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Fixed Point Result for Weakly Isotone Increasing Mappings in Ordered Complex Valued Generalized Metric Spaces

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ABSTRACT

The aim of this paper is to establish common fixed point theorem and periodic point result in the setting of ordered Complex valued generalized metric spaces using pair of weakly isotone increasing mappings.

Keywords: Complex valued generalized metric space, weakly increasing mapping, weakly isotone increasing mapping.

2010 MSC: 47H10, 54H25.

1. Introduction

Metric fixed point theory is widely recognized to have been originated in the work of S. Banach in 1922 [5]. Over the years metric fixed point theory has developed in different directions. A comprehensive account of this development provided in the handbook entitled by Kirk and Sims [18]. On the other hand, Dass and Gupta [10] generalized the Banach's contraction mapping principle by using a contractive condition of rational type. Fixed point theorems for contractive type conditions satisfying rational inequalities in metric spaces have been developed in a number of works ([8],[9],[14],[16],[17],[19],[21]).

Also there are large efforts for generalizing metric spaces by changing the form and interpretation of the metric function like 2-metric space [12], Probabilistic metric space [24, 25], Fuzzy metric space [13], Cone metric space [15], G-metric space [20] etc. Recently, Azam et al. [3] introduced the notion of Complex valued metric space as a generalization of metric space and Cone metric space where the metric function takes values from the field of complex numbers, thus opening the scope of the concepts from complex analysis for incorporation in the metric space structure. Fixed point theory has been studied in this space in a suitable number of papers, some of which we mention in ([4],[7],[26],[27],[28]). Very recently, Abbas et al [1] introduced the notion of complex valued generalized metric spaces and studied the existence of fixed points and common fixed points for two mappings satisfying contractive condition of rational type, without exploiting any type of commutativity condition.

In this paper, we prove some common fixed point result for pair of weakly isotone increasing mappings in the context of ordered complex valued generalized metric spaces. As an application, periodic point is also established.

2. Preliminaries

Consistent with Azam et al. [3] and Rouzkard et al. [23], the following definitions and result will be needed in the sequel.

Let C be the set of complex numbers and let $z_1, z_2 \in C$. Define a partial order $\leq$ on C as follows: $z_1 \leq z_2$ if and only if $\operatorname{Re}(z_1) \leq \operatorname{Re}(z_2), \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2)$.

It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

(1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$

(2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$

(3) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$

(4) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$

In particular, we will write $z_1 \lesssim z_2$ if one of (1), (2) and (3) satisfied and we will write $z_1 < z_2$ if only (3) is satisfied.

Some elementary properties of the partial order $\leq$ on $\mathbb{C}$ are the following:

(i) If $0 \leq z_1 \leq z_2$, then $|z_1| \leq |z_2|$.

(ii) $z_1 \leq z_2$ is equivalent to $z_1 - z_2 \leq 0$.

(iii) If $z_1 \leq z_2$ and $r \geq 0$ is a real number, then $rz_1 \leq rz_2$.

(iv) If $0 \leq z_1$ and $0 \leq z_2$ with $z_1 + z_2 \neq 0$, then $\dfrac{z_1^2}{z_1 + z_2} \leq z_1$.

(v) $0 \leq z_1$ and $0 \leq z_2$ do not imply $0 \leq z_1 z_2$.

(vi) $0 \leq z_1$ does not imply $0 \leq \dfrac{1}{z_1}$. Moreover, if $0 < z_1$ and $0 \leq \dfrac{1}{z_1}$, then $\operatorname{Im}(z_1) = 0$.

Now we give the definition of complex valued generalized metric space.

Definition 2.1: Let X be a non empty set. If a mapping $d : X \times X \to \mathbb{C}$ satisfies:

(a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct $u, v \in X$ each one is different from x and y.

Then d is called a complex valued generalized metric on X and (X, d) is called a complex valued generalized metric space

Example 2.1: [1] Let $X = \{-1, 1, -i, i\}$. Defined $d : X \times X \to \mathrm{C}$ as follows:

$$d(1, -1) = d(-1, 1) = 3e^{i\theta},$$

$d(-1,i) = d(i,-1) = d(1,i) = d(i,1) = e^{i\theta}$,

$d(1,-i) = d(-i,1) = d(-1,-i) = d(-i,-1) = d(i,-i) = d(-i,i) = 5e^{i\theta}$

$d(1,1) = d(-1,-1) = d(i,i) = d(-i,-i) = 0$.

It is to verify that (X,d) is a complex valued generalized metric space when $\theta \in \left[0, \frac{\pi}{2}\right]$.

Note that

$$3e^{i\theta} = d(1,-1) > d(1,i) + d(i,-1) = 2e^{i\theta}.$$

So d is not a complex valued metric.

Let X be a complex valued generalized metric space and $A \subseteq X$. A point $x \in X$ is called an interior point of a set A whenever there exists $0 < r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) < r\} \subseteq A$.

A subset A in X is called open whenever each point of A is an interior point of A. The family $F = \{B(x,r) : x \in X, 0 < r\}$ is a sub-basis for a Hausdorff topology τ on X.

A point $x \in X$ is called an limit point of A whenever for every $0 < r \in \mathbb{C}$, $B(x,r) \cap (A \setminus x) \neq \phi$. A subset is $B \subseteq X$ called closed whenever each limit point of B belongs B.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 < c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x_m) < c$, then x is called the limit point of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathbb{C}$, with $0 < c$, there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0, d(x_n, x_m) < c$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then it is called a complete complex valued generalized metric space.

Lemma 2.1: Let X be a complex valued generalized metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x_m)| \to 0$ as $n \to \infty$.

Lemma 2.2: Let X be a complex valued generalized metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $n \to \infty$.

The following definition is due to Altun and Erduran ([2]).

Definition 2.2: [2] Let $(X, \leq)$ be a partially ordered set. A pair (f, g) of self-maps of X is said to be weakly increasing if $fx \leq gfx$ and $gx \leq fgx$ for all $x \in X$, then we have $fx \leq f^2x$ for all $x \in X$ and in this case, we say that f is weakly increasing map.

Note that two weakly increasing mappings need not be non-decreasing. There exist some examples to illustrate this fact in [11].

Definition 2.3: [22] Let $(X, \leq)$ be a partially ordered set and let $f, g : X \to X$ be two mappings. The pair (f, g) is weakly isotone increasing if for all $x \in X$ we have $fx \leq gfx \leq fgfx$.

Remark 2.1: If $f, g : X \to X$ are weakly increasing, then the pair (f, g) is weakly isotone increasing. A point x in X said to be a fixed point of a self map f on X if $fx = x$. A fixed point problem is to find some x in X such that $fx = x$ and we denote it by $FP(f, X)$. A point $x \in X$ is called a common fixed point of pair (f, g) if $x = fx = gx$, where f and g are two self-maps on X. A common fixed point problem is to find some x in X such that $x = fx = gx$, and we denote it by $CFP(f, g, X)$. A nonempty subset W of a partially ordered set X is said to be totally ordered if every two elements of W are comparable.

3. Main Results

Theorem 3.1: Let $(X,\leq)$ be a partial ordered set such that there exist a complete complex valued generalized metric d on X and (S,T) a pair of weakly isotone increasing mapping on X. Suppose that, for every comparable $x, y \in X$ we have either

$$\begin{aligned} d(Sx,Ty) \leq\ & a_1 \frac{d(x,Sx)\left[d(y,Ty)+d(y,Sx)\right]}{d(x,y)+d(Sx,y)} \\ & +a_2 \frac{d(y,Sx)\left[d(x,Sx)+d(y,Ty)\right]}{d(x,y)+d(Sx,y)} \\ & +a_3 \frac{d(y,Ty)\left[1+d(x,Sx)\right]}{1+d(x,y)} + a_4 \frac{d(x,Sx)d(y,Ty)}{d(x,y)} \\ & +a_5\left[d(x,Sx)+d(y,Ty)\right]+a_6 d(x,y) \end{aligned} \tag{3.1}$$

in case $d(x,y)+d(Sx,y)\neq 0$ and $1+d(x,y)\neq 0$ with $a_i,(i=1,2,3,4,5,6)\geq 0$ and $\sum_{i=1}^{6} a_i <1, a_6<1$ or $d(x,y)+d(Sx,y)=0$ and $1+d(x,y)=0$ implies $d(Sx,Ty)=0$.

If S or T is continuous or for any non decreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \leq z$ for all $n\in N$, then S and T have a common fixed point . Moreover the set of common fixed points of S or T is totally ordered if and only if S or T have one and only one common fixed point.

Proof: First, we shall show that if S or T has a fixed point, then it is s common fixed point of S and T. Let u be an arbitrary point in X. If $u=Su$ or $u=Tu$ then the proof can be easily finished using contractive condition (3.1). Indeed let $u=Su$ then we have

$$\begin{aligned} d(u,Tu) &= d(Su,Tu) \\ & \leq a_1 \frac{d(u,Su)\left[d(u,Tu)+d(u,Su)\right]}{d(u,u)+d(Su,u)} \\ & +a_2 \frac{d(u,Su)\left[d(u,Su)+d(u,Tu)\right]}{d(u,u)+d(Su,u)} \\ & +a_3 \frac{d(u,Tu)\left[1+d(u,Su)\right]}{1+d(u,Su)} + a_4 \frac{d(u,Su)d(u,Tu)}{d(u,u)} \\ & +a_5\left[d(u,Su)+d(u,Tu)\right]+a_6 d(u,u) \\ & \leq (a_3+a_5)d(u,Tu) \\ & < d(u,Tu) \end{aligned}$$

i.e. $u=Tu$

Similarly, if $u=Tu$ we obtain that $u=Su$.

So we assume that $u\neq Su$ and $u\neq Tu$. Now we define a sequence $\{x_n\}$ in X, as follows:

$x_{2n+1}=Sx_{2n}$ and $x_{2n+2}=Tx_{2n+1}$ for $n=0,1,2,\cdots$

We can also suppose that the successive term of $\{x_n\}$ are different, otherwise we have again finished. Since pair (S,T) is weakly isotone increasing, we have

$$x_1 = Sx_0 \le TSx_0 = Tx_1 = x_2 \le STSx_0 = STx_1 = Sx_2 = x_3$$

$$x_3 = Sx_2 \le TSx_2 = Tx_3 = x_4 \le STSx_2 = STx_3 = Tx_4 = x_5$$

And continuing this process, we get

$$x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \cdots \tag{3.2}$$

Since the successive terms of $\{x_n\}$ are comparable therefore replacing x by x_{2n} and y by x_{2n+1} in (3.1) we have,

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\le a_1 \frac{d(x_{2n}, Sx_{2n})\left[d(x_{2n+1}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})\right]}{d(x_{2n}, x_{2n+1}) + d(Sx_{2n}, x_{2n+1})} \\
&+ a_2 \frac{d(x_{2n+1}, Sx_{2n})\left[d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})\right]}{d(x_{2n}, x_{2n+1}) + d(Sx_{2n}, x_{2n+1})} \\
&+ a_3 \frac{d(x_{2n+1}, Tx_{2n+1})\left[1 + d(x_{2n}, Sx_{2n})\right]}{1 + d(x_{2n}, x_{2n+1})} \\
&+ a_4 \frac{d(x_{2n}, Sx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1})} \\
&+ a_5\left[d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})\right] + a_6 d(x_{2n}, x_{2n+1}) \\
&= a_1 \frac{d(x_{2n}, x_{2n+1})\left[d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})\right]}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+1})} \\
&+ a_2 \frac{d(x_{2n+1}, x_{2n+1})\left[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\right]}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+1})} \\
&+ a_3 \frac{d(x_{2n+1}, x_{2n+2})\left[1 + d(x_{2n}, x_{2n+1})\right]}{1 + d(x_{2n}, x_{2n+1})} \\
&+ a_4 \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} \\
&+ a_5\left[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\right] + a_6 d(x_{2n}, x_{2n+1}) \\
&\le (a_1 + a_3 + a_4 + a_5)\, d(x_{2n+1}, x_{2n+2}) + (a_5 + a_6)\, d(x_{2n}, x_{2n+1})
\end{aligned}$$

which implies that

$$d(x_{2n+1}, x_{2n+2}) \le h d(x_{2n}, x_{2n+1}) \text{ for all } n \ge 1$$

where $h = \dfrac{a_5 + a_6}{1 - (a_1 + a_3 + a_4 + a_5)} < 1$.

Similarly it can be shown that

$$d\left(x_{2n}, x_{2n+1}\right) \leq hd\left(x_{2n-1}, x_{2n}\right) \text{ for all } n \geq 1$$

Therefore for all $n \geq 1$

$$d\left(x_n, x_{n+1}\right) \leq hd\left(x_{n-1}, x_n\right)$$

Consequently

$$d\left(x_n, x_{n+1}\right) \leq hd\left(x_{n-1}, x_n\right) \leq \cdots \leq h^n d\left(x_0, x_1\right)$$

for all $n \geq 1$.

Now for any $m > n$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots\ldots + d(x_{m-1}, x_m).$$

$$\leq h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \ldots\ldots\ldots\ldots\ldots + h^{m-1} d(x_0, x_1)$$

$$\leq \frac{h^n}{1-h} d(x_0, x_1)$$

Therefore

$$|d(x_n, x_m)| \leq \frac{h^n}{1-h} |d(x_0, x_1)|.$$

which implies that

$$|d(x_n, x_m)| \to 0 \text{ as } n, m \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, the sequence $\{x_n\}$ converges to a point u in X.

Now, if S or T is continuous, then it is clear that $Su = u = Tu$.

If neither S or T is continuous, then by given assumption we have, $x_n \leq u$ for all $n \in N$.

We claim that u is a fixed point of S.

If not, then $d(u, Su) = z > 0$.

We have from (3.1)

$$\begin{aligned}
z &\leq d\left(u, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + d\left(x_{n+2}, Su\right) \\
&= d\left(u, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + d\left(Su, Tx_{n+1}\right) \\
&\leq d\left(u, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + a_1 \frac{d\left(u, Su\right)\left[d\left(x_{n+1}, Tx_{n+1}\right) + d\left(x_{n+1}, Su\right)\right]}{d\left(u, x_{n+1}\right) + d\left(Su, x_{n+1}\right)} \\
&\quad + a_2 \frac{d\left(x_{n+1}, Su\right)\left[d\left(u, Su\right) + d\left(x_{n+1}, Tx_{n+1}\right)\right]}{d\left(u, x_{n+1}\right) + d\left(Su, x_{n+1}\right)} + a_3 \frac{d\left(x_{n+1}, Tx_{n+1}\right)\left[1 + d\left(u, Su\right)\right]}{1 + d\left(u, Su\right)}
\end{aligned}$$

$$+a_4\frac{d(u,Su)d(x_{n+1},Tx_{n+1})}{d(u,Su)}+a_5\left[d(u,Su)+d(x_{n+1},Tx_{n+1})\right]+a_6d(u,x_{n+1})$$

$$=d(u,x_{n+1})+d(x_{n+1},x_{n+2})+a_1\frac{d(u,Su)\left[d(x_{n+1},x_{n+2})+d(x_{n+1},Su)\right]}{d(u,x_{n+1})+d(Su,x_{n+1})}$$

$$+a_2\frac{d(x_{n+1},Su)\left[d(u,Su)+d(x_{n+1},x_{n+2})\right]}{d(u,x_{n+1})+d(Su,x_{n+1})}+a_3\frac{d(x_{n+1},x_{n+2})\left[1+d(u,Su)\right]}{1+d(u,Su)}$$

$$+a_4\frac{d(u,Su)d(x_{n+1},x_{n+2})}{d(u,Su)}+a_5\left[d(u,Su)+d(x_{n+1},x_{n+2})\right]+a_6d(u,x_{n+1})$$

and so

$$|z|\le\left|d(u,x_{n+1})\right|+\left|d(x_{n+1},x_{n+2})\right|+a_1\frac{\left|d(u,Su)\right|\left[\left|d(x_{n+1},x_{n+2})\right|+\left|d(x_{n+1},Su)\right|\right]}{\left|d(u,x_{n+1})+d(Su,x_{n+1})\right|}$$

$$+a_2\frac{\left|d(x_{n+1},Su)\right|\left[\left|d(u,Su)\right|+\left|d(x_{n+1},x_{n+2})\right|\right]}{\left|d(u,x_{n+1})+d(Su,x_{n+1})\right|}+a_3\left|d(x_{n+1},x_{n+2})\right|$$

$$+a_4\left|d(x_{n+1},x_{n+2})\right|+a_5\left[\left|d(u,Su)\right|+\left|d(x_{n+1},x_{n+2})\right|\right]+a_6\left|d(u,x_{n+1})\right|$$

Taking the limit as $n\to\infty$, we have

$$|z|\le a_1\left|d(u,Su)\right|+a_2\left|d(u,Su)\right|+a_5\left|d(u,Su)\right|$$
$$\le(a_1+a_2+a_5)\left|d(u,Su)\right|$$
$$\le(a_1+a_2+a_5)|z|$$

which is a contradiction. Hence $u=Su$. Therefore $Su=Tu=u$.

Next we prove that the common fixed point of S and T is unique.

Now suppose that the set of common fixed point of S and T is totally ordered. Assume on contrary that u and v are distinct common fixed point of S and T. Replace x by u and y by v in (3.1), we have

$$d(u,v)=d(Su,Tv)$$

$$\le a_1\frac{d(u,Su)\left[d(v,Tv)+d(v,Su)\right]}{d(u,v)+d(Su,v)}+a_2\frac{d(v,Su)\left[d(u,Su)+d(v,Tv)\right]}{d(u,v)+d(Su,v)}$$

$$+a_3\frac{d(v,Tv)\left[1+d(u,Su)\right]}{1+d(u,v)}+a_4\frac{d(u,Su)d(v,Tv)}{d(u,v)}$$

$$+a_5\left[d(u,Su)+d(v,Tv)\right]+a_6d(u,v)$$

$$=_1\frac{d(u,u)\left[d(v,v)+d(v,u)\right]}{d(u,v)+d(u,v)}+a_2\frac{d(v,u)\left[d(u,u)+d(v,v)\right]}{d(u,v)+d(u,v)}$$

$$+a_3\frac{d(v,v)\left[1+d(u,u)\right]}{1+d(u,v)}+a_4\frac{d(u,u)d(v,v)}{d(u,v)}$$

$$+a_5\left[d(u,u)+d(v,v)\right]+a_6d(u,v)$$

$$\le a_6d(u,v)$$

which implies that $\left|d(u,v)\right| \le a_6 \left|d(u,v)\right|$, a contradiction. Hence $u = v$.

Conversely, if S and T have only one common fixed point then the set of common fixed point of S and T being singleton is totally ordered.

This completes the proof.

In Theorem 3.1 if we take S = T, we get the following corollary.

Corollary 3.1: Let $(X,\le)$ be a partial ordered set such that there exist a complete complex valued generalized metric d on X and T be a weakly isotone increasing mapping on X. Suppose that, for every comparable $x, y \in X$ we have either

$$\begin{aligned} d(Tx,Ty) \le\, & a_1 \frac{d(x,Tx)\left[d(y,Ty)+d(y,Tx)\right]}{d(x,y)+d(Tx,y)} \\ & +a_2 \frac{d(y,Tx)\left[d(x,Tx)+d(y,Ty)\right]}{d(x,y)+d(Tx,y)} \\ & +a_3 \frac{d(y,Ty)\left[1+d(x,Tx)\right]}{1+d(x,y)} + a_4 \frac{d(x,Tx)\,d(y,Ty)}{d(x,y)} \\ & +a_5\left[d(x,Tx)+d(y,Ty)\right]+a_6 d(x,y) \end{aligned} \tag{3.2}$$

in case $d(x,y)+d(Tx,y) \ne 0$ and $1+d(x,y) \ne 0$ with $a_i,(i=1,2,3,4,5,6) \ge 0$ and $\sum_{i=1}^{6} a_i < 1, a_6 < 1$ or $d(x,y)+d(Tx,y) = 0$ and $1+d(x,y) = 0$ implies $d(Tx,Ty) = 0$.

If T is continuous or for any non decreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily has $x_n \le z$ for all $n \in N$, then T has a fixed point . Moreover the set of fixed points of T is totally ordered if and only if T has one and only one fixed point.

Next, we prove a periodic point result as an application.

A point p of T is also a fixed point of T^n for every $n \in \mathrm{N}$. However, the converse is false. For example, consider $X = [0,1]$, and defined T by $Tx = 1 - x$. Then T has a unique fixed point $\frac{1}{2}$ and every even iterate of T is the identity map, which has every point of [0,1] as a fixed point.

On the other hand, if $X = [0,\pi]$, $Tx = \cos x$, then every iterate of T has the same fixed point as T.

If a map T satisfies $F(T) = F(T^n)$ for each $n \in \mathrm{N}$, where $F(T)$ is the set of fixed point of T, then it is said to have property P [6]. The set $O(x,\infty) = x, Tx, T^2x, \ldots\ldots\ldots$ is called the orbit of x.

Theorem 3.2: Let $(X,\le)$ be a partial ordered set such that there exist a complete complex valued generalized metric d on X. Let T be a self map on X as in Corollary 3.2. If $O(x,\infty)$ is totally ordered, then T has property P.

Proof: From Corollary 3.2, T has a fixed point. Let $u \in F(T^n)$. Now from (3.2), we have

$$d(u,Tu) = d\left(T\left(T^{n-1}u\right), T\left(T^{n}u\right)\right)$$

$$\leq a_1 \frac{d\left(T^{n-1}u, TT^{n-1}u\right)\left[d\left(T^{n}u, TT^{n}u\right) + d\left(T^{n}u, TT^{n-1}u\right)\right]}{d\left(T^{n-1}u, T^{n}u\right) + d\left(TT^{n-1}u, T^{n}u\right)}$$

$$+ a_2 \frac{d\left(T^{n}u, TT^{n-1}u\right)\left[d\left(T^{n-1}u, TT^{n-1}u\right) + d\left(T^{n}u, TT^{n}u\right)\right]}{d\left(T^{n-1}u, T^{n}u\right) + d\left(TT^{n-1}u, T^{n}u\right)}$$

$$+ a_3 \frac{d\left(T^{n}u, TT^{n}u\right)\left[1 + d\left(T^{n-1}u, TT^{n-1}u\right)\right]}{1 + d\left(T^{n-1}u, T^{n}u\right)} + a_4 \frac{d\left(T^{n-1}u, TT^{n-1}u\right) d\left(T^{n}u, TT^{n}u\right)}{d\left(T^{n-1}u, T^{n}u\right)}$$

$$+ a_5\left[d\left(T^{n-1}u, TT^{n-1}u\right) + d\left(T^{n}u, TT^{n}u\right)\right] + a_6 d\left(T^{n-1}u, T^{n}u\right)$$

$$\leq a_1 \frac{d\left(T^{n-1}u, T^{n}u\right)\left[d\left(T^{n}u, TT^{n}u\right) + d\left(T^{n}u, T^{n}u\right)\right]}{d\left(T^{n-1}u, T^{n}u\right) + d\left(T^{n}u, T^{n}u\right)}$$

$$+ a_2 \frac{d\left(T^{n}u, T^{n}u\right)\left[d\left(T^{n-1}u, T^{n}u\right) + d\left(T^{n}u, TT^{n}u\right)\right]}{d\left(T^{n-1}u, T^{n}u\right) + d\left(T^{n}u, T^{n}u\right)}$$

$$+ a_3 \frac{d\left(T^{n}u, TT^{n}u\right)\left[1 + d\left(T^{n-1}u, T^{n}u\right)\right]}{1 + d\left(T^{n-1}u, T^{n}u\right)} + a_4 \frac{d\left(T^{n-1}u, T^{n}u\right) d\left(T^{n}u, TT^{n}u\right)}{d\left(T^{n-1}u, T^{n}u\right)}$$

$$+ a_5\left[d\left(T^{n-1}u, T^{n}u\right) + d\left(T^{n}u, TT^{n}u\right)\right] + a_6 d\left(T^{n-1}u, T^{n}u\right)$$

$$\leq a_1 \frac{d\left(T^{n-1}u, u\right)\left[d(u,Tu) + d(u,u)\right]}{d\left(T^{n-1}u, u\right) + d(u,u)}$$

$$+ a_2 \frac{d(u,u)\left[d\left(T^{n-1}u, u\right) + d(u,Tu)\right]}{d\left(T^{n-1}u, u\right) + d(u,u)}$$

$$+ a_3 \frac{d(u,Tu)\left[1 + d\left(T^{n-1}u, u\right)\right]}{1 + d\left(T^{n-1}u, u\right)} + a_4 \frac{d\left(T^{n-1}u, u\right) d(u,Tu)}{d\left(T^{n-1}u, u\right)}$$

$$+ a_5\left[d\left(T^{n-1}u, u\right) + d(u,Tu)\right] + a_6 d\left(T^{n-1}u, u\right)$$

$$\leq \left(a_1 + a_3 + a_4 + a_5\right) d(u,Tu) + \left(a_5 + a_6\right) d\left(T^{n-1}u, u\right)$$

which implies that

$$d(u,Tu) \leq \frac{a_5 + a_6}{1 - (a_1 + a_3 + a_4 + a_5)} d\left(T^{n-1}u, u\right)$$

or

$$d(u,Tu) \leq k d\left(T^{n-1}u, u\right)$$

where $k = \dfrac{a_5 + a_6}{1-(a_1+a_3+a_4+a_5)} < 1$.

Obviously $0 \le k < 1$ and we have

$$d(u,Tu) = d(Tu,T^n u)$$
$$\le kd(T^{n-1}u,T^n u) \le k^2 d(T^{n-2}u,T^{n-1}u) \le \cdots \le k^n d(u,Tu)$$

Since $0 \le k < 1$ implies $d(u,Tu)=0$ and $u = Tu$. This completes the proof.

Remark: In our result we use products and quotients of the metric values which is permissible in the structure of complex numbers. But this is not always the case with other generalizations of metric spaces as, for example, in cone metric, where the metric is real Banach space valued, we cannot use products and quotients of metric values.

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