways. On the one hand the tables give separately the values of $R\zeta(½ + it)$ and $I\zeta(½ + it)$, so that one can evaluate.

$\arg \zeta(1/2 + it) = \operatorname{arc\,tg} \dfrac{I\zeta(1/2 + it)}{R\zeta(1/2 + it)} \pm m\pi$

As $\zeta(1/2) = -1{,}460355$, it seems a correct conventional thing to adopt above that

$\arg \zeta(1/2) = \pi$

This is the initial value used to draw the graph of $\arg \zeta(½ + it)$ given in fig.1.

On the other hand, the tables tabulate the values of $\theta = \arg \Gamma(¼ + it/2)$ and thus provide an alternative way to calculate $\arg \zeta(½ + it)$ thanks to (3..5).

The numerical details of this checks were given in ref. (7) and the double agreement of the tables with (3.5) is perfect.

2) Formula (3.5) can be deduced from the Cauchy-Riemann equations for analytic functions. See ref (7)
3) It can be deduced from the argument "principle" when applied to ξ (s). See § 8 below.
4) It can be deduced from the functional equation of the zeta function. See § 5 below.
5) It can be deduced from the approximate functional equation of the zeta function. See § 6 below.
6) It can be deduced from an observation made by Titchmarsh. See § 7 below.

4. Comparison of $N_o(t)$ with N(t)

Once we feel fairly sure by these six checks about the validity of (3.5) we compare it with the result of the contour integration performed by Stieltjes, von Mangoldt and Backlund in order to determine N(T), the quantity of zeros in the critical strip between ordinates t = 0 and t = T.

In ref[2], Chapter IX, § 9.3, line 13 p.179 we find the following.:

Theorem 1. - The positive ordinates of the zeros of the zeta function in the critical strip are defined by the condition

$n\pi = \Delta \arg s(s-1) + \Delta \arg \pi^{-s/2} + \Delta \arg \Gamma(s/2) + \Delta \arg \zeta(s)$

(4.1)

$$= \pi - t/2 \log \pi + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it)$$

Stated in a slightly different form, there are zeros in the critical strip every time that

$$(4.2) \quad \boxed{\pm m\pi = - t/2 \log \pi + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it)}$$

where we have replaced n-1 by m.

But (4.2) and (3.5) coincide for every t. Hence

$$\boxed{N(t) = N_0(t)}$$

which is the Riemann hypothesis.

As a further check, Prof. Gerd Faltings, after having read ref[5] said he had been unable to find any mistake in it. (except the typographical ones, of course)

5. Development of alternative 4)

Here we develop alternative 4), mentioned above.

In the functional equation of the zeta function, which we write as:

$\zeta(s) \Gamma(s/2) = \pi^{s-½} \Gamma((1-s)/2) \zeta(1-s)$

we choose s = ½ + it, and equate the arguments of both members. We obtain:

$$(4.1) \arg \zeta(½ + it) + \arg \Gamma(¼ + i t/2) = t \log \pi + \arg \Gamma(¼ - i t/2) + \\ + \arg \zeta(½ - it) \pm 2n\pi$$

But

$$\arg \zeta(½ + it) = -\arg \zeta(½ - it)$$

$$\arg \Gamma(¼ - i t/2) = -\arg \Gamma(¼ - i t/2)$$

Replacing these values in (4.1) we obtain:

$$(4.2) \quad 2 \arg \zeta(½ + it) + 2 \arg \Gamma(¼ + it/2) = t \log \pi \pm 2n\pi$$

which is again (3.5)

6. Development of alternative 5)

Here we develop alternative 5) mentioned above.

As known (see for instance ref (2)), the approximate functional equation of the zeta function can be written (when we choose s = (½ + it) as:

(6.1) $\zeta(½ + it) = = \sum_{1}^{m} \frac{1}{n^{1/2+it}} + e^{-i2\theta} \sum_{1}^{m} \frac{1}{n^{1/2-it}} + e^{-i\theta} R_m$

where:

$m = \left[\sqrt{\frac{t}{2\pi}} \right]$ $R_m = (-1)^m \left(\frac{t}{2\pi}\right)^{-1/4} \sum_{r=0}^{\infty} \left(\frac{t}{2\pi}\right)^{-r/2} \phi_r \left(2\sqrt{\frac{t}{2\pi}} - 2\left[\sqrt{\frac{t}{2\pi}} \right] - 1 \right)$

$\theta = - t/2 \log \pi + \arg \Gamma(¼ + it/2)$

As far as the form of the functions Φ_r, it can be consulted in ref. (2), but it is irrelevant for what follows.

We remark now that $e^{-i\theta} R_m$ can be written as :

$$e^{-i\theta} R_m = M e^{i\alpha_m} + e^{-i2\theta} M e^{-i\alpha_m}$$

In fact, this is the same that

$$R_m = Me^{i(\theta + \alpha m)} + M e^{-i(\theta + \alpha m)} = 2M \cos(\theta + \alpha_m)$$

and plainly:

$$M = \frac{R_m}{2} \qquad \operatorname{Cos}(\theta + \alpha_m) = \frac{R_m}{|R_m|}$$

These equations determine both M and α_m. Hence we can write that

(6.2) $\zeta(1/2 + it) = \sum_{1}^{m} \frac{1}{n^{1/2+it}} + Me^{i\alpha_m} + e^{-2i\theta} \left\{ \sum_{1}^{m} \frac{1}{n^{1/2-it}} + Me^{-i\alpha_m} \right\}$

and the zeros of $\zeta(½ + it)$ are given by the formula

(6.3) $0 = \sum_{1}^{m} \frac{1}{n^{1/2+it}} + Me^{i\alpha_m} + e^{-2i\theta} \left\{ \sum_{1}^{m} \frac{1}{n^{1/2+it}} + Me^{-i\alpha_m} \right\}$

If $t \neq t_v$, with t_v = a zero of the right bracket, we can divide both numbers by this bracket obtaining thus:

(6.4) $e^{-2i\theta} = \frac{\sum_{1}^{m} \frac{1}{n^{1/2+it}} + Me^{i\alpha_m}}{\sum_{1}^{m} \frac{1}{n^{1/2-it}} + Me^{-i\alpha_m}}$

We observe now that the quotient at right is formed by two conjugate complex numbers, so that its modulus is exactly one :

Let

(6.5) $\beta = \arg \left\{ \sum_{1}^{m} \frac{1}{n^{1/2+it}} + Me^{i\alpha_m} \right\}$

Then the above quotient has argument 2 β, and the former equation adopts this simple form:

$e^{-2i\theta} = -e^{2i\beta}$

the roots of which are

(6.6) $2(\theta + \beta) = \pm \pi \pm 2k\pi$

Now we evaluate β in terms of $\alpha_\zeta = \arg \zeta(½ + it)$.

(6.2) can be written as $\zeta(½ + it) = M_1 e^{i\beta} + e^{-i2\theta} M_1 e^{-i\beta}$

(6.7) $= M_1 \left\{ e^{i\beta} + e^{-i(2\theta+\beta)} \right\}$

where

$$M_1 = \left| \sum_1^m \frac{1}{n^{1/2+it}} + Me^{i\alpha_m} \right|$$

Let us put:

(6.8) $\zeta(½ + it) = M_\zeta e^{i\alpha_\zeta}$

Replacing in (6.7)) we get:

$$M_\zeta e^{i\alpha_\zeta} = M_1 \left\{ e^{i\beta} + e^{-i(2\theta+\beta)} \right\}$$

and

$$\frac{M_\zeta}{M_1} e^{i(\alpha_\zeta - \beta)} = 1 + e^{-i(2\theta+2\beta)}$$

Taking logarithms:

$$\log \frac{M_\zeta}{M_1} + i(\alpha_\zeta - \beta) = \log\left|1 + e^{i2(\theta+\beta)}\right| - i\left(\operatorname{arctg} \frac{\sin 2(\theta+\beta)}{1+\cos 2(\theta+\beta)} \pm n\pi\right)$$

Equating imaginary parts :

(6.9) $$\alpha_\zeta - \beta = -\operatorname{arc\,tg} \frac{\sin 2(\theta+\beta)}{1+\cos 2(\theta+\beta)} \pm n\pi$$

But according to (6.6)

$$\sin 2(\beta + \theta) = \sin(\pm\pi) = 0$$
$$\cos 2(\beta + \theta) = \cos(\pm\pi) = -1$$

so that the quotient

$$\frac{\sin 2(\theta+\beta)}{1+\cos 2(\theta+\beta)}$$

is of the type 0/0. Once the indetermination is saved by L'Hopital rule, we find $-\infty$ as its true value. Hence we deduce

(6.10) $$\operatorname{arc\,tg.} \frac{\sin 2(\vartheta+\beta)}{1+\cos 2(\vartheta+\beta)} = \pm \pi/2 \pm r\pi$$

Thus (6.9) transforms itself to

(6.11) $$\alpha_\zeta - \beta = \pm \pi/2 \pm m\pi$$

Replacing this value of β in (6.6), we obtain that there are zeros on the critical line every time that

(6.12) $$2(\theta + \alpha_\zeta) = \pm 2k\pi$$

or

$$\theta + \alpha_\zeta = \pm k\pi$$

which is again (3.5).

7.Titchmarsh's observation.

In p.181 § 9.4 of ref. (2), he states:

"The behaviour of the function S (t) = 1/π arg ζ (½ + it)) appears to be very complicated. It must have a discontinuity k where t passes through the ordinate of a zero of ζ (s) of order k ... Between the zeros, N(t) is constant, so that the variation of S(t) must just neutralize that of the other terms" (in the expression of N(t)).

A glance at fig.1 shows that the behaviour of π S(t) is very simple: it consists of the curve t/2 log π - arg Γ (¼ + it/2) broken at the points t = γ_i. But Titchmarsh did not dispose of Haselgrove's tables when writing his book.

The last statement is particularly interesting: between zeros, the variation of S(t) must neutralize that of the other terms in the formula of N(t).

That is to say, that according to (3.5), between consecutive zeros must hold that:

$$\Delta\, t/2 \log \pi \;-\; \Delta \arg \Gamma\ (1/4 + it/2) \;=\; \Delta \arg \zeta\,(1/2 + it)$$

or

$$(t_1 - t_2)\log \pi - \left\{ \arg\Gamma\left(1/4+it_1/2\right) - \arg\Gamma\left(1/4+it_2/2\right)\right\} =$$

$$= \arg \zeta\left(1/2+it_1\right) - \arg\zeta\left(1/2+it_2\right)$$

If we regard t_1 as a variable and t_2 = constant, then we have:

(7.1) $\quad t \log \pi \;-\; \arg \Gamma\ (1/4 + it/2) = \arg \zeta\,(1/2 + it) + \text{constant}$

Due to the fact that $\arg\ \zeta\ (1/2 + it)$ has jumps of $r\pi$ at each zero of multiplicity r, it is easily seen that the constant must be $\pm\, m\,\pi$.

But then (7.1) is nothing but (3.5)!

8. Application of the argument principle

As known, the "argument principle"states that the quantity N of zeros of an analytic function f(s) inside a contour C where it has no poles, is equal to the variation of arg f(s) along C divided by 2π.

When applied to the function

$$\xi^*(s) = \pi^{-s/2}\,\Gamma(s/2)\ \zeta(s)$$

with functional equation

$$\xi^*(s) = \xi^*(1-s)$$

it gives

$$N = \frac{1}{2\pi}\,\Delta_c\ \arg \xi^*(s)$$

Next, we choose as contour C a thin strip of width δ surrounding the critical line between ordinates t = ± T, and after we let δ→0,. Then we obtain for the quantity N_0 of zeros along the critical line:

$$N_0 = \frac{1}{2\pi}\left\{\arg\xi\left(1/2+it\right) \;-\arg\xi\left(1/2-it\right)\right\}$$

$$= \frac{1}{\pi}\arg\xi\,(1/2+it)$$

$$= \frac{1}{\pi}\left\{-t\log\pi + \arg\Gamma\left(1/4+it/2\right) + \arg\zeta\left(1/2+it\right)\right\}$$

which is again (3.5)

9. Answer to some objections

<u>Objection 1:</u> The fact that $N_0(T)$ be equal to N(T) does not prove the Riemann hypothesis. It could happen that for a given $T = \gamma_i$ could exist two zeros at this level: one on σ = ½ and other in σ = 1/3, for instance.

Answer – When in the expression for N(T), the variable T changes from

$T = \gamma_i - 0$ to $T = \gamma_i + 0$, N(T) changes exactly in the same quantity than $N_0(T)$does. Hence due to this very special form of N(T) and $N_0(T)$, such a possibility is entirely excluded.

<u>Objection 2:</u> In the proof of § .6 it is assumed that ζ (1/2 + it) = 0 which is the thing that must be proved.

Answer - When one wishes to find the zeros of a polynomial P(x) at some stage of the reasoning one must put P(x) =0, or one is solving other problem.

It is obvious then that if one wishes to find the zeros of the zeta function on the critical line, one must put in some place that ζ (1/2 + it) = 0, as was put in (§.6)

Objection 3: In equality (3.5) it is not proved that n be a bounded quantity.
Answer – It makes no sense to determine if n is bounded or not; in every case, equality (3.5) holds.
Objection 4: The same objection that 3.- , with the quantity r of (6.10)
Answer – In any case, (6.11) is valid.
Objection 5: Concerning the β of (6.5), it seems that lots of ordinates of zeros could give rise to the same β.
Answer - Who denies such a thing? And what importance has this fact in relation to the calculation that is performed after (6.5)?
Objection 6: One γ_i could make in (6.4)

$$\sum_{n\leq m} \frac{1}{n^{1/2-it}} + M e^{\alpha m} = 0 \qquad (\rho_i = \beta_0 + \gamma_i)$$

in which case it is not counting at all, it seems.
Answer – When passing from (6.3) to (6.4) it was clearly stated that
$t \neq t_i$ = zero of the above expression. We can not assume a case that was discarded beforehand.

10. Proof using the *Z*(*t*) function

In connection with paragraph.6, it is well known (ref.[2]) that Hardy's zeta function *Z*(*t*), defined as:

(10.1) $Z(t) = e^{i\theta}\zeta(1/2+it)$

where

$$\theta = \frac{t}{2}\log \pi + \arg\Gamma(1/4 + it/2)$$

has the remarkable feature of being real for real *t*. This enables us to reduce the problem of finding the imaginary zeros of the ζ-function to the more easy of determination of the real roots of the *Z*-function.

It is obvious from (10.1) that *Z*(*t*) has the same zeros than $\varsigma(1/2+it)$, and that

(10.2) $$|Z(t)| = |\zeta(1/2+it)|.$$

Besides, we can write (10.1) as:

(10.3) $Z(t) = e^{i\theta}|\zeta(1/2+it)|e^{i\alpha_\zeta}$

where

$$\alpha_\zeta = \arg\zeta(1/2+it).$$

Now, (10.3) can be expressed as:

(10.4) $Z(t) = e^{i(\theta+\alpha_\zeta)}|\zeta(1/2+it)|.$

But as *Z*(*t*) is a real quantity, necessarily must hold:

$$Z(t) = \cos(\theta+\alpha_\zeta)|\zeta(1/2+it)|.$$

Now, if *Z*(*t*) vanishes, one or two of the factors at right must also vanish. Namely, must hold at least one of the following alternatives:

(A) $\cos(\theta+\alpha_\zeta) = 0$

or

(B) $|\zeta(1/2+it)| = 0.$

Any zero of (A) must be also a zero of (B), because, in contrary, we are led to a contradiction.

In fact, assume that a zero in (A) is not in (B), then, according to (10.4), it is a zero of *Z*(*t*), and so, of ζ(1/2+*it*), against was assumed.

Does it mean that, due to (10.4) the zeros of $Z(t)$ are double ones? Not at all.

The zeros of (A) are given at the points where $\vartheta+\alpha_\zeta$ has discontinuities of the second kind (jumps), while the zeros of (B) occur where there are discontinuities of the first kind (abrupt changes in the derivative of the continuous curve $|\zeta(1/2+it)|$.

We conclude that the zeros of the ζ-function are given at least by the equation:

$$\theta+\alpha_\zeta=\pm\frac{\pi}{2}\pm n\pi \qquad (n = 0, 1, 2, 3, \ldots)$$

Replacing values:

(10.5) $$+\frac{t}{2}\log\pi+\arg\Gamma(1/4+it/2)+\arg\zeta(1/2+it)=\pm\frac{\pi}{2}\pm n\pi$$

or

(10.6) $$t\log\pi+2\arg\Gamma(1/4+it/2)+2\arg\zeta(1/2+it)=\pm\pi\pm 2n\pi$$

Now, we remark that, as indicated by fig. 1, the function $\zeta(1/2+it)$ has a jump of value π when $t = \gamma_\zeta$ = imaginary part of a zero.

In other words

$$\arg\zeta(1/2+i(\gamma_\zeta+0))+\arg\zeta(1/2+i(\gamma_\zeta-0))=\pi$$

if γ_ζ is a simple zero: so that the $\pi/2$ that appears in (.5) is half of the jump.

We arrive finally to the following theorem:

The $Z(t)$ function has zeros at least every time that (10.5) is fulfilled.

If $t = \gamma_\zeta$, then the $\pm\,\pi/2$ quantity there represents half of the jump there. If $t = \gamma_\zeta \pm 0$, the term can be suppressed. Then (10.5) coincides with our former (3.5).

11. The graph of arg $\zeta(1/2+it)$

This graph (Fig.1) was published for the first time in a paper I wrote in 1984 entitled "The argument of the zeta function into the critical line" in "Bulletin of Number theory" Vol. VIII, aug. 1984 Nro. 2 p.6-29. For the sake of completeness I reproduce here how I constructed it, step by step, with the help of Haselgrove´s tables (ref [6]).

Besides, at the final part of the paper (also reproduced here), I show how the formula (3.5) can be obtained also starting with the Cauchy Riemann equations for analytic functions (variant 2) mentioned in §1.3

In table I of his tables. Haselgrove gives us the values of $R\zeta(1/2+it)$ *and* $I\zeta(1/2+it)$ in the interval $0\le t\le 100$ By using the formula

$$\arg\zeta(1/2+it)=\text{are tg }\frac{I\zeta(1/2+it)}{R\zeta(1/2+it)}\pm n\pi$$

we can draw the graph of $\arg\zeta(1/2+it)$ in that interval, without any trouble, (at least in principle).

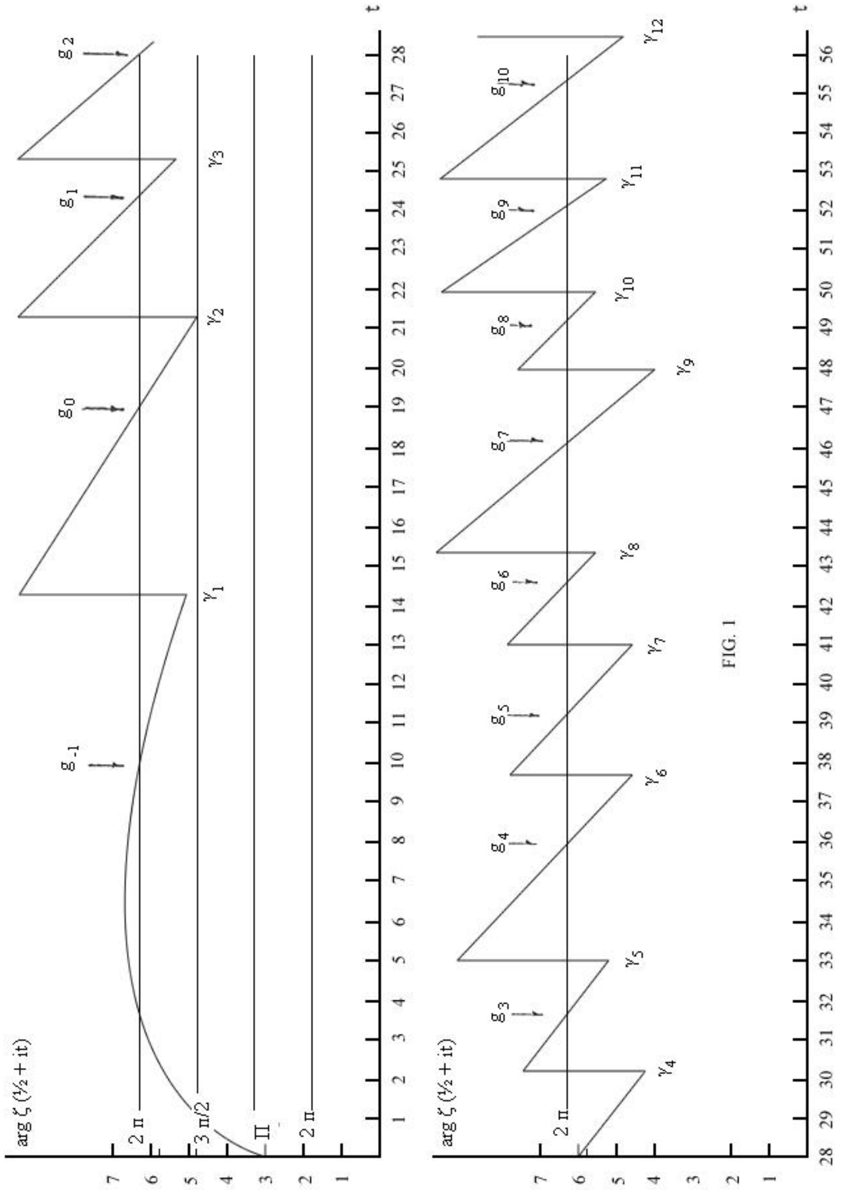


FIG. 1

12. Construction of the graph of arg. $\zeta\left(1/2\right)+it$ using Haselgrove´s tables.

However some cares must be kept in this job.

First of all, we must agree about he value of $\arg\zeta\left(1/2+it\right)$. As $\zeta\left(1/2\right)=-1,460355$ (this value is already taken form Table I), we agree to choose arg $\zeta\left(1/2\right)=\pi$ Hence we admit the following.

CONVENTION: arg $\zeta\left(1/2\right)=\pi$

If we adopt any other consistent convention, for instance

arg $\zeta\left(1/2\right)=\pm\left(2m+1\right)\pi$ this merely implies that the graph is displaced upwards or downward, without any change on its form.

For t = 0,5, we have $R\zeta\left(1/2+it\right)=-0,459303$ and $I\zeta\left(1/2+it\right)=-0,961254$.

Hence the angle of the argument lies is the third quadrant and we have.

$$\arg\zeta\left(1/2+0,5i\right)=arc\ \ tg\frac{-0,961254}{-0,459303}=$$

$= arc\ tg\ 2,09285\ = \pi + 1,12505 = 4,26664$

When $t \square 0,85$ the real part vanishes, and arc tg passes from $+\infty - \infty$, hence we asign to the angle the value $3\pi / 2$.

In the interval $0,85 \le t \le 3,45$ the argument lies in the fourth quadrant. For $t \le 3,45$ the argument lies in the first quadrant reaching a máximum about t = 6,3. Then it decreases and when t = 9,667, I = 0 and we are again in $\arg = 2\pi$. In the interval $9,667 < t < 14,134725 = \gamma_1$ ($\gamma_1 =$ ordinate of the first zero), the argument lies in the fourth quadrant.

All this can be conveniently tabulated, and we have:

Table A – Values of arg $\zeta(1/2+it)$ in the range $0 \le t < \gamma_1$

t	I/R	$\operatorname{arc} tg\ I/R\ = \arg\ \zeta(1/2+it)$
0	0	$+\pi$
0,5	2,09285	4,2666
0,85	+00,	3 π /2
1,00	-5,0168	4,9092
1,5	-1,3790	5,3420
2,0	-0,7074	5,6675
2,5	0,3713	5,9277
3,0	0,1481	6,1362
3,45	0	2 π = 6,2832
5,04	+0,3292	6,6002
7,0	+0,3879	6,652
9,0	+0,1325	6,4149
9,667	0	2 π
10,0	-0,07465	6,2076
12,0	-0,7334	5,6512
14,10	-5,759	4,8833

When $t = \gamma_{1'}$ due to the presence of the zero, we must expect a jump of height π, because γ_1 is a simple zero. (Were γ_1 a doublé zero, the jump would be of 2π and so on.

These are consequences of the argument principle). Let us see if this can be checked numerically. When $t = 14,10$ as shown in the above table, $\arg = 4,883$ and the angle is situated in the fourth quadrant, because I is negative and R is positive.

When $t = 14,20 > \gamma_1$ the angle is situated in the second quadrant, because I is positive and R is negative, and we have:

$$\arg \zeta(1/2 + i14,2) = arc\ tg \frac{+0,051597}{-0,06816} = arc\ tg(-7,570) = 7,985$$

We have then:

$$\arg \zeta(1/2 + i14,2) - \arg\ \zeta(1/2 + i14,1) = 7,985 - 4,883 = 3,102 \cong \pi$$

Once we have surpassed γ_1, we can go on in the same way than before, taking account that $\gamma_2\ \ \gamma_3\ \ \gamma_4$ ...are in every case simple zeros.

We have then the following table:

Table B – Values of $\arg\zeta\left(1/2+it\right)$ in the range $\gamma_1 <$ t $< \gamma_{12} = 56,446$

t	I/R	$arctg\ I/R\ \zeta\left(1/2+it\right)$
15	4,7906	7,6482
16	1,2963	7,1969
17	0,4600	6,7143

t	I / R	$\arg\ \zeta\left(1/2+it\right)$
$g_0 = 17,846$	-0	2π
18,5	-0,3621	5,9358
19,5	-1,2617	5,3826
20,0	-2,4756	5,0963
20,65	$-\infty\ +\infty$	$3\pi / 2$
21,00	4,7553	4,5051
$\gamma_2 = 21,022$	Jump of π	
21,5	1,7797	7,3421
23,0	0,1112	6,3939
$g_1 = 23,2$	0	2π
24,9	2,2963	5,1332
$\gamma_3 = 25,010$	Jump of π	
25,2	-4,8782	8,0550
25,5	$-\infty\ +\infty$	$5\pi / 2$
26,0	2,6684	7,4942
$g_2 = 27,670$	0	2π
28,0	-0,2505	6,038
29,75	$-\infty\ +\infty$	$3\pi / 2$
$\gamma_4 = 30,425$	Jump of π	
30,5	1,4721	7,2577
$g_3 = 31,75$	0	2π
32,9	-1,4520	5,3155
$\gamma_5 = 32,935$	Jump of π	
33,0	-1,7456	8,4168
33,65	∞	$5\pi / 2$
$g_4 = 35,48$	0	2π
37,25	$-\infty\ +\infty$	$3\pi / 2$
37,4	7,7924	4,5848
$\gamma_6 = 36,586$	Jump of π	
38,0	1,2749	7,1888
$g_5 = 38,999$	0	2π
40,7	331,406	□ $3\pi / 2$
$\gamma_7 = 40,919$	Jump of π	

41,0	3,4672	7,5732
$g_6 = 42,364$	0	2π
43,2	-1,0343	5,481
$\gamma_8 = 43,327$	Jump of π	
44,0	158,75	7,847
$g_7 = 45,593$	0	2π
47,5	2,8401	4,3739
$\gamma_9 = 48,005$	Jump of π	
48,2	0,5748	6,8049
$g_8 = 48,711$	0	2π
$\gamma_{10} = 49,774$	Jump of π	
49,5	-1,0533	5,4719
50,0	-4,0483	8,097
$g_9 = 51,734$	0	2π
52,7	1,6383	5,2602
$\gamma_{11} = 52,970$	Jump of π	
53,1	-8,1712	7,9758
$g_{10} = 54,675$	0	2π
$\gamma_{12} = 56,446$	Jump of π	
$g_{11} = 57,545$	0	2π
$\gamma_{13} = 59,347$	Jump of π	
$g_{12} = 60,352$	0	2π
$\gamma_{14} = 60,832$	Jump of π	
$g_{13} = 63,102$	0	2π

Between these values there are certain remarkable points, which can serve us as reference points. $I\zeta\left(1/2+it\right)$ vanishes, of course, at the γ_i abscisas; but it also wanishes at interme-diate points, the so called "Gram points" g_n defined by:

$$\arg\left\{\pi^{-\frac{it}{2}}\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right\}=n\pi$$

At these points:

$$\arg\zeta\left(\frac{1}{2}+ig_n\right)=arc\ tg\frac{I}{R}=0$$

So that the argument must be 2π or a mutiple of it: In the range of the graph, al-ways holds: $\arg\zeta\left(\frac{1}{2}+ig_n\right)=2\pi$ but this needs not to be true for larger values of g_n

3- Comments about the graph (Fig.1)

At first glance, the most impressive feature of the graph is its saw-tooth appearance.

A second remarkable feature is that the derivatives at both sides of the jumps have the same value. In other words, the argument curve consists of a single "smooth" curve broken just at the jumps. The sequence of the arcs centered about the g_n allows for the form of the curve.

It follows that it would be very convenient if we could find the equation of the continuous smooth curve, or at least, of its slope.
This problem is solved in the following paragraph.
4- An equation for $\arg\zeta\left(1/2+it\right)$

As is well known, the $\xi\left(s\right)$ function

$$\xi\left(s\right)=\frac{1}{2}s\left(s-1\right)\pi^{-\frac{s}{2}}\Gamma\left(s/2\right)\zeta\left(s\right)$$

is real when $s=1/2+it$. (t: real number)
Now, if a product of complex numbers equals a real number then we must have:
arg of product $=\pm 2\ k\ \pi$ k = 0,1,2,3,...
if the real number is positive; or
arg of product = $\pi\ \pm\ 2\ k\ \pi$
if the real number is negative.

So we can write:
arg of product $=\ \pm 2\ m\ \pi$ m = 0,1,2,3,...
When this is applied to $\xi\left(\frac{1}{2}+it\right)$, we obtain that:

$$\arg\left\{\pi^{-it/2}\Gamma\left(1/4\ +\ it/2\right)\zeta\left(1/2\ +\ it\right)\right\}=\pm\ m\ \pi$$

From which follows:

$[A]$ $$\arg\zeta\left(1/2+it\right)=\arg\pi^{-\ it/2}-\arg\Gamma\left(1/4+it/2\right)\pm m\pi$$

$[A]$ $$\boxed{\arg\zeta\left(1/2+it\right)=-\vartheta\left(t\right)\mp m\ \pi}$$

(by formula 3,31 of Haselgrove´s introduction to his Tables)
This equation $[A]$ is very clearly the piecewise equation for $\zeta\left(1/2+it\right)$.
The $\vartheta(t)$ function can be numerically evaluated from its asymptotic expansion

$$\vartheta\left(t\right)\square\ \frac{t}{2}\log\frac{t}{2\pi}-\frac{t}{2}-\frac{B}{2}+\frac{1}{2}\sum_{r*1}^{\infty}\frac{B_r\left(1-2^{1-2r}\right)}{2r\left(r-1\right)}\frac{1}{t^{2r-1}}$$

(this is Haselgrove´s formula 3,32, and $\vartheta\left(o\right)=0$ as he points out)
But we can dispense of this calculus as he gives (Table I) in the same range $t<100$ the quantity $\theta\left(t\right)=\frac{1}{\pi}\vartheta\left(t\right)$.
This circumstance enables us to perform a quantity of checks in connection with the former calculations, in order to verify the truth of $[A]$ and our own tables 1 and 2.
We have, for instance, that $\vartheta\left(t\right)$ reaches its máximum in the range
$0<t<\gamma_1$ when t = 6,3.
When t = 14,1 $\theta\left(t\right)$ has the value -0,5547. Hence

$$\vartheta\left(t\right)=\pi.\left(-0554726\right)=-1,7427$$

The value of $\arg\zeta\left(1/2+i\ 14,1\right)$ was given in our Table A. above, as 4,883. Hence:

$$\arg\zeta\left(1/2+i\ 14,1\right)+\vartheta\left(14,1\right)=4,8833-1,7427=3,1406\cong\pi$$

in accordance with equation $\left[A\right]$

We choose now (at random) t = 20

From our table B above $\frac{I}{R}=-2,4756$ so that $\arg\zeta\left(1/2+i\ 20\right)=5,0963$

On the other hand, from Haselgrove´s table we obtain:

$\theta\left(t\right)=0,3778$ $\qquad\qquad$ $\vartheta\left(t\right)=\pi.0,3778=1,1869$

$\arg\zeta\left(1/2+i\ 20\right)+\vartheta\left(20\right)=5,0963+1,1869=6,2832=2\pi$

The interested reader can perform as many numerical checks as he wish in order to convince himself of the truth of equation $\left[A\right]$ and of our graph of $\arg\zeta\left(1/2+it\right)$

5. The meaning of equation $\left[A\right]$

From the analytical standpoint, we can summaryze the preceding numerical-theoretical results as follows:

- In the range $0<t<\gamma_1$ we have

$\arg\zeta\left(1/2+it\right)=-\vartheta\left(t\right)+\pi$

- In the range $\gamma_1<t<\gamma_2$ we have

$\arg\zeta\left(1/2+it\right)=-\vartheta\left(t\right)+2\pi$

- In the range $\gamma_2<t<\gamma_3$ we have

$\arg\zeta\left(1/2+it\right)=-\vartheta\left(t\right)+3\pi$

- In the range $\gamma_n<t<\gamma_{n+1}$ we have

$\arg\zeta\left(1/2+it\right)=-\vartheta\left(t\right)+\left(n+1\right)\pi$

These statements can he interpreted or presented under other forms, for instance:

- The number of different zeros $N_0\left(T\right)$ in the critical line in the interval $0<t<T$ is:

$$n=N_0\left(T\right)=\left[\frac{1}{\pi}\left\{\arg\zeta\left(1/2+it\right)+\vartheta\left(T\right)\right\}\right]$$

where $\left[U\right]$ is the greatest integer function.

- There are zeros on the critical line every time that

$\arg\zeta\left(1/2+it\right)+\vartheta\left(t\right)=n\pi$

(independently of the multiplicity of each zero, which is given by the height ot the jump of $\arg\zeta\left(1/2+it\right)$).

12.- **All the imaginary zeros of the zeta function are simple ones**

We have seen that

$$\arg\zeta\left(1/2+i\gamma_n\right)\ =\ -\vartheta\left(t\right)+\left(n+1\right)\pi \qquad n=1,2,3,...$$

in the range $\gamma_n<t<\gamma_{n+1}$

Then we have:

$$\arg\zeta\left(1/2+i\left(\gamma_n+0\right)\right)-\arg\zeta\left(1/2+i\left(\gamma_n-0\right)\right)=-\theta\left(\gamma_n+0\right)+\theta\left(\gamma_n-0\right)+\pi$$

But according to Haselgrove´s formula 3.32 reproduced above, $\theta(t)$ is a continuous function, and then lim $-\theta\left(\gamma_n+0\right)+\theta\left(\gamma_n-0\right)=0$ so that we get finally that the jump of the argument at $t=\gamma_n$ is always equal to π, whatever be n.

REFERENCES

[1]. H.M Edwards. Riemann’s zeta function − Academic Press − New York − London. 1974.

[2]. E. C. Titchmarsh − The theory of the Riemann zeta function. Oxford − Clarendon Press. 1951.

[3]. A. Peretti. Some new formulas in the theory of the zeta function (II). BNT Vol. III aug. 1978 N° 2 p. 1-18.

[4]. A. Peretti. Some new formulas in the theory of the Riemann zeta function (IV) BNT. Vol XV (1991) p. 46-106.

[5]. A. Peretti. The functions N(T) and N_0(T) of the Riemann zeta function − Italian Journal of Pure and Applied Math. N° 10 (2001) p. 163-168.

[6]. C. B. Haselgrove, J. C. P. Miller − Tables of the Riemann zeta function. Royal Society Mathematical Tables. Vol. 6 Cambridge University Press. New York, 1960, 80 pp. MR M 15-23 (22#8679)

[7]. A. Peretti − The argument of the zeta function on the critical line. B.N.T Vol. VIII, aug. 1984. N°2, p. 6-29

[8]. E. Landau: Handbuch der Verteilung der Primzahlen. Teubner, Leipzig (1909) − Chapter XIX, § 88 p. 364