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RESEARCH ARTICLE

POSITIVE DEFINITE SOLUTION OF NONLINEAR MATRIX EQUATIONS

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ABSTRACT

In this paper, we study the nonlinear matrix equations(1).First, we obtain the condition for the existence of positive definite solution for the equations. Secondly, we propose a fixed point iterative method for solving the equations. Finally, numerical example is given to demonstrate the efficiency of the iterative method.

Keywords：Positive solution， fixed point iteration method， nonlinear matrix equations.

1. INTRODUCTION

In this paper, we consider the nonlinear matrix equation:

$$\begin{cases} X + A^* Y^{-1} A = E \\ Y + B^* X^{-1} B = F \end{cases} \tag{1}$$

where A, B are n order Hermitian positive definite matrix, E, F, X, Y are n order positive definite matrix.

Another form of promotion is as follows:

$$X^s + A^* X^{-t_1} A + B^* X^{-t_2} B = I \tag{2}$$

Nonlinear matrix equations with the form (2) have many applications in engineering, dynamic programming, ladder networks statistics and so on. Several authors have studied the necessary and sufficient conditions of the existence of Hermitian positive definite (HPD) solutions of similar kinds of nonlinear matrix equations. In [3], the case $s = t_1 = t_2 = 1$ is considered and different itreative methods for computing the HPD solutions are proposed. In [4], the case $s = t_1 = 1,\ 0 < t_2 \le 1$ has been studied for computing the HPD solutions are proposed . In [5], author considered the matrix equation $X + A^* X^{-t_1} A + B^* X^{-t_1} B = I\ \ (0 < t_1, t_2 \le 1)$ and proposed three different kinds of iterative methods to compute the HPD solutions . In[6], the authors considered the matrix equation

$X^s + \sum_{i=1}^{m} A_i^* X^{t_i} A_i = I$ with $s > 0,\ 0 < t_i \le 1$ and studied the existence and the uniqueness of the HPD solution.

In this paper, we discuss $X^s + A^* X^{-t_1} A + B^* X^{-t_2} B = I$ with $s, t_1, t_2 > 0$. We propose necessary and sufficient conditions for the existence of HPD solutions. Based on the Banach fixed point theorem, the existence and the uniqueness of the Hermitian positive definite solution are studied. This paper is organized as follows: In Section 2, we give some lemmas and definition that will be needed to develop this work. Then in Section 3, we get some theorems of nonlinear matrix equations (1). Finally, numerical example is presented to illustrate the performance and the efficiency of the algorithm.

2.PRELIMINARIES

Lemma 2.1 [1] If $A \ge B > 0$ (or $A > B > 0$), then $A^\alpha \ge B^\alpha > 0$ (or $A^\alpha > B^\alpha > 0$) for all $\alpha \in (0,1]$, and $B^\alpha \ge A^\alpha > 0$ (or $B^\alpha > A^\alpha > 0$) for all $\alpha \in [-1,0)$.

Lemma 2.2[2] If P and Q are Hermitian matrices of the same order with $PQ = QP$, then $P^\alpha > Q^\alpha > 0$ for all $\alpha \in (0,+\infty)$.

D**efinition2.3**[8]If $A \in C^{n\times n}$, $A^* = A$, then A is Hermitian matrix；If $A^* A = I$, then A is unitary matrix.

D**efinition2.4**[8] **If** $A \in C^{n\times n}$ is Hermitian matrix, for any nonzero vector x we have $x^* A x > 0$ (or $x^* A x \ge 0$), then A is positive definite matrix(or Semi - definite matrix).

In this paper, for $A, B \in C^{n\times n}$, $A > 0 (\ge 0)$ shows A is positive definite matrix(or Semi-definite matrix). $A - B > 0 (A - B \ge 0)$ shows $A - B$ is positive definite matrix(or Semi-definite matrix).

3.SOME RESULTS

Theorem3.1 If X, Y are the positive definite solution of nonlinear matrix equations(1), then:

$$F \ge B^* E^{-1} B + A E^{-1} A^*, \quad E \ge B F^{-1} B^* + A^* F^{-1} A.$$

Proof: For $X + A^* Y^{-1} A = E$, where $X \le E$, then from lemma 2.1 we can get $X^{-1} \ge E^{-1}$, $A^* X^{-1} A \ge A^* E^{-1} A$.From the nonlinear matrix equations (1) we can get $Y = F - B^* X^{-1} B$ $\le F - B^* E^{-1} B$, we have $F \ge B^* E^{-1} B$ because Y is positive definite matrix.

Similarly we have $Y \le F$,and according to Lemma 2.1 we can get $Y^{-1} \ge F^{-1}$ and $A^* Y^{-1} A \ge A^* F^{-1} A$. According to the nonlinear matrix equations (1) we can get $X = E - A^* Y^{-1} A \le E - A^* F^{-1} A$,Since X is a positive definite matrix, we can get $E \ge A^* F^{-1} A$.

For $X + A^* Y^{-1} A = E$, both sides left by A^{-*} and right by A^{-1} at the same time, then we can get $A^{-*} X A^{-1} + Y^{-1} = A^{-*} E A^{-1}$, and then we get $Y^{-1} \le A^{-*} E A^{-1}$, so we have $A E^{-1} A^* \le Y \le F - B^* E^{-1} B$;

For $Y + B^* X^{-1} B = F$, both sides left by B^{-*} and right by B^{-1} at the same time, then we get $B^{-*} Y B^{-1} + X^{-1} = B^{-*} F B^{-1}$, and then we get $X^{-1} \le B^{-*} F B^{-1}$, so we have $B F^{-1} B^* \le X \le E - A^* F^{-1} A$;

In summary, we can get： $F \ge B^* E^{-1} B + A E^{-1} A^*$, $E \ge A^* F^{-1} A + B F^{-1} B^*$.

In this paper, we consider the iterative algorithm of nonlinear matrix equations (1) and give the convergence theorem.

Algorithm 3.1

(1) we give the initial matrix $Y_0 = \frac{1}{2}F$;

(2) For $k = 0,1,2\cdots$, we do iteration on $X_{k+1} = E - A^* Y_k^{-1} A$, $Y_{k+1} = F - B^* X_{k+1}^{-1} B$.

(3) Then we check the convergence.

Theorem 3.2 If $F \ge 2\left(B^* E^{-1} B + AE^{-1}A^*\right), E \ge 2\left(A^* F^{-1} A + BF^{-1}B^*\right)$,then the sequence $\{X_k\}$, $\{Y_k\}$ by Algorithm 3.1 converges to the minimal solution X_s of the nonlinear matrix equations (1).

Proof: Then we prove it by mathematical induction method.

When $k = 0$, we have $Y_0 = \frac{1}{2}F$, $E \ge 2\left(A^* F^{-1} A + BF^{-1}B^*\right)$, then we can get

$$X_1 = E - A^* Y_0^{-1} A = E - 2A^* F^{-1} A \ge 2BF^{-1}B^* > BF^{-1}B^*,$$

We have $X_1 = E - A^* Y_0^{-1} A = E - 2A^* F^{-1} A < E - A^* F^{-1} A < E$, then

$$BF^{-1}B^* < X_1 < E - A^* F^{-1} A.$$

For $BF^{-1}B^* < X_1 < E$, we can get $E^{-1} < X_1^{-1} < B^{-*} F B^{-1}$ by Lemma2.1,

So

$$\frac{1}{2}F < Y_1 = F - B^* X_1^{-1} B < F - B^* E^{-1} B < F.$$

For $F \ge 2\left(B^* E^{-1} B + AE^{-1}A^*\right)$, then we have $Y_0 = \frac{1}{2}F > AE^{-1}A^* + B^* E^{-1} B > AE^{-1}A^*$.

Thus we have:

$$AE^{-1}A^* < Y_0 = \frac{1}{2}F < Y_1 = F - B^* X_1^{-1} B < F - B^* E^{-1} B < F.$$

When $k = 1$, because $\frac{1}{2}F < Y_1 < F$, so

$$BF^{-1}B^* < X_1 = E - 2A^* F^{-1} A < X_2 = E - A^* Y_1^{-1} A < E - A^* F^{-1} A < E.$$

For $BF^{-1}B^* < X_1 < X_2 < E$, we can get:

$$AE^{-1}A^* < B^* E^{-1} B + AE^{-1}A^* < \frac{1}{2}F < Y_1 = F - B^* X_1^{-1} B < Y_2 = F - B^* X_2^{-1} B < F - B^* E^{-1} B < F$$

$$AE^{-1}A^* < Y_1 < Y_2 < F - B^* E^{-1} B.$$

If $k = n-1$, we can get the following two inequalities:

$$BF^{-1}B^* < X_1 < X_2 < \cdots < X_{n-1} < X_n < E - A^* F^{-1} A < E \;;$$

$$AE^{-1}A^* < Y_0 < Y_1 < Y_2 < \cdots < Y_{n-1} < Y_n < F - B^* E^{-1} B < F.$$

Then for $k = n$, we can have

$$BF^{-1}B^* < X_n < E - A^* Y_{n-1}^{-1} A < E - A^* Y_n^{-1} A = X_{n+1} < E - A^* F^{-1} A < E \;;$$

$$AE^{-1}A^* < Y_n = F - B^* X_n^{-1} B < Y_{n+1} = F - B^* X_{n+1}^{-1} B < F - B^* E^{-1} B < F \;;$$

In summary, from the mathematical induction method we can get $\{X_k\}$, $\{Y_k\}$ is monotonically increasing and has upper bound, so $X_k \to X_s$, $Y_k \to Y_s$.

4.NUMERICAL EXAMPLE

In this section, we give numerical example to illustrate the efficiency of the Algorithm 3.1. All computations are performed on Intel(R) Core(TM) i53210M CPU @ 2.50GHz comput-er.All the tests are performed by MATLAB version 7.0.

Example For nonlinear matrix equations

$$\begin{cases} X + A^{*}Y^{-1}A = E \\ Y + B^{*}X^{-1}B = F \end{cases},$$

where $A = \begin{bmatrix} 0.025 & -0.045 & 0.039 & 0.026 \\ -0.019 & 0.057 & -0.038 & 0.039 \\ 0.044 & -0.073 & 0.093 & -0.034 \\ 0.021 & 0.016 & -0.013 & 0.023 \end{bmatrix}$, $B = \begin{bmatrix} 0.215 & -0.045 & 0.039 & 0.024 \\ -0.015 & 0.127 & -0.038 & 0.007 \\ 0.024 & -0.027 & 0.293 & -0.034 \\ 0.012 & 0.016 & -0.012 & 0.021 \end{bmatrix}$

$$F = \begin{bmatrix} 0.2814 & 0.3094 & 0.0136 & 0.2646 \\ 0.4150 & 0.0842 & 0.6927 & 0.6872 \\ 0.4840 & 0.4657 & 0.7015 & 0.8033 \\ 0.7324 & 0.4582 & 0.2903 & 0.9779 \end{bmatrix}, F = \begin{bmatrix} 1.3368 & 1.0514 & 0.1340 & 0.2622 \\ 0.5679 & 0.0185 & 0.9129 & 0.3605 \\ 0.8753 & 0.5136 & 0.2795 & 0.4311 \\ 0.8136 & 0.5252 & 0.4175 & 0.3085 \end{bmatrix}$$

Fig 4.1 is depicts $n = 50$ of change curve between iteration steps k and residual y by Algorithm 3.1.

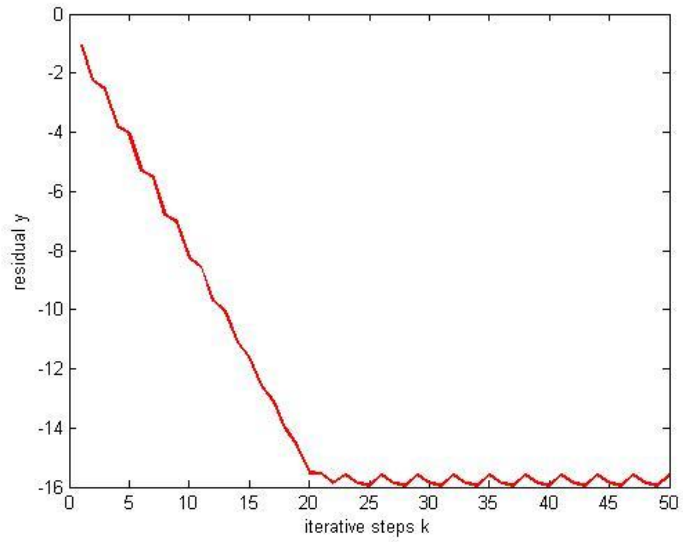

Figure 4.1 the relationship of the iterative steps k and residual y

Using algorithm 4.1 to solve the nonlinear matrix equations (4.1), we obtain the minimal solution.

$$X_s = \begin{bmatrix} 0.2859 & 0.3062 & 0.0147 & 0.2618 \\ 0.3941 & 0.1122 & 0.6641 & 0.7085 \\ 0.5030 & 0.4430 & 0.7241 & 0.7839 \\ 0.7220 & 0.4668 & 0.2816 & 0.9862 \end{bmatrix},$$

$$Y_s = \begin{bmatrix} 0.4733 & 0.8289 & 0.7181 & 0.0877 \\ 0.9028 & 0.1663 & 0.5692 & 0.4435 \\ 0.4511 & 0.3939 & 0.4608 & 0.3663 \\ 0.8045 & 0.5208 & 0.4453 & 0.3025 \end{bmatrix}.$$

From the example, we can get that the Algorithm 3.1 solves the nonlinear matrix equations (1) and it is feasible and effective.

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