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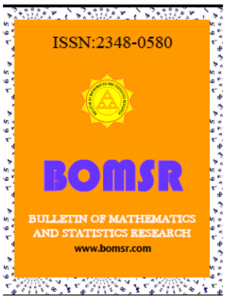
RESEARCH ARTICLE

Some partial convexity results of heat equation in space forms

Weiling Du[1], Yunhua Ye[2]
[1,2]School of Mathematics, Jiaying University, Meizhou, Guangdong, 514015, China
E-mail:429515582@qq.com



ABSTRACT

In his paper, we are concerned with partial convexity of smooth solutions to heat equation. We prove that partial convexity of these solutions to the heat equation are preserved in space forms with nonnegative sectional curvature. Consequently we give a proof that the convex cones of these solutions Γ_k (see the definition in Section 2) are invariant cones.
Keywords: Heat equation; Partial convexity; Space form.

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1. Introduction

The convexity has been studied for a long time in partial differential equations and it is intimately related to geometric properties of solutions to partial differential equations. There are macroscopic and microscopic convexity principle in general to yield convex solutions. The macroscopic convexity principle developed from 1980s, which was obtained by Korevaar [12], Kennington [11] and for the general nonlinear partial differential equations by Alvarez-Lasry-Lions[1]. However this method has difficulties in some geometric partial differential equations on compact manifold. The microscopic convexity principle concentrates on establishing the constant rank theorem for convex solutions to partial differential equations. It is a powerful tool in producing convex solutions to partial differential equations via the continuity methods. Caffarelli-Friedman [4] proved a constant rank theorem for convex solutions of quasilinear elliptic equations in R^2, a similar result was also discovered by Yau [16] at the same time. Korevaar-Lewis [13] generalized these results to R^n. The microscopic convexity principle has been generalized to a variety of fully nonlinear differential equations. For the case of parabolic convexity, Svante-Johan [17] and Lions-Musiela [14] independently found necessary and sufficient conditions for preservation of convexity of parabolic equation. Convexity plays an important role in geometric flow. For example, Huisken [8] proved that the mean curvature flow deforms initial surface with positive curvature into a point, while the curvature remains positive along the mean curvature flow.

In addition to convexity, partial convexity is also an interesting subject in analysis and geometry. So far as partial convexity of solution is concerned, there are usually two definitions of partial convexity of a function u: one is that the sum of the smallest k eigenvalues of the Hessian matrix $\{D^2u\}$ of u is positive; the other is that there exists a positive integer k, such that $\sigma_l(D^2u) > 0(\text{or} \ge 0)$ for $1 \le l \le k$, where $\sigma_l(D^2u)$ is the l-th elementary symmetric function of the eigenvalues of D^2u. In the following we will use k convexity to mean the former definition and *partial* k convexity to mean the latter one. In this paper, we will mainly prove that these partial convexity properties are preserved for smooth solutions to the heat equation in space forms with nonnegative sectional curvature and therefore we will give a direct proof of the fact that these convex cones $\Gamma_k = \{\lambda = \lambda(D^2u) \in R^n : \sigma_l(\lambda) > 0, 1 \le l \le k\}$ (see the detail definition in Section 2) are invariant cones along the heat equation.

We first recall some results concerning partial convexity. A famous result is that in 1976, Brascamp-Lieb [3] established the logarithmic concavity of the fundamental solution of diffusion equation with convex potential in bounded convex domain in R^n. As a consequence, they proved the logarithmic concavity of the first eigenfunction of Laplacian equation in convex domain. In geometry, the assumption on the curvature of surface, such as positive Ricci curvature or positive curvature operator in some sense can be interpreted as partial convexity conditions. For geometric evolution equation, invariant cones play important roles and hence geometric quantities satisfying partial convexity properties can be used to construct invariant cones. For example, Huisken-Sinestrari [10] classified the compact 2-convex hypersurfaces in R^n using the technique of mean curvature flow. They proved that if $F_0 : M^n \to R^{n+1}$ be a smooth immersion of a closed n-dimensional hypersurface, with $n \ge 3$ and if $M_0 = F_0(M)$ is two convex, i.e., $\lambda_1 + \lambda_2 > 0$ everywhere on M_0; then there exists a mean curvature flow with surgeries starting from M_0 which terminates after a finite number of steps. As corollary, they classified all closed hypersurface with two positive curvature operator. For Ricci flow, there are plenty of such results. For example, Hamilton [6] ([7]) proved that if a compact 3-manifold (4-manifold) M^n admits a Riemannian metric g_0 with positive Ricci curvature (positive curvature operator), then this metric can be deformed to a metric g of constant positive sectional curvature. In addition to these, Chen [5] and Böhm-Wilking [2] studied the classification of compact Riemannian manifolds with 2-positive curvature operator via Ricci flow. They proved that if (M^n, g) has 2-positive curvature operator, then the normalized Ricci flow evolves the initial metric g to a constant curvature limit metric. In their proof, they constructed a pinching family with initial cone being the cone of 2-positive curvature operator. With the existence of such pinching family, they could prove the convergence of the normalized Ricci flow to a constant curvature limit metric.

As significance and broad applications of the partial convexity as illustrated in the above, the partial convexity property of solution to differential equations is well worth studying and hence we consider this subject in this paper. We consider a model of smooth solution to the heat equation in space form M^n with nonnegative sectional curvature:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{in} \quad M^n \times [0,T), \\ u|_{t=0} = u_0(x) & \text{on} \quad M^n. \end{cases} \tag{1}$$

We will prove some partial convexity results for solutions of this heat equation. The first main theorem is the following partial convexity result for an n-dimensional space form M^n with nonnegative sectional curvarture.

Theorem 1. Let M^n be a compact space form with nonnegative sectional curvature s and u be the smooth solution to the heat equation (1) on M^n. If the initial data u_0 is partially k convex, i.e., for $1 \le l \le k$, $\sigma_l(D^2 u_0) > 0$. Then the solution u is also partially k convex, i.e., for any $t > 0$ and $1 \le l \le k$, we have $\sigma_l(D^2 u) > 0$.

Concerning the relation of the two definitions of partial convexity for solutions to heat equation, we have the following corollary of Theorem 1.

Corollary 1. Let M^n and $u(x,t)$ satisfy the assumptions of Theorem 1. If the initial data u_0 is partially k convex, i.e., there exists a positive integer k, such that $\sigma_l(D^2 u_0) > 0$ for $1 \le l \le k$. Then for any $t \ge 0$ the solution $u(x,t)$ is $n - [\frac{k}{2}]$ convex, i.e., the sum of smallest $n - [\frac{k}{2}]$ eigenvalues of $D^2 u(x,t)$ is positive, where $[m]$ denotes the largest integer not exceeding m.

The paper is organized as follows. We first recall the definitions and some fundamental facts on the elementary symmetric functions σ_k in Section 2. Then we prove partial convexity for the smooth solutions of heat equation is preserved in general space forms with nonnegative sectional curvature, i.e., we prove Theorem 1 and corollary 1 in Section 3. We finally give some concluding remarks in Section 4.

2. Preliminary

In this section, we recall the definition and some basic properties of the elementary symmetric functions of $\lambda = (\lambda_1, \cdots, \lambda_n)$.

Definition 1. For any $k = 1, 2, \cdots, n$, we set

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \quad \forall \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in R^n.$$

We also set $\sigma_0(\lambda) \equiv 1$ and $\sigma_k(\lambda) = 0$ for $k > n$.

For a symmetric matrix W, we define by letting $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \cdots, \lambda_n(W))$ are the eigenvalues of the symmetric matrix.

In addition, we define

$$\Gamma_k = \{\lambda \in R^n : \sigma_1(\lambda) > 0, \sigma_2(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}.$$

Obviously Γ_k contains the positive cone $\Gamma_n = \{\lambda \in R^n : \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_n > 0\}$. Γ_k is symmetric in the sense that if $\lambda \in \Gamma_k$, then any permutation of λ also lies in Γ_k.

Let us denote by $\sigma_k(\lambda | i)$ the sum of the terms of $\sigma_k(\lambda)$ not containing the factor λ_i. We list some basic properties of elementary symmetric functions which will be frequently used in the following calculation.

Proposition 1. For any $k = 0, 1, \cdots, n$, $i = 1, 2, \cdots, n$, and $\lambda \in R^n$, the following identities hold:

$$\frac{\partial \sigma_{k+1}}{\partial \lambda_i}(\lambda) = \sigma_k(\lambda \mid i), \quad (2)$$

$$\sigma_{k+1}(\lambda) = \sigma_{k+1}(\lambda \mid i) + \lambda_i \sigma_k(\lambda \mid i), \quad (3)$$

$$\sum_{i=1}^{n} \sigma_k(\lambda \mid i) = (n-k)\sigma_k(\lambda), \quad (4)$$

$$\sum_{i=1}^{n} \lambda_i \sigma_k(\lambda \mid i) = (k+1)\sigma_{k+1}(\lambda). \quad (5)$$

Proposition 2. 1. If $\lambda \in \Gamma_k$ for $k \in \{1,2,\cdots,n\}$, then we have $\sigma_h(\lambda \mid i) > 0$, for any $h \in \{0,1\cdots,k-1\}$ and $i \in \{1,2,\cdots,n\}$.

2. Let $W = \{W_{ij}\}$ be a symmetric matrix such that its eigenvalues belong to Γ_k, and set $F(W) = \sigma_k^{\frac{1}{k}}(W)$, that is

$$\sum_{i,j,k,l=1}^{n} \frac{\partial^2 F}{\partial W_{ij} \partial W_{kl}}(W)\eta_{ij}\eta_{kl} \le 0, \quad \text{for any } \{\eta_{ij}\} \in R^{n\times n}. \quad (6)$$

3. Let $W = \{W_{ij}\}$ be a symmetric matrix such that its eigenvalues belong to Γ_{k-1}, and set $F(W) = \frac{\sigma_k}{\sigma_{k-1}}(W)$, then F is concave on Γ_{k-1}, that is

$$\sum_{i,j,k,l=1}^{n} \frac{\partial^2 F}{\partial W_{ij} \partial W_{kl}}(W)\eta_{ij}\eta_{kl} \le 0, \quad \text{for any } \{\eta_{ij}\} \in R^{n\times n}. \quad (7)$$

Proposition 1 is standard which can be directly checked. For the proof of Proposition 2, the readers can consult [9] for example.

3. Proof of Theorem 1 and Corollary 1

In this section, we will prove Theorem 1 that partial k convexity of smooth solutions to the heat equation (1) is preserved in general space forms with nonnegative sectional curvature.

Since we are working in space forms and need to commute covariant derivatives, we first recall Ricci identities in Riemannian geometry. Let M^n be a Riemannian manifold with nonnegative constant sectional curvature and R_{ijkl} be its Riemann curvature tensor. Let u be a smooth function on M^n. If we denote the first and second covariant derivatives of u with respect to the frame $\{e_i\}$ by u_i and u_{ij} etc., then we have the following Ricci identities

$$u_{ijk} - u_{ikj} = -\sum_{l=1}^{n} u_l R_{likj} \quad (8)$$

and

$$u_{ijkl} - u_{ijlk} = \sum_{m=1}^{n} u_{im} R_{mjkl} + \sum_{m=1}^{n} u_{mj} R_{mikl}. \quad (9)$$

For Riemannian manifold with constant sectional curvature s, we can see from the definition of Riemann curvature tensor that

$$R_{ijkl} = s\cdot(g_{ik}g_{jl} - g_{il}g_{jk}) = s\cdot(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad (10)$$

where we have chosen normal coordinates at a fix point such that the metric takes the form $g_{ij}=\delta_{ij}$.

Combining (8)-(10), we can rewrite the Ricci identities in space form with sectional curvature s in the following form

$$u_{ijk}-u_{ikj}=-s\cdot u_l\cdot\sum_{l=1}^{n}(\delta_{lk}\delta_{ij}-\delta_{lj}\delta_{ik})=s\cdot(u_j\delta_{ik}-u_k\delta_{ij}), \tag{11}$$

and

$$\begin{aligned} u_{ijkl}-u_{ijlk} &=\sum_{m=1}^{n}u_{im}\cdot s\cdot(\delta_{mk}\delta_{jl}-\delta_{ml}\delta_{jk})+\sum_{m=1}^{n}u_{mj}\cdot s\cdot(\delta_{mk}\delta_{il}-\delta_{ml}\delta_{ik}) \\ &=s\cdot(u_{ik}\delta_{jl}-u_{il}\delta_{jk}+u_{jk}\delta_{il}-u_{jl}\delta_{ik}). \end{aligned} \tag{12}$$

Now we can give the following

Proof of Theorem 1.

We prove by induction method. We need to make some simplifications. By perturbation argument,

We first prove the case for $k=1$. We calculate by using the heat equation (1) and obtain

$$\frac{\partial}{\partial t}(\Delta u)=\Delta(\frac{\partial u}{\partial t})=\Delta(\Delta u).$$

From the above calculation and strong maximum principle we know that $\square u$ is positive along the heat equation (1).

For the general case. Suppose $\sigma_m(D^2u)>0$ holds for $m\le l-1$ and therefore D^2u lies in the convex cone Γ_{l-1}. For the case $m=l$.

We consider the following auxiliary function

$$P=\sigma_l(D^2u)\sigma_{l-1}(D^2u).$$

The Hessian quotient P is concave on the convex cone Γ_{l-1}, which we have indicated previously in Proposition 2.

We compute the evolution equation for P. Noting that

$$\Delta P=\sum_{s=1}^{n}\frac{\partial^2 P}{\partial u_{ij}\partial u_{kl}}\nabla^s u_{ij}\nabla_s u_{kl}+\sum_{s=1}^{n}\frac{\partial P}{\partial u_{ij}}\nabla^s\nabla_s u_{ij},$$

therefore we have

$$\frac{\partial P}{\partial t}-\Delta P=\sum_{i,j=1}^{n}\frac{\partial P}{\partial u_{ij}}(\frac{\partial u_{ij}}{\partial t}-\Delta(u_{ij}))-\sum_{s=1}^{n}\frac{\partial^2 P}{\partial u_{ij}\partial u_{kl}}\nabla^s u_{ij}\nabla_s u_{kl}. \tag{13}$$

Since D^2u lies in the convex cone Γ_{l-1}, the auxiliary function P is concave and therefore the last term in (13) ≥ 0. We need to calculate the first term on the right hand side of (13). By adopting normal coordinates, we obtain from the Ricci identities (11) and (12) that

$$
\begin{aligned}
\Delta(u_{ij}) &= \sum_{k=1}^{n} u_{ijkk} = \sum_{k=1}^{n} (u_{ikj} + s\cdot u_j\delta_{ik} - s\cdot u_k\delta_{ij})_k \\
&= \sum_{k=1}^{n} u_{ikjk} + s\cdot u_{ij} - s\cdot\Delta u\delta_{ij} \\
&= \sum_{k=1}^{n} u_{ikkj} + \sum_{k=1}^{n} s\cdot(u_{ij}\delta_{kk} - u_{ik}\delta_{kj} + u_{kj}\delta_{ik} - u_{kk}\delta_{ij}) + s\cdot u_{ij} - s\cdot\Delta u\delta_{ij} \\
&= \sum_{k=1}^{n} (u_{kki} + s\cdot(u_i\delta_{kk} - u_k\delta_{ik}))_j + (n+1)s\cdot u_{ij} - 2s\Delta u\delta_{ij} \\
&= (\Delta u)_{ij} + 2snu_{ij} - 2s\Delta u\delta_{ij}.
\end{aligned} \tag{14}
$$

Combining (13), (14) and the heat equation (1), we obtain

$$
\begin{aligned}
\frac{\partial P}{\partial t} - \Delta P &= \sum_{i,j=1}^{n} \frac{\partial P}{\partial u_{ij}} \left(\frac{\partial u}{\partial t} - \Delta u\right)_{ij} - \sum_{s=1}^{n} \frac{\partial^2 P}{\partial u_{ij}\partial u_{kl}} \nabla^s u_{ij} \nabla_s u_{kl} \\
&\quad - 2sn\sum_{i,j=1}^{n} \frac{\partial P}{\partial u_{ij}} u_{ij} + 2s\Delta u \sum_{i,j=1}^{n} \frac{\partial P}{\partial u_{ij}} \delta_{ij} \\
&\geq -2snP + 2s\Delta u \sum_{i=1}^{n} \frac{\partial P}{\partial u_{ii}},
\end{aligned} \tag{15}
$$

where we have used the concavity of the Hessian quotient P on the convex cone Γ_{l-1} and equality (5) in Proposition 1.

If we denote the eigenvalues of the matrix D^2u by $\lambda = \lambda(D^2u)$, we then calculate the second term in (15) as follows:

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{\partial P}{\partial u_{ii}} &= \sum_{i=1}^{n} (\sigma_l(\lambda)\sigma_{l-1}(\lambda))_i \\
&= \sum_{i=1}^{n} \sigma_{l-1}(\lambda|i)\sigma_{l-1}(\lambda) - \sigma_l(\lambda)\sigma_{l-2}(\lambda|i)\sigma_{l-1}(\lambda)^2 \\
&= (n-l+1)\sigma_{l-1}(\lambda)^2 - (n-l+2)\sigma_l(\lambda)\sigma_{l-2}(\lambda)\sigma_{l-1}(\lambda)^2 \\
&= (n-l+1) - (n-l+2)\sigma_{l-2}(\lambda)\sigma_{l-1}(\lambda)\cdot P,
\end{aligned} \tag{16}
$$

where we have used equalities (2) and (4) in Proposition 1.
Substituting (16) into (15), we get

$$
\begin{aligned}
\frac{\partial P}{\partial t} - \Delta P &\geq 2s(n-l+1)\sigma_1(\lambda) - 2(n-l+2)s\sigma_1(\lambda)\sigma_{l-2}(\lambda)\sigma_{l-1}(\lambda)\cdot P - 2snP \\
&\geq -2s\cdot\{(n-l+2)\sigma_1(\lambda)\sigma_{l-2}(\lambda)\sigma_{l-1}(\lambda) + n\}\cdot P,
\end{aligned} \tag{17}
$$

where in the last inequality we have used the assumption that the sectional curvature of M^n s is nonnegative.

Since M^n is assumed to be a compact space form and the induction assumption that $\sigma_{l-1}(\lambda) > 0$ is preserved along the heat equation, we know there exists a constant C_0, such that

$$(n-l+2)\sigma_1(\lambda)\sigma_{l-2}(\lambda)\sigma_{l-1}(\lambda) + n \leq C_0.$$

We take a new auxiliary function $\tilde{P} = e^{2sC_0 t}P$. Then from the differential inequality (17), we compute

$$\begin{aligned}\frac{\partial \tilde{P}}{\partial t}-\Delta \tilde{P} &= e^{2sC_0t}(\frac{\partial P}{\partial t}-\Delta P)+2sC_0e^{2sC_0t}P \\ &\geq 2se^{2sC_0t}[C_0-((n-l+2)\sigma_1(\lambda)\sigma_{l-2}(\lambda)\sigma_{l-1}(\lambda)+n)]\geq 0.\end{aligned} \tag{18}$$

Since $\tilde{P}=P>0$ at the initial time $t=0$, we conclude from the strong maximum principle for parabolic equation that for any $t>0$, we have $\tilde{P}>0$ or equivalently $P=e^{-2sC_0t}\tilde{P}>0$ and hence $\sigma_l(D^2u)>0$.

In the following part of this section, we prove Corollary 1.

Proof of Corollary 1. We divide the proof into 3 steps.

Step 1. We claim that if $\lambda_n\geq\cdots\geq\lambda_1$ and $\sigma_l(\lambda)>0(1\leq l\leq k)$, then we have $\lambda_1+\cdots+\lambda_{n-[\frac{k}{2}]}>0$. We prove the claim by induction on n and k. For the case $n=3$, when $k=1$, it is obvious; when $k=2$, it is by Proposition 2 that $\sigma_1(\lambda\,|\,3)>0$, i.e., $\lambda_1+\lambda_2>0$, the claim follows for $n=3$. Suppose the claim holds for the case of dimension $n-1$ and $k-1$. Then for the case of $\sigma_l(\lambda)>0(1\leq l\leq k)$, we use proposition 2 once more to obtain that $\sigma_{l-1}(\lambda\,|\,n)>0(1\leq l\leq k)$. This means that $\lambda_1,\cdots,\lambda_{n-1}$ satisfy the assumption of the claim and by induction hypothesis, we therefore obtain

$$\lambda_1+\cdots+\lambda_{n-1-[\frac{k-1}{2}]}>0. \tag{19}$$

We divide k into two cases:

Case 1: k is an even integer, then $n-1-[\frac{k-1}{2}]=n-1-\frac{k-2}{2}=n-\frac{k}{2}=n-[\frac{k}{2}]$ and the claim follows;

Case 2: k is an odd integer, then $n-1-[\frac{k-1}{2}]=n-1-\frac{k-1}{2}=n-\frac{k+1}{2}$ whereas $n-[\frac{k}{2}]=n-\frac{k-1}{2}$. The induction hypothesis (19) and $\lambda_n\geq\cdots\geq\lambda_1$ immediately lead to $\lambda_1+\cdots+\lambda_{n-[\frac{k}{2}]}>0$. Therefore the claim also follows.

Step 2. To finish the proof of Corollary 1, we first prove the following Lemma 1.

Lemma 1. Let $u(x,t)$ be a smooth solution to the heat equation (1) in $R^n\times[0,T)$. We denote by D^2u the Hessian matrix $\{u_{ij}\}$ of the solution u and assume $D^2u\in C_{exp}$. If the initial data u_0 is k convex, i.e., the sum of smallest k eigenvalues of the matrix D^2u_0 is nonnegative (positive), then k convexity will be preserved for solutions by the heat equation, i.e., for any $t>0$, the sum of smallest k eigenvalues of $D^2u(x,t)$ is nonnegative (positive).

Sketch of the Proof of Lemma 1. We need to make some simplifications. By perturbation argument, we may assume the sum of smallest k eigenvalues of Hessian for D^2u_0 of the initial data u_0 is positive, otherwise we may consider $u(x,t)+\varepsilon(|\,x\,|^2\;2n-t)$ instead and let $\varepsilon\to 0$.

We may assume the sum of smallest k eigenvalues of $D^2u(x,t)$ vanishes at some space-time point, otherwise we are done. We may assume t_0 be the first vanishing time and assume the vanishing point be attained at (x_0,t_0). we can also rotate the coordinates such that the matrix D^2u is diagonal and its eigenvalues satisfy $u_{nn}\geq u_{n-1n-1}\geq\cdots\geq u_{11}$ at this point.

Therefore at the space-time point (x_0,t_0), we have

$$\frac{\partial}{\partial t}(u_{11}+\cdots+u_{kk})-\Delta(u_{11}+\cdots+u_{kk})$$
$$=(\frac{\partial u}{\partial t}-\Delta u)_{11}+\cdots+(\frac{\partial u}{\partial t}-\Delta u)_{kk}=0,$$

where we have used the heat equation (1).

From the above calculation and maximum principle we know that the sum of the smallest k eigenvalues of D^2u is nonnegative along the heat equation (1). From the strict parabolic maximum principle, the case of strict inequality in Lemma 1 follows.

Step 3. We use Lemma 1 in step 2 to finish the proof. If the initial data u_0 is partially k convex, then from Theorem 1, we know that for any $t\geq 0$, $u(x,t)$ is partially k convex, i.e., $\sigma_l(D^2u)>0(1\leq l\leq k)$. From step 1, we know that the sum of smallest $n-[\frac{k}{2}]$ eigenvalues of Hessian matrix D^2u is positive. We can also use Lemma 1 in step 2 to finish the proof: we first use Step 1 to see that the sum of smallest $n-[\frac{k}{2}]$ eigenvalues of the matrix D^2u_0 is positive and then use Lemma 1 in step 2 to conclude the corollary.

As another consequence of the concavity of Hessian quotient, we give an estimate of $\sigma_l(D^2u)$ in terms of Δu and $\sigma_{l-1}(D^2u)$ for solution of heat equation in Euclidean space.

Theorem 1. Suppose $u(x,t)$ is a smooth solution to heat equation (1) with initial data u_0 in R^n. If for any $2\leq l\leq k$, there exists a positive constant ε_l, such that u_0 satisfies

$$\sigma_l(D^2u_0)\geq\varepsilon_l \Delta u_o\cdot\sigma_{l-1}(D^2u_0),$$

then for any $t\in[0,T)$, the same inequality holds true for u.

Proof. The proof is similar to that of Theorem 1. Consider the auxiliary function

$$h=P-\varepsilon_l\Delta u=\sigma_l(D^2u)\sigma_{l-1}(D^2u)-\varepsilon_l\sigma_1(D^2u),$$

then what we need is to calculate the evolution equation for this auxiliary function.

$$\begin{aligned}\frac{\partial h}{\partial t}-\Delta h&=\frac{\partial}{\partial t}(P-\varepsilon_l\Delta u)-\Delta(P-\varepsilon_l\Delta u)\\&=(\frac{\partial}{\partial t}P-\Delta P)-\varepsilon_l(\frac{\partial}{\partial t}(\Delta u)-\Delta(\Delta u))\\&=\sum_{i,j=1}^{n}\frac{\partial P}{\partial u_{ij}}(\frac{\partial u}{\partial t}-\Delta u)_{ij}-\sum_{s=1}^{n}\frac{\partial^2 P}{\partial u_{ij}\partial u_{kl}}\nabla^s u_{ij}\nabla_s u_{kl}\\&\geq 0,\end{aligned}\tag{20}$$

where we have used the heat equation (1) and the concavity of Hessian quotient $P=\sigma_l(D^2u)\sigma_{l-1}(D^2u)$ on the convex cone Γ_{l-1}. The result is then a direct consequence of the parabolic maximum principle.

4. Concluding Remarks

In the present paper, we only prove partial convexity of heat equation in space form. We will use the concavity property of the Hessian operator and Hessian quotient operator in Proposition 2 to study the general case in arbitrary Riemanian manifold in the future.

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