Vol.5.Issue.3.2017 (July-Sept)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com
Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

VALUE CONCENTRATION OF ADDITIVE AND MULTIPLICATIVE FUNCTION DEFINED ON RANDOM PERMUTATION

MOHD. FAREEDUDDIN & E. KESHAVA REDDY
Department of Mathematics, JNTUA, Anantapur, Andhra Pradesh, INDIA

ABSTRACT

Let R_n be the symmetric set of permutations σ defined on $n \geq 1$ letters. Every $\sigma \in R_n$ has a unique representation by the product of independent cycles K , $\sigma = K_1, K_2, ...K_W$ where $W = W(\sigma)$ represent the number of cycles.For determine the value concentration of completely additive function and multiplicative functions defined on random permutations, the analog of the Kolmogorov-Rogozin inequality and Voronoi summability are analysed.

Keywords- Permutations, Ewens sampling, Kolmogorov-Rogozin inequality, Voronoi summability.

1. Introduction

The family of probability computes on the symmetric set R_n of permutations on $\{1, 2, ..., n\}$, influenced by the Ewens sampling formula are given by

$$V_{n,\theta}(\bar{K}) := \frac{n!}{\theta_{(n)}} \prod_{j=1}^{n} \left(\frac{\theta}{j}\right)^{K_j} \left(\frac{1}{K_j!}\right), \bar{K} := (K_1, K_2, ..., K_n) \in Z^{+^n}, \quad (1)$$

forthe partition $n = 1K_1 + 2K_2 + ... + nK_n$, $n \in N$, and 0 otherwise, where $\theta > 0$, and $\theta_{(n)} = \theta(\theta+1).....(\theta+n-1)$. The quantity $V_{n,\theta}(\bar{K})$ can also be viewed as the probability measure on the class of conjugate elements $\sigma \in R_n$, all having $K_j(\sigma) = K_j$ cycles of length $j, 1 \leq j \leq n$. The probability measure $V_{n,\theta}$ is persuaded by the measure $V'_{n,\theta}$ on R_n, that assigns a mass proportional to $\theta^{W(\sigma)}$ for $\sigma \in R_n$, where $W(\sigma) = K_1(\sigma) + ... + K_n(\sigma)$ represents the overall number of cycles of σ. This can be observed that

$$V_{\theta}^{'}(\sigma)=\theta^{W(\sigma)}\left(\sum_{\tau\in R_n}\theta^{W(\tau)}\right)^{-1}=\frac{\theta^{W(\sigma)}}{\theta_{(n)}}. \tag{2}$$

Thus, the probability measure on R_n and leave the similar notation $V_{n,\theta}$ for it.

The case $\theta=1$relates to the measure induced by the uniform probability $(1/n!)\#\{\sigma\in R_n:...\}$ on R_n. If $K_j(\sigma)=1$ for some $\frac{n}{2}<j\leq n,$ then $K_i(\sigma)=0$ for all $\frac{n}{2}<i\leq n, i\neq j$.

This influences deliberation of additive and multiplicative functionson R_n. A function $H:R_n\to S$ is known as additive if $H_j(0)=0$ and $H(\sigma)=\sum_{j=1}^{n}h_j(K_j(\sigma)),\sigma\in\bar{K}$. The measure $V_{n,\theta}$ can be represented using independent Poisson random variables ζ_j with $D(\zeta_j)=\frac{\theta}{j}$ as

$$V_{n,\theta}(\bar{K})=Q(\gamma_1=K_1,....,\gamma_n=K_n\mid\zeta=1\gamma_1+....+n\gamma_n) \tag{3}$$

Let $\sigma\in R_n$ be a random permutation and

$$\sigma=X_1,X_2,...X_W=:\prod_{X/\sigma}X \tag{4}$$

be its unique up to order expression by the result of the cycles X. Represent

$$V_n(...)=(n!)^{-1}\#\{\sigma\in R_n..\}$$

Consider $\delta(\sigma)$ represent the number of the cycles involving σ. Then,

$$V_n(\delta(\sigma)-\log n<X\sqrt{\log n})\to\psi(X):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{X}e^{-u^2/2}du \tag{5}$$

Where the limit is taken as $n\to\infty$. Observing the asymptotic distribution of the set theoretic order of the random permutation σ. The sum $c(\sigma)$ of the natural logarithms of the different cycle lengths $L(X)$, where X/σ also obeys the normal limit law. It was attained that

$$V_n(c(\sigma)-(1/2)\log^2 n<(1/\sqrt{3})X\ \log^{3/2}n)\to\psi(X) \tag{6}$$

2. Random Permutation

Let R_n be the symmetric group. The R_n is comprised of everyprobable functions that bijectively plot the group of first n integers $\{1,2,.....,n\}$ into itself. This is referred as permutations. All permutation σ belonging to the symmetric group R_n can be representedas an oriented graph with n vertices that are labeled by natural numbers $1,2,.....,n$ and n edges, all edge conforming to a set $(j,\sigma(j))$, beginning at vertex j and pointing to vertex $\sigma(j)$. Such graphs are characterized by the property that all edge has only one outgoing edge and one incoming edge.

Let the classes of additive and multiplicative functions on permutations whose values are computed by the decomposition of permutations into cyclical components. These functions are defined as follows. Suppose consider n real numbers $\hat{H}(1),\hat{H}(2),.....\hat{H}(n),$ then for every permutation $\sigma\in R_n$, compute a sum $H(\sigma)$ over all cycles in the graph of σ so that for every cycle of length j we add one summand $\hat{H}(j)$. Or equivalently,

$$H(\sigma)=\hat{H}(1)\alpha_1(\sigma)+\hat{H}(2)\alpha_2(\sigma)+,.....+\hat{H}(n)\alpha_n(\sigma) \tag{7}$$

Where $\alpha_j(\sigma)$ is the total of cycles of length j in permutation σ. The resulting additive function $\delta(\sigma)$ is then equivalent to the whole number of cycles of in the graph of σ. Then the number of permutations $\sigma \in R_n$ fulfilling inequality $\frac{\delta(\sigma)-\log n}{\sqrt{\log n}} < X$ divided by the overall total of permutations $|R_n| = n!$ converges to $\frac{1}{\sqrt{2\pi}} \leftarrow \int_{-\infty}^{X} e^{-u^2/2} du$ as $n \to \infty$.

Similarly, define multiplicative functions on the symmetric group R_n. Suppose consider n complex numbers $\hat{F}(1), \hat{F}(2), \ldots\ldots, \hat{F}(n)$. Then for every permutation $\sigma \in R_n$, assign a product $F(\sigma)$ over all cycles belonging to the oriented graph of σ that include one multiplicand $\hat{F}(j)$ corresponding to every cycle of size j belonging to σ. In other words

$$f(\sigma) = \hat{F}(1)^{\alpha_1(\sigma)}, \hat{F}(2)^{\alpha_2(\sigma)}, \ldots\ldots, \hat{F}(n)^{\alpha_n(\sigma)} \tag{8}$$

Where assume $0^0 = 1$. For this instance of $\lambda \in R_{15}$ we have

$$f(\lambda) = \hat{F}(1)^3, \hat{F}(2) \; , \hat{F}(3)^2 \, \hat{F}(4) \tag{9}$$

Suppose $p(\sigma)$ is a non-negative multiplicative function, which is not identically equal to zero. Then define a probabilistic measure $V_{n,p}$ on R_n by the formula

$$V_{n,p}(\sigma) = \frac{p(\sigma)}{\sum_{s \in R_n} p(\tau)} \tag{10}$$

The simplest and the most natural choice is to put $\hat{p}(j) \equiv 1$, which lead to the uniform probability measure

$$V_n^{(1)}(\sigma) = \frac{1}{n!}$$

Thus result can be expressed in probabilistic terms as a limit theorem

$$V_n^{(1)}\left(\frac{W(\sigma)-\log n}{\sqrt{\log n}} < X\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-u^2/2} du \text{ , as } n \to \infty \tag{11}$$

stating that the number of cycles $W(\sigma)$ in permutation σ chosen with equal probability among all the permutations of the symmetric group R_n is asymptotically normally distributed.

More generally if all $\hat{p}(j)$ are equal $\hat{p}(j) \equiv \theta > 0$ then $\hat{p}(j) = \theta^{W(\sigma)}$, thus this is called as a Ewens probability measure

$$V_n^{(\theta)}(\sigma) = \frac{\theta^{W(\sigma)}}{\sum_{s \in R_n} \theta^{W(\tau)}}$$

$$= \frac{\theta^{W(\sigma)}}{\theta(\theta+1)\ldots(\theta+n-1)} \tag{12}$$

Let us represent by $G_n^p(F)$ a weighted mean of a multiplicative function $F: R_n \to \mathrm{E}$ with respect to the measure $V_{n,p}(\sigma)$:

$$G_n^p(F) = \sum_{\sigma \in R_n} F(\sigma) V_{n,p}(\sigma)$$

$$= \frac{\sum_{\sigma \in R_n} F(\sigma) p(\sigma)}{\sum_{\sigma \in R_n} p(\sigma)} \tag{13}$$

3.Value Concentration of Additive Function on Random Permutation

Theorem 3.1

Let

$$P_n(m) \le Cm(E_n(m))^{-1/2}$$

Where $E_n(m) = o(m^2)$ as $n \to \infty$, the trivial estimate $P_n(m) \le 1$ is better. Observe that the Kolmogorov-Rogozin theorem applied for the sum $R_n := b(1)\gamma_1 + ... + b(n)\gamma_n$, where γ_j are the independent Poisson random variables yields the estimate

$$\sup_{X \in S} Q(X \le R_n < X + m) \le C_1 m(E_n(m;0))^{-1/2} \tag{14}$$

Thus, with a successful choice of λ, concentration estimate for $H(\sigma) - \lambda H^1(\sigma)$ is comparable with that for R_n

Proof for theorem 3.1

The proof for the theorem is split into various lemmas 3.2, 3.3, 3.4, 3.5 and 3.6

Lemma 3.2: Let $g : R_n \to C$ be a completely multiplicative function defined by

$$g(\sigma) = \prod_{j=1}^{n} a(j)^{K_j(\sigma)}, \quad 0^0 := 1 \tag{15}$$

with $a(j) \in C, |a(j)| \le 1$ for each $1 \le j \le n$. Then

$$\frac{1}{n!}\left|\sum_{\sigma \in R_n} g(\sigma)\right| \le C_2 \exp\left\{-\frac{1}{4}\min_{|u| \le \pi} \sum_{j=1}^{n} \frac{1 - \rho(a(j)e^{iuj}}{j}\right\}$$

Proof:

Apply Fourier transform of the distribution $V_n(H(\sigma) < X)$. In this case, $a(j) = e^{2\pi i a(j)t}, t \in S$, the first step will be analysis of the trigonometrical polynomial

$$y(u,t) := \sum_{j=1}^{n} \frac{1 - \cos 2\pi(b(j)t - uj)}{j} \tag{16}$$

Consider the values $u(t)$ giving

$$\min_{-1/2 \le u \le 1/2} y(u,t) = y(u(t),t), t \in [-1,1] \tag{17}$$

Use conditions for implicit functions while dealing with the stationary points, and observe that $u(t)$ is well defined continuous function in some nontrivial neighborhood of the point $t = 0, u(0) = 0$. Beyond it, if numerous values of $u(t)$ appear for a fixed t, select the least of them and so get the function $u(t)$ defined on the whole interval $[-1,1]$ and taking values in $[-1/2,1/2)$. It indicates that $u(t)$ is connected to a homomorphism of the additive sets S and $S/Z = T$. For convenience, identify T with the interval $[0,1)$ and yield addition modulo one. Notice that the group T is the whole metric space in terms of the metric defined via the distance to the nearest integer $\| X \| = \{X\} \wedge (1 - \{X\})$ which is not a norm. Besides, if $-1/2 \le u(t) < 0$, redefine this value to

$1+u(t)$ and so get the function $u:[-1,1]\to T$ maintaining continuity at the point $t=0$. Then observe the Cauchy equations with respect to $\| u\|$.

Lemma 3.3: Let $V:[-1,1]\to T$ be continuous at the point $t=0$ and $V(0)=0$. Suppose that, for some $0<\xi<1/18$,

$$\| V(t_1+t_2)-V(t_1)-V(t_2)\|\le \xi \tag{18}$$

Whenever $t_1, t_2, t_1+t_2\in[-1,1]$. Then

$$\| V(t)-\lambda t\|\le 3\xi \tag{19}$$

for some $\lambda\in S$ and all $t\in[-1,1]$

Proof

State $W:S\to T$ by

$$W(2K+t)=2Ku(1)+V(t)\bmod 1,\quad t\in[-1,1]$$

This function outspreads $V(t)$ to the real line, excluding at $t=1$, where $W(1)=2V(1)+V(-1)\bmod 1$. The function $L(t):=W(t)-V(1)t \bmod 1$ is 2-periodical. Thus estimating for all $X, Z\in S$,

$$e:=\| W(X+Z)-W(X)-W(Z)\|=\| L(X+Z)-L(X)-L(Z)\| \tag{20}$$

confine to the values $X, Z\in[-1,1)$. If also $X+Z\in[-1,1)$, then by equation (6) $e\le n$. If $X+Z\ge 1$, we get

$$\begin{aligned} e &=\| 2V(1)+V(X+Z-2)-V(X)-V(Z)\| \\ &\le e=\| V(X+Z-2)+V(X-1)-V(Z-1)\|+\| V(X-1)+V(1)-V(X)\|+ \\ &\quad |V(Z-1)+V(1)-V(Z)\|\le 3\xi \end{aligned} \tag{21}$$

Lemma 3.4: For $0\le X\le 1/2$ and $0\le\Theta\le 10$, we have $l(\Theta X,0)\le l(X,0)+C_3$

Proof

Let $s:=e^{-1/n}$ and

$$\phi(z):=\sum_{j=1}^{\infty}\frac{1-\cos 2\pi jz}{j}s^j=\log\frac{|1-se^{2\pi iz}|}{1-s}$$

$$=\frac{1}{2}\log\left(1+\frac{4s}{(1-s)^2}\sin^2\pi z\right)$$

By virtue of

$$\sum_{j=1}^{n}\frac{1-s^j}{j}+\sum_{j>n}\frac{s^j}{j}\le 1+\int_n^{\infty}\frac{s^X}{X}dX<1+e^{-1}, n\ge 1 \tag{22}$$

We have $|l(z,0)-\phi(z)|<3$. Thus it remains to check if $\phi(\theta X)\le\phi(X)+C_4$ in the region presumed in the lemma. Then

$$C(\Theta):=\max_{0\le X\le 1/2}\frac{\sin^2\pi\Theta X}{\sin^2\pi X}\le C_5$$

for $0\le\Theta\le 10$. Thus

$$\phi(\Theta X)\le\frac{1}{2}\log\left(1+\frac{4rC(\Theta)}{(1-s)^2}\right)\le\phi(X)+\frac{1}{2}\log(\max(1,C_5)) \tag{23}$$

aschosen. As a result, lemma 3.4 is verified.

Lemma 3.5: Let D be a constant such that, for a continuous at the point $t=0$ function $U:[-1,1]\to T, U(0)=0$, then

$$D(U(t),t) \le D \quad (24)$$

for all $t \in [-1,1]$. Then, for some $\lambda \in S$,

$$D(\lambda t,t) \le 20D + 2C_3, \quad t \in [-1,1] \quad (25)$$

and

$$E_n(1,\lambda) \le (10D + C_3)C_6 \quad (26)$$

Proof:

Set

$$\alpha = \sup\{\| U(t_1+t_2) - U(t_1) - U(t_2) \| : t_1, t_2, t_1+t_2 \in [-1,1]\}$$

If $\alpha = 0$, then $\| U(t) - \lambda t \| = 0$ and inequality (25) follows from (24). If $\alpha > 0$, chose $t_1, t_2, t_1+t_2 \in [-1,1]$ so that

$$\beta := \| U(t_1+t_2) - U(t_1) - U(t_2) \| \ge \frac{9}{10}\alpha .$$

For arbitrary $t \in [-1,1]$ with $\xi = \alpha$, we have $\beta_1 := \| U(t\ ) - \lambda t \| \le 9\alpha \le 10\beta$. Since the first inequality is trivial for $\alpha \ge 1/18$, avoid the condition on α. Now, by virtue of Lemma 3.4, $d(\beta_1,0) \le d(\beta,0) + C_3$. The inequality

$$1 - \cos(X_1 + ... + X_K) \le K((1-\cos X_1) + ... + (1-\cos X_K)) \quad (27)$$

and equation (24) obtain

$$\begin{aligned} d(\beta,0) &\le 3(d(U(t_1+t_2), t_1+t_2) + d(U(t_1),t_1) + d(U(t_2),t_2)) \\ &\le 9D. \end{aligned}$$

Again by equation (27) obtain bound equation (25):

$$\begin{aligned} d(\lambda t,t) &\le 2d(U(t),t) + 2d(\beta_1,0) \\ &\le 2D + 2d(\beta\ ,0) + 2C\ _3 \le 20D + 2C_3 \end{aligned} \quad (28)$$

Incorporating the trigonometrical polynomial $d(\lambda t,t)$ over the interval [0,1] and by the inequality $1 - \sin X / X \ge c_1 \min\{1, X^2\}$, where $X \in S$ and $c_1 > 0$ is an absolute constant. Thus, Lemma 3.5 is verified.

Lemma 3.6: If $\hat{X} \in [-1,1]$ is a set of positive Lebesgue measure, symmetric to the origin and containing it, then we have

$$\hat{X}^{s} = \{X_1 + ... + X_s : X_1...., X_s \in \hat{X}\} \supset [-1,1] \quad (29)$$

In case that $S = [12 / meas(\hat{X})]$

Proof

For $m > 0$, it avails to deal with $P_n(1)$, only and then apply the result for $H(\sigma)/m$. Then

$$P_n(1) \le \frac{C_7}{n!}\int_{-1}^{1}\left|\sum_{\sigma\in R_n} e^{2\pi i t H(\sigma)}\right| dt \le C_8 \int_{-1}^{1}\left\{-\frac{1}{4}\min d(U,t)\right\} dt \quad (30)$$

Set

$$\hat{X}_K = \left\{t \in [-1,1] : \min_{U\in T} d(U,t) \le K\right\}, \quad K = 1,2,...$$

These sets are nonempty measurable, symmetric with respect to the origin, and having the Lebesgue measure $\mu_K := meas(\hat{X}_K) > 0$. For $X = \hat{X}_K$, then the set sum $\hat{X}^s$ involves the interval $[-1,1]$ if $s = [12/\mu_K]$ i.e., this means that every $t \in [-1,1]$ has an expression $t = t_1 + ... + t_s$ such that $d(U_d, t_d) \le K$ with some $U_d \in T$, $1 \le d \le s$. Thus, by equation (27) obtain

$$d(U,t) \le Ks^2$$

for any $t \in [-1,1]$ and $U = U_1 + ... + U_s$ mod 1, $U \in T$. The same holds for the function $U(t)$ discussed above. So, by Lemma 3.5,

$$E_n(1,\lambda) \le C_6(10Ks^2 + C_3) \le C_9 K \mu_K^{-2}$$

for some $\lambda \in S$, or equivalently,

$$\mu_K \le C_{10}(K / E_n(1))^{1/2}$$

This and equation (5) deduces

$$P_n(1) \le C_8 \sum_{K \ge 1} e^{-K/4} \mu_{K+1} \le C_{11}(E_n(1))^{-1/2} \tag{31}$$

Therefore, lemma 3.6 is proved.

4.Theorem for Multiplicative Function on Random Permutation

Theorem 4.1:

Let $g : R_n \to C$ be a multiplicative function filling the condition $|g(\sigma)| \le 1$ for all $\sigma \in R_n$. Assume that the measure defining multiplicative function $a(\sigma)$ is such that $0 < a^- \le \hat{a}(j) \le a^+$. Then

$$\Delta_n := \left| D_n^a(g) - \exp\left\{ \sum_{j=1}^{n} \hat{a}(j) \frac{\hat{a}(j) - 1}{j} \right\} \right|$$

$$\le c_1 \left(\sum_{j=0}^{n} q_j \right)^{-1} \sum_{K=1}^{n} \left| \hat{g}(j) - 1 \right| q_{n-K} + \frac{1}{n^{a^-}} \sum_{K=1}^{n} \left| \hat{g}(j) - 1 \right| K^{a^- - 1} + \frac{1}{n} \sum_{K=1}^{n} \left| \hat{g}(j) - 1 \right| \tag{32}$$

for $a^- < 1$ and

$$\Delta_n \le c_1 \left(\left(\sum_{j=0}^{n} q_j \right)^{-1} \sum_{K=1}^{n} \left| \hat{g}(j) - 1 \right| q_{n-K} + \frac{1}{n} \sum_{K=1}^{n} \left| \hat{g}(j) - 1 \right| \left(1 + \log \frac{n}{k} \right) \right) \tag{33}$$

for $a^- \ge 1$, where $c_1 = c_1(a^-, a^+)$ is appositive constant which depends on a^- and a^+ only, and

$$q_n = \frac{1}{n!} \sum_{\sigma \in R_n} a(\sigma) = [z^n] \exp\left\{ \sum_{j=1}^{\infty} \frac{\hat{a}(j)}{j} z^j \right\}$$

Observe that if function $H(\sigma)$ is additive then function $\exp(ith(\sigma))$ is multiplicative which defines that the characteristic function of an additive function with respect to measure is a mean value of a multiplicative function. It follow that the estimate for the mean values of multiplicative functions allows us to get information on the distribution of the values of additive functions.

Let denote

$$W(n) = \sum_{K=1}^{n} \hat{a}(K) \frac{\hat{H}_n(K)}{K}$$

$$C_n = \sum_{K=1}^{n} \hat{a}(j) \frac{\hat{H}_n(j)}{j} \left(\frac{q_{n-j}}{q_n} - 1 \right)$$

$$M_{n,q} = \sum_{K=1}^{n} \frac{\left| \hat{H}_n(K) \right|^q}{K}$$

$$M'_{n,2} = \sum_{j=1}^{n} \frac{\hat{H}_n(j)^2}{j} \left| \frac{q_{n-j}}{q_n} - 1 \right|$$

Henceforth assume that $\hat{H}_n(K)$ satisfies the normalizing condition

$$\sum_{K=1}^{n} \hat{a}(K) \frac{\hat{H}_n(j)^2}{K} = 1$$

Proof

Since $\hat{g}(K)$ for $K > n$ do not influence the n-th Taylor coefficient D_n for the generating function $D(z)$, consider that $\hat{g}(K) = 1$ for $K > n$. Then the inequality

$$\left| R(d;j) \right| = \left| [z^{j-1}] q(z) d'(z) \right| \le a^+ \sum_{K=1}^{n} \left| \hat{a}(K) - 1 \right| q_{j-K} \tag{34}$$

By the condition $h(z) = d(z)$, then get inequality

$$\left| \frac{D_n}{q_n} - d(e^{-1/n}) \right| = \left| \frac{D_n}{q_n} - \exp\left\{ \sum_{K=1}^{n} \frac{a_K (\hat{g}(K) - 1)}{K} e^{-K/n} \right\} \right|$$

$$\le c \left(\frac{1}{nq_n} \sum_{K=1}^{n} \left| \hat{g}(K) - 1 \right| q_{n-K} + \frac{1}{n^\theta} \sum_{j=1}^{n} \frac{j^{\theta-1}}{q(e^{-1/j})} \sum_{k=1}^{j} \left| \hat{g}(K) - 1 \right| q_{n-K} + \frac{1}{q(e^{-1/n})} \sum_{j>n} \frac{e^{-j/n}}{j} \sum_{k=1}^{j} \left| \hat{g}(K) - 1 \right| q_{j-K} \right)$$

Where $\theta = \min\{1, a^-\}$ and $c = c(a^+, a^-)$. After computing simpler estimates for the second and the third term in the sum on the right hand side of the above inequality, inequality observed in the formulation of the theorem. Further, consider the second term. Varying the order of summation, then

$$\frac{1}{n^\theta} \sum_{j=1}^{n} \frac{j^{\theta-1}}{q(e^{-1/j})} \sum_{k=1}^{j} \left| \hat{g}(K) - 1 \right| q_{n-K} = \frac{1}{n^\theta} \sum_{k=1}^{n} \left| \hat{g}(K) - 1 \right| \sum_{n \ge j \ge K} \frac{j^{\theta-1}}{q(e^{-1/j})} q_{j-K}$$

$$<< \frac{1}{n^\theta} \sum_{k=1}^{n} \left| \hat{g}(K) - 1 \right| \sum_{n \ge j \ge K} \frac{j^{\theta-1}}{q(e^{-1/j})} q_{j-K} \frac{q(e^{-1/(j-k)})}{j-K}$$

$$+ \frac{1}{n^\theta} \sum_{k=1}^{n} \left| \hat{g}(K) - 1 \right| K^{\theta-1} \sum_{2k \ge j \ge K} \frac{q_{j-K}}{q(e^{-1/j})} \tag{35}$$

As a Taylor series with positive coefficients function $q(X)$ is increasing for increasing values of X. Hence for $j > K$ we have $q(e^{-1/(j-K)}) \le q(e^{-1/j})$ and $q(e^{-1/j}) \ge q(e^{-1/K})$. By these inequalities to calculate the last estimate we obtain,

$$\frac{1}{n^{\theta}}\sum_{j=1}^{n}\frac{j^{\theta-1}}{q(e^{-1/j})}\sum_{k=1}^{j}\left|\hat{g}(K)-1\right|q_{j-K}$$

$$<<\frac{1}{n^{\theta}}\sum_{K=1}^{n}\left|\hat{g}(K)-1\right|\sum_{n\geq j\geq 2K}j^{\theta-2}+\frac{1}{n^{\theta}}\sum_{K=1}^{n}\left|\hat{g}(K)-1\right|^{\theta-1}\frac{1}{q(e^{-1/K})}\sum_{r=0}^{K}q_{r}$$

$$<<\frac{1}{n^{\theta}}\sum_{K=1}^{n}\left|\hat{g}(K)-1\right|\left(K^{\theta-1}+\int_{K}^{n}X^{\theta-2}dX\right) \tag{36}$$

The third term can be defined as

$$\frac{1}{q(e^{-1/n})}\sum_{j>n}\frac{e^{-j/n}}{j}\sum_{k=1}^{j}\left|\hat{g}(K)-1\right|q_{j-K}=\frac{1}{q(e^{-1/n})}\sum_{K=1}^{n}\left|\hat{g}(K)-1\right|\sum_{j>n}\frac{e^{-j/n}}{j}q_{j-K}$$

$$\frac{1}{n}\sum_{K=1}^{n}\left|\hat{g}(K)-1\right|q_{n-K}\frac{e^{-K/n}}{q(e^{-1/n})}\sum_{j>n}^{n}e^{-(j-K)/n}q_{j-K}\leq\sum_{k=1}^{n}\left|\hat{g}(K)-1\right|$$

The inequality in statement of the theorem observed as

$$\left|\sum_{K=1}^{n}\frac{a_K(\hat{g}(K)-1)}{K}e^{-K/n}-\sum_{K=1}^{n}\frac{a_K(\hat{g}(K)-1)}{K}\right|\leq\frac{1}{n}\sum_{k=1}^{n}a_K\left|\hat{g}(K)-1\right| \tag{37}$$

to estimate the quantity under exponent in the inequality.

The theorem is verified.

Theorem 4.2

For any fixed $\infty\geq q>\max\{1,1/a^{-}\}$, there exists such a positive $\gamma=\gamma(a^{-},a^{+},q)$ that if $\upsilon\leq\gamma$, then

$$\frac{D_N}{q_N}=\exp\{M_N(1)\}\left(1+\sum_{j=1}^{N}a_j\frac{\hat{g}(j)-1}{j}\left(\frac{q_{N-j}}{q_N}-1\right)+O(\upsilon^2)\right) \tag{38}$$

Proof

The values of $\hat{g}(j)$ with $j>N$ do not influence the value of D_N, then deduce that $\hat{g}(j)=1$ for all $j>N$. Consider $R(h;n)$ with

$$h(z)=U_N(z)=d(z)-d(e^{-1/N})\sum_{j=1}^{\infty}a_j\frac{\hat{g}(j)-1}{j}z^j \tag{39}$$

instead of $h(z)=d(z)$ then

$$R(U_N;n)=[z^{n-1}]q(z)U_N^{'}(z)$$

$$=R(d;n)-[z^{n-1}]d(e^{-1/N})q(z)\sum_{j=1}^{\infty}a_j(\hat{g}(j)-1)z^{j-1}$$

$$=\sum_{j=1}^{n}a_j(\hat{g}(j)-1)(D_{n-j}-d(e^{-1/N})q_{n-j})$$

Applying here the estimate $D_{n-j}=q_{n-j}d(e^{1/(n-j)})(1+O(\upsilon))$, then get

$$R(U_N;n)<<\sum_{j=1}^{n}\left|\hat{g}(j)-1\right|\left|d(e^{1/(n-j)})-d(e^{-1/N})q_{n-j}\right|$$

$$+\upsilon\sum_{j=1}^{n}\left|\hat{g}(j)-1\right|q_{n-j}\left|d(e^{1/(n-j)})\right| \tag{40}$$

The second sum of the above estimate has already been already shown to be $O(\upsilon^2 nq_n \left|d(e^{-1/n})\right|)$.

Hence,

$$R(U_N;n) << \left(\frac{1}{n}\sum_{j=1}^{n-1}\left|d(e^{-1/j}) - d(e^{-1/N})\right|^p q_j^p\right)^{1/p} + \upsilon^2 nq_n \left|d(e^{-1/n})\right|$$

$$<< n\upsilon^2 \left|d(e^{-1/N})\right| \left(\frac{1}{n}\sum_{j=1}^{n-1}\exp\left\{pa^+\upsilon\left|\log\frac{N}{j\vee 1}\right|\right\}q_j^p\right)^{1/p} + \upsilon^2 nq_n \left|d(e^{-1/n})\right| \quad (41)$$

Since $e^{\beta|\log X|} \le X^{\beta} + X^{-\beta}$ if $\beta > 0$, further estimate

$$R(U_N;n) << n\upsilon^2 \left(\frac{1}{n}\sum_{j=1}^{n-1} q_j^p \left(\left(\frac{N}{j\vee 1}\right)^{pa^+\upsilon} + \left(\frac{j}{N}\right)^{pa^+\upsilon}\right)\right)^{1/p} + \upsilon^2 nq_n \left|d(e^{-1/n})\right|$$

$$<< n\upsilon^2 qn \left|d(e^{-1/n})\right| \left(\left(\frac{N}{n}\right)^{pa^+\upsilon} + \left(\frac{n}{N}\right)^{pa^+\upsilon}\right) + \upsilon^2 nq_n \left|d(e^{-1/n})\right| \quad (42)$$

After plugging this estimate into the inequality for Voronoi mean, we obtain the estimate

$$\frac{1}{qN}[z^N]q(z)U_N(z) - U_N(e^{-1/N}) << \upsilon^2 \left|d(e^{-1/n})\right|$$

This after recalling the definition of U_N becomes

$$\frac{D_N}{q_N} = d(e^{-1/N})\left(1 + \sum_{j=1}^{N} a_j \frac{\hat{g}(j)-1}{j}\left(\frac{q_{N-j}}{q_N} - e^{-j/N}\right) + O(\upsilon^2)\right)$$

Applying here estimate

$$d(e^{-1/N}) = \exp\{M_N(e^{-1/N})\} = \exp\{M_N(1)\}\left(1 + M_N(e^{-1/N}) - M_N(1) + O(\upsilon^2)\right) \quad (43)$$

Thus the theorem is verified.

5. Conclusion

This Paper discusses a value concentration of additive and multiplicative function on random permutation. The analog of the Kolmogorov-Rogozin inequality is used for determine the value concentration of completely additive function defined on random permutations. For the multiplicative function, the Voronoi summability are used to analyse the value concentration of multiplicative functions on random permutation.

REFERENCES

[1]. Babu, G. J. (2006). Probabilistic number theory and random permutations: Functional limit theory. *The Riemann Zeta function and related themes, R. Balasubramanian and K. Srinivas (eds.). Ramanujam Mathematical Society–Lecture Notes Series*, (2), 19-27.

[2]. Manstavičius, E. (1996). Additive and multiplicative functions on random permutations. *Lithuanian Mathematical Journal*, *36*(4), 400-408.

[3]. Manstavičius, E. (1998). The law of the iterated logarithm for random permutations. *Lithuanian Mathematical Journal*, *38*(2), 160-171.

[4]. Manstavičius, E. (2007). Moments of additive functions on random permutations. *ActaApplicandaeMathematicae*, *97*(1), 119-127.

[5]. KARGINA, T. (2009). Asymptotic distributions of the number of restricted cycles in a random permutation. *Liet. mat. rink. LMD darbai*, *50*.

[6]. Bakshajeva, T., & Manstavičius, E. (2014). On statistics of permutations chosen from the Ewens distribution. *Combinatorics, Probability and Computing*, *23*(06), 889-913.

[7]. Manstavičius, E. (2008). Asymptotic value distribution of additive functions defined on the symmetric group. *The Ramanujan Journal*, *17*(2), 259-280.

[8]. Babu, G. J., Manstavicius, E., &Zacharovas, V. (2005). Limiting processes with dependent increments for measures on symmetric group of permutations. *Probability and number theory—Kanazawa*, *49*, 41-67.

[9]. Manstavicius, E. (1997). A Tauber theorem and multiplicative functions on permutations. *Number theory in progress*, *2*, 1025-1038.

[10]. Wieand, K. L. (1998). *Eigenvalue in distribution of random matrices in the permutation group and compact lie groups* (Doctoral dissertation, Department of Mathematics, Harvard University).