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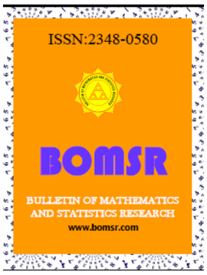
RESEARCH ARTICLE

LOCALLY HAUSDORFF SPACES

MALLIKA MITRA*, NASIMA AKHTER, SUBRATA MAJUMDAR

*mallika@vu.edu.bd



ABSTRACT

In this paper locally Hausdorff spaces of three kinds have been defined as generalizations of a Hausdorff space. Their interrelationships and properties of the first kind have been studied. It has been shown that the class of such spaces of the first kind is closed under formation of arbitrary products, quotient spaces and disjoints union-spaces. It has been proved that this class is precisely the class of T_1-spaces.Two conjectures have been made regarding one the structure of this class in terms of Hausdorff spaces and cofinite spaces, and the other about the largest topology of a space in this class.

Four characterizations of a Hausdorff space have been proved two of which describe the axiom T_2 as an obvious generalization of T_1. At the end, Hausdorffifications of a locally Hausdorff space of the first kind have been described.

Keywords: Locally Hausdorff space, kc-space, us-space, S_1-space, S_2-space,T^a-space,T^b-space, Hausdorffification, Anti-Hausdorff space.

AMS Classification (2010) : 54 D10, 54 G12, 54 D15.

1. Introduction

Many authors have studied topological spaces with separation axioms between T_1 and T_2 and also those which are weaker than T_1 ([1], [2], [3], [4], [5], [6], [7], [8], [9] ,[10], [11], [12], [13], [14]). In this paper we have defined three kinds of locally Hausdorff spaces. We have used definitions and terminology of [12] in general.

Definition 1.1

A topological space X will be called a **locally Hausdorff space of the first kind** if each point x in X is contained in a closed set F_x which is Hausdorff.

Definition 1.2

A topological space X will be called a **locally Hausdorff space of the second kind** if each point x in X is contained in an open set G_x which is Hausdorff.

Definition 1.3

A topological space X will be called a **locally Hausdorff space of the third kind** if each point x in X is contained in an open set G_x such that $\overline{G}_x$ is Hausdorff.

Relationship among the above three classes of spaces and properties of the spaces of the first kind have been studied. In particular, it has been proved that the topology of each space in this class contains the cofinite topology. This class has been proved to be identical with class of T_1-spaces. We recall ([2], [4], [11]) that a topological space X is called a **T_1-space** if any one of the equivalent conditions (i), (ii) and (iii) hold, where (i), (ii) and (iii) are as follows:

i) For each pair of distinct elements x, y in X, there exists an open set G_{xy} such that $x \in G_{xy}$ and $y \notin G_{xy}$

ii) For each pair of distinct elements x, y in X, there exist open set G_{xy} and H_{yx} such that $x \in G_{xy}$, $y \notin G_{xy}$ and $y \in H_{yx}$, $x \notin H_{yx}$.

iii) For each x in X, {x} is closed.

While (i) and (ii) occur in definitions, (iii) is usually proved to be their equivalent (see [2] p-47, [12] p-100, [11] p-100).

The locally Hausdorff spaces of the second and the third kinds will be called T_a-spaces and T_b-spaces respectively. It has been shown that $T_2 \Rightarrow T_b \Rightarrow T_a \Rightarrow T_1$, a, b $\in$ [1, 2], where the last implication is not reversible.

We have stated a conjecture on the structure of a locally Hausdorff space of the first kind using Hausdorff spaces and cofinite spaces as building blocks and formation of products, union-spaces, quotient spaces and subspaces as glues.

Four characterizations of a T_2-spaces have been given one of which describes T_2 as a natural generalization of T_1. At the end of this paper, new topologies have been introduced in a locally Hausdorff space of the first kind, i.e., a T_1-space, which makes the latter Hausdorff. Throughout the section 2 and section 3, a locally Hausdorff space of the first kind will simply be called locally Hausdorff.

Example 2.1

Every Hausdorff space is locally Hausdorff.

Example 2.2

Let X be an infinite set and let C_X be the cofinite topology on X. Then (X, C_X) is locally Hausdorff. A proof of this statement directly from the definition is as follows:

Let a∈X and b,c,d be three distinct points of X each of which is different from a. Then {a,b,c,d} is a closed subset, say F, of X. Here a∈X. Then $\{a,b\} \cup F^c$, $\{a,c\} \cup F^c$, $\{a,d\} \cup F^c$, $\{b,c\} \cup F^c$, $\{b,d\} \cup F^c$, $\{c,d\} \cup F^c$ are open subsets of X. And so, V_1={a,b}, V_2={a,c}, V_3={a,d}, V_4={b,c}, V_5={b,d}, V_6={c,d} are open sets in F.

Now a and b are separated by V_2 and V_5, a and c are separated by V_1 and V_6, a and d are separated by V_1 and V_6, b and c are separated by V_1 and V_6, b and d are separated by V_3 and V_4, c and d are separated by V_2 and V_5.

Thus F is Hausdorff. Hence X is locally Hausdorff.

Comment 2.1

Surprisingly, this space (X, C_X) is anti-Hausdorff (see [10],[13]), i.e., no two distinct points of it can be separated by disjoint open sets.

Example 2.1

Every T_1-space is locally Hausdorff.

In fact the converse is true too.

Thus we have:

Theorem 2.1

A topological space is locally Hausdorff if and only if it is T_1.

Proof: Let X be a topological space. First suppose X is T_1. Then for each x in X, {x} is closed. Now, {x} is Hausdorff. For, if not, there exist distinct points y, z ∈{x}, such that y and z cannot be separated by disjoint open sets. Since no two distinct points exist in {x}, this is absurd. Hence x is contained in the Hausdorff closed subspace {x}. Thus X is locally Hausdorff.

Conversely, suppose X is locally Hausdorff. Let x_1, x_2 be two distinct points in X. Since X is locally Hausdorff, there exists closed Hausdorff subspaces F_1 and F_2 of X such that $x_1 \in F_1$, $x_2 \in F_2$. If both x_1, x_2 belong to either F_1 or F_2, say F_1, then there exist disjoint open sets G_1 and G_2 in F_1 such that$x_1 \in G_1$, $x_2 \in G_2$.Now $G_1=V_1 \cap F_1$ and $G_2=V_2 \cap F_1$ for some open sets V_1, V_2 in X. Then, $x_1 \in V_1$, $x_1 \notin V_2$, $x_2 \in V_2$, $x_2 \notin V_1$. Now, if both x_1, x_2 do not belong to F_1 or F_2, then we may assume that $x_1 \in F_1$, $x_1 \notin F_2$,$x_2 \in F_2$, $x_2 \notin F_1$. So, $x_1 \in F_2^c, x_1 \notin F_1^c$, $x_2 \in F_1^c, x_2 \notin F_2^c$ where F_1^c and F_2^c are open sets in X.

Thus in each case the T_1-condition is satisfied.

Comment 2.2

In view of Theorem-2.1, the space (X, C_X) is easily seen to be locally Hausdorff since each singleton subset of X is closed.

Comment 2.3

The topology of an infinite locally Hausdorff space contains the cofinite topology. This is so, because for an infinite locally Hausdorff space X, every singleton subset, and hence, every finite subset is closed. In fact, a topological space (X, $\Im$) is locally Hausdorff if and only if $\Im$ contains the cofinite topology. This is obvious.

Although, as mentioned earlier, the topology of an infinite locally Hausdorff space X contains the cofinite topology on X, it need not be equal to the cofinite topology. The usual space R^n being Hausdorff is locally Hausdorff with the topology strictly finer than the cofinite topology.

We give below two examples of locally Hausdorff spaces which are not Hausdorff but still whose topologies are strictly finer than the relevant cofinite topologies :

Example 2.3

Let A be an infinite set, and for each $\alpha \in A$, let X_α be an infinite set with the cofinite topology. Let X= $\prod_{\alpha \in A} X_\alpha$ be the product space.

Let G= $\prod_{\alpha \in A} G_\alpha$, where $G_\alpha =\{x_{\alpha_1}, \ldots, x_{\alpha_1}\}^c$. Then each G_α^c is finite but G^c is not finite. Also G is open in X.

Example 2.4

Let X_1 be an infinite set with the cofinite topology $\Im_1$, and let X_2 be a finite set disjoint from X_1 with the discrete topology. Let X= ($X_1 \cup X_2$, $\Im$), where $\Im$ is the topology on X generated by $\Im_1 \cup \Im_2$, i.e., X is the union space of X_1 andX_2 . Then $\Im$ is strictly finer than the cofinite topology on X, since the (finite) subsets of X_2 are open in X but their complements in X are not finite. X is not Hausdorff, since the distinct points in X_1 cannot be separated by disjoint open sets in X.

In particular, we may take X_1=lN or $\mathbb{Z}$ and X_2={i, 2i, 3i, …, 10i} or $\{\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7}\}$, or $\{\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{1}{7},\frac{1}{11}\}$

Comment 2.4

I. Let X be a compact locally Hausdorff space. Then, for each point x∈X there exists a compact Hausdorff subspace K of X such that $x\in K$. This is so because every closed subset of a compact space is compact.

II. Let X be a compact locally Hausdorff space, then, every point x∈X is contained in both a closed Hausdorff space of X and a compact Hausdorff subspaces of X.

Example: Let X= [a,b] with the usual topology and Y=iz ={ip|p∈Z} with the cofinite topology. Then X×Y is a non-trivial compact locally Hausdorff spaces.

2. Here we prove a number of theorems yielding newer classes of locally Hausdorff spaces.

Theorem 3.1

Every subspace of a locally Hausdorff space is locally Hausdorff .

Proof: Let X be a locally Hausdorff space and let Y be asubspace of X. Let y_1, y_2 be two distinct points in Y. By Theorem-1, X is T_1, and so, there exist open sets G_1 and G_2 in X such that $y_1\in G_1$, $y_1 \notin G_2$, $y_2\in G_2$, $y_2 \notin G_1$. Let $V_1 = Y \cap G_1$, $V_2 = Y \cap G_2$.Then $y_1\in V_1$, $y_1 \notin V_2$, $y_2\in G_2$, $y_2 \notin V_1$. Hence Y is T_1 and so, Y is locally Hausdorff.

Corollary-1

Let A and B be two locally Hausdorff subspaces of a topological space X. Then, $A\cap B$ is locally Hausdorff

Theorem 3.2

Let X be a topological space and let A and B be two locally Hausdorff subspaces of X. Then, $A\cup B$ is locally Hausdorffif either both A and B are open or both A and B are closed.

Proof: Suppose both A and B are open. Let $x, y \in A\cup B$. If both x and y belong to A then there exists open sets G and H in A such that $x\in G, x\notin H, y\in H, y\notin G$. Since A is open in X, G and H too are open in X. Similarly, if both $x, y \in B$, then $\exists$ open sets G, H in X such that $x\in G, x\notin H$, $y\in H, y\notin G$.

Lastly, if one of x and y belongs to A and the other belongs to B, say $x\in A, x\notin B$, $y\in B, y\notin A$, then we will still have the above situation with G=A, H=B. Thus $A\cup B$ is T_1 and hence locally Hausdorff.

Now suppose both A and B are closed. Let $x\in A\cup B$. Then $x\in A$, or $x\in B$, suppose $x\in A$. A being locally Hausdorff, there exists a closed subset F of A such that $x\in F$, and F is Hausdorff. Since A is closed in X, F is closed in X, and so in $A\cup B$. Similarly, if $x\in B$, then there is a closed subset P which is Hausdorff and is closed in $A\cup B$. Thus $A\cup B$ is locally Hausdorff.

Theorem 3.3

Let $(X_1, \Im_1)$ and $(X_2, \Im_2)$ be two disjoint locally Hausdorffspaces. Let $\Im = \langle \Im_1 \cup \Im_2 \rangle$ be the topology generated by $\Im_1 \cup \Im_2$ on X= $X_1 \cup X_2$. Then (X, $\Im$) is locally Hausdorff.

Proof: We shall show that (X, $\Im$) is T_1. Let x,y be two distinct points in X. If x,y both belong to X_1or X_2, say to X_1. Then there exist open sets V, W in X_1, and hence in X, $x\in V, x\notin W$, $y\in W, y\notin V$. If x and y belong to X_1 and X_2 respectively, then $x\in X_1$, $x\notin X_2$, $y\in X_2$, $y\notin X_1$, and X_1, X_2 are open sets in X. Thus X is T_1, and so, locally Hausdorff.

We call (X, $\Im$) the union space of $(X_1, \Im_1)$ and $(X_2, \Im_2)$.

An exactly similar proof will yield

Theorem 3.4

Let (X_α, $\Im_\alpha$) be a non-empty family of locallyHausdorff spaces, then (X, $\Im$), where X=$\underset{\alpha}{\cup} X_\alpha$ and $\Im = \left\langle \underset{\alpha}{\cup} \Im_\alpha \right\rangle$, the topology generated by $\underset{\alpha}{\cup} \Im_\alpha$, is locally Hausdorff.

Theorem 3.5

Let {X_α, $\Im_\alpha$} be a non-empty family of topological space, and let X= $\underset{\alpha\in\wedge}{\Pi} X_\alpha$, the product space. Then X is locally Hausdorff spaces if and only if each X_α is locally Hausdorff.

Proof: We first assume that each X_α is locally Hausdorff and prove that X= $\underset{\alpha\in\wedge}{\Pi} X_\alpha$ is locally Hausdorff too. Let $x = \{x_\alpha\} \in X$. Since each X_α is locally Hausdorff, for each α, there exists a closed Hausdorff subspace F_α of X_α such that $x_\alpha \in F_\alpha$. Then F = $\underset{\alpha\in\wedge}{\Pi} F_\alpha$ is a closed subspace of X. Hence X is locally Hausdorff.

Conversely suppose that X is locally Hausdorff. We shall show that each X_α is locally Hausdorff. Let $\Pi_\alpha : X \to X_\alpha$ be the projection map. Let $x_\alpha \in X_\alpha$. Then $x_\alpha = \Pi_\alpha(x),$ for some $x \in X$. Let {F_{α_i}} be the collection of all closed subsets of X_α which contains x_α. Let, $F_i = \Pi_\alpha^{-1}(F_{\alpha_i}),$ for each i. Then F_i ' s are closed subsets of X and contains x. Since X is locally Hausdorff there exists i_0 such that F_{i_0} is Hausdorff . Then $F_{i_0} = \Pi_\alpha(F_{\alpha_{i_0}})$ is Hausdorff. For, if $x_{\alpha_{i_0}}$, $y_{\alpha_{i_0}}$ be two distinct elements of F_{i_0}, consider the elements $x_{\alpha_{i_0}}$, $y_{\alpha_{i_0}}$ such that $x_{\alpha_{i_0}} \neq y_{\alpha_{i_0}}$ where $\Pi_\beta(x_{i_0}) = \Pi_\beta(y_{i_0})$, $(\alpha \neq \beta)$. Then $x_{i_0} \neq y_{i_0}$, and so, X being locally Hausdorff, there exist disjoint sub basic open sets G_{i_0}, H_{i_0} such that $x_{i_0} \in G_{i_0}, y_{i_0} \in H_{i_0}$. Then $\Pi_\alpha(G_{i_0}) = G_{\alpha_{i_0}}$ and $\Pi_\alpha(H_{i_0}) = H_{\alpha_{i_0}}$ are disjoint open sets in X_α with $x_{\alpha_{i_0}} \in G_{\alpha_{i_0}}, y_{\alpha_{i_0}} \in H_{\alpha_{i_0}}$. Hence $F_{\alpha_{i_0}}$ is Hausdorff. Thus X_α is locally Hausdorff.

This result was proved in [6], [9] in a different manner.

Theorem 3.6

Let X be a locally Hausdorff space and R an equivalence relation on X. Then X/R is locally Hausdorff.

Proof: Let $\Pi: X \to X/R$ be the projection map. Let $\bar{x} \in X/R$ .Let $x \in \Pi^{-1}(\bar{x})$. Then there exists a closed subset F of X such that x∈F and F is Hausdorff. Since F^c is open in X and Π is both open and continuous, $\Pi(F^c)$ is open in X/R. Hence $(\Pi(F))^c$ $=\Pi(F^c)$ is open in X/R. Thus $\Pi(F)$ is closed in X/R. Let $\bar{x}, \bar{y}$ be two distinct points in $\Pi(F)$. Let $y \in \Pi^{-1}(\bar{y}), z \in \Pi^{-1}(\bar{z})$. Then y, z are distinct points in F, and so, there exist disjoint open sets G, H in F such that $y \in G, z \in H.$ Then $\Pi(G)$ and $\Pi(H)$ are disjoint sets in $\Pi(F)$ and $\bar{y} \in \Pi(G), \bar{z} \in \Pi(H).$ Thus $\Pi(F)$ is Hausdorff. Hence X/R is locally Hausdorff.

Theorems 3.1, 3.3, 3.4, 3.5 and 3.6 can be summarized as follows:

Theorem 3.7

The class of all locally Hausdorff spaces is closed under the formation of products, union subspaces and quotient spaces

We shall now give here two examples which will prove that the product of locally Hausdorff spaces at least one of which is not Hausdorff may or may not be Hausdorff.

Example 3.1

Let X= $\mathbb{R}$ with the usual topology and Y is $\mathbb{Z}$ with the cofinite topology. Then Y is not Hausdorff but X×Y is Hausdorff.

Example 3.2

Let X= $\mathbb{Z}$ with the cofinite topology and Y=i $\mathbb{Z}$={i n |n∈ $\mathbb{Z}$} with the cofinite topology. Then X×Y islocally Hausdorff but not Hausdorff. For if (x, y) and (x/, y/) are two distinct points in X×Y, every pair of open sets G and H in X×Y with (x, y)∈G,(x/, y/)∈H, there are open sets $G_x, G_x^{/}$ in X, $H_y, H_y^{/}$ in V such that (x, y)∈ $G_x \times H_y$, (x/, y/)∈ $G_{x'} \times H_{y'}$. Since X, Y have the corresponding cofinite topologies $G_x \cap G_x^{/} \neq \Phi$, $H_y \cap H_y^{/} \neq \Phi$. Hence $G \cap H \neq \Phi$. Thus X×Y is not Hausdorff.

Remark: The proof in example 3.1 shows that, if X is a Hausdorff space and Y is any topological space, then X×Y is Hausdorff, i.e., the product of a Hausdorff space with any space is Hausdorff.

The class of all locally Hausdorff spaces, i.e., T_1-spaces can be characterized as follows:

Theorem 3.8.

A topological spaces (X, $\Im$) islocally Hausdorff spaceif and only if either (X, $\Im$) is finite discrete space or X is infinite and $\Im$ contains the cofinite topology on X.

We now state a conjecture regarding the structure of a locally Hausdorff space and hence regarding the classification of all locally Hausdorff spaces.

Conjecture 3.1

The class of all locally Hausdorff topological spaces is formed from (i) the class of all Hausdorff spaces and (ii) the class of all infinite sets with the corresponding cofinite topologies with the help of (1) Products, (2) Union spaces, (3) Subspaces and (4) Quotient spaces.

3. Relationship among the various kinds of locally Hausdorff spaces.

Proposition 4.1

Every locally Hausdorff space of the third kind is a locally Hausdorff space of the first kind.

Proof: This is obvious.

Proposition 4.2

Every locally Hausdorff space of the third kind is a locally Hausdorff space of the second kind.

Proof: Let X be a locally Hausdorff space of the third kind. Then, for each x∈X, there exists an open set G in X with x∈G such that $\overline{G}$ is Hausdorff. Then G is Hausdorff, and so X is locally Hausdorff of the second kind.

Proposition 4.3

A locally Hausdorff space of the first kind need not be a locally Hausdorff space of the second or third kind.

Proof: Let X be an infinite set with the cofinite topology. Then X is locally Hausdorff space of the first kind. Let V be an open set in X with x∈V. Then V is infinite. The topology of V is the cofinite topology since $V - G \subseteq X - G$ which is finite. So V is not Hausdorff. Hence X is not locally Hausdorff space of the second kind. Now $\overline{V}$ =X, which is not Hausdorff. So, X is NOT locally Hausdorff space of the third kind.

Proposition 4.4

Every locally Hausdorff of the second kind is a locally Hausdorff space of the first kind.

Proof: Let X be a locally Hausdorff of the second kind. Let x, y$\in$X and $x \neq y$. $\exists$ open sets G_x and H_y in X such that $x \in G_x, y \in H_y$, and both G_x and H_y are Hausdorff. For all $x_1, x_2 \in G_x \ \exists G_{x_1}, G_{x_2}$ (open in G_x) such that $x_1 \in G_{x_1}, x_2 \in G_{x_2}$, $G_{x_1} \cap G_{x_2} = \Phi$. Here G_{x_1} and G_{x_2} are open in X, since $G_{x_1} = V_{x_1} \cap G_x, G_{x_2} = V_{x_2} \cap G_x$, for some open sets V_{x_1}, V_{x_2} in X.

If any of x and y belongs to $G_x \cap H_y$, then x ,y both belong to G_x or to H_y , and so, x & y can be separated by disjoint open sets in G_x (or H_y), and hence by open sets in X . If none of $x \in G_x$, $x \notin H_y$ and $y \in H_y$, $y \notin G_x$.

Since $T_2 \Rightarrow T_1$, each pair of points x, y in X satisfies the condition for T_1.Thus, X is locally Hausdorff of the 1st kind.

Comment: We thus see that the classes of all locally Hausdorff spaces of the second and third kind are proper subclasses of the class of all locally Hausdorff spaces of the first kind i.e., of T_1-space.

Thus, $T_2 \Rightarrow T_b \Rightarrow T_a \Rightarrow T_1$. The last implication is not reversible. We have not been able to decide whether the first two implications are reversible or not. We know ([1]) that {Hausdorff spaces}$\subset${kc-spaces}$\subset${us-spaces}$\subset${T_1-spaces}, the inclusions being strict everywhere. Here a **kc-space** is one in which every compact subset is closed, and a space in which every convergent sequence has a unique limit is called a **us-space**. These terms were introduced by Wilansky [1]. Two more classes of spaces viz., S_1-spaces and S_2-spaces have been proved to exist between T_1 and T_2 by Aull [3]. An **S_1-space** is a us-space in which every convergent sequence has a subsequence which does have a side point i.e., a limit point which is not the limit of a sequence. An **S_2-space** is a us-space in which no convergent sequence has a **side point**.

We do not know the relationship of T_a –spaces and T_b-spaces with kc-spaces, us-spaces, S_1-spaces and S_2-spaces.

4. Maximal and Minimal T_1-topologies

Theorem 5.1

The collection of all T_1-topologies on a non-empty set X has a smallest member, viz., the cofinite topology C_X on X.

Proof: Let X be a non-empty set.

Let C be the collection of all T_1-topologies and $\Im = \bigcap_\alpha \Im_\alpha$. Then $\Im$ is a T_1-topology. For if $x \in X$, $x \notin V_\alpha$ for each $V_\alpha \in \Im_\alpha$. Let $V \in \Im$. Then, $V \in \Im_\alpha, \forall \alpha$. Hence $x \notin V$. Thus {x} is closed in (X, $\Im$). Hence $\Im$ is a T_1-topology.

Since each T_1-topologyon X contains the cofinite topology C on X, and since C itself is a T_1-topology. So $\Im$ =C.

Definition 5.1.

For a non-empty class {$\Im_\alpha$} of topologies on X, the topology $\Im$ generated by $\bigcup_\alpha \Im_\alpha$ will be called the **union topology** on X defined by {$\Im_\alpha$}.

It is clear that if each $\Im_\alpha$ is a T_1-topology, $\Im$, too, is a T_1-topology.

Assuming the truth of the conjecture 3.1, we see that the following is true:

Conjecture 5.1.

The largest T_1-topology on a non-empty set X is the union topology defined by the class of all Hausdorff topologies on X.

5. Hausdorffification of the locally Hausdorff space of the first kind

Here we alter the topology of a locally hausdorff space of the first kind to make it Hausdorff.

(A) Let (X, $\Im$) be locally Hausdorffspace of thefirst kind. We form a topology $\overline{\Im}$ on X from $\Im$ as follows:

For every pair of disjoint points x, y in X, which cannot be separated by disjoint open sets, we choose a pair of open sets $G_x , H_y \in \Im$ such that $x \in G_x, x \notin H_y$, $y \in H_y, y \notin G_x$. Such G_x , H_y exist since X is locally Hausdorff, and hence T_1. Let S be obtained from $\Im$ by replacing $G_x \, and \, H_y$ by $U_{x,y} = G_x - H_y, V_{y,x} = H_y - G_x$ respectively, for each such pair x, y $\in$X. Let $\overline{\Im}$ denote the topology generated by S on X. Then $(X, \overline{\Im})$ is a Hausdorff space. $(X, \overline{\Im})$ will be called a Hausdorffification of the first kind for the space (X, $\Im$).

Comment: Since $\overline{\Im}$ depend on the choice G_x and H_y, $\overline{\Im}$ is not unique.

However, we give below another kind of Hausdorffification which yields a unique Hausdorff topology $\overline{\Im}'$ on X :

(B) Let (X, $\Im$) be locally Hausdorff space of the first kind. Let A be the set of all pairs {x, y} in X such that x and y cannot be separated by disjoint open sets in X. Since X is T_1, for pair {x, y} in A, there exist open sets $G_{x,y}$ and $H_{y,x}$, in X such that $x \in G_{x,y}, y \notin G_{x,y}, y \in H_{y,x}, x \notin H_{y,x}$. Let $V_{x,y} = \cup G_{x,y}$ and $W_{y,x} = \cup H_{y,x}$, the union being taken over all such special $G_{x,y}$'s and $H_{y,x}$'s respectively. Clearly, x$\in V_{x,y}$, y$\notin V_{x,y}$, and $y \in W_{y,x}, x \notin W_{y,x}$, and$V_{x,y}$ and $W_{y,x}$ are the largest such open sets.

Let V ={$V_{x,y}$ –$W_{y,x}$| {x, y}$\in$ A}, W={$W_{y,x}$ -$V_{x,y}$ |{x, y}$\in$ A} . Let $\overline{\Im}$ be the topology on X which is generated by $\Im \cup$ V$\cup$ W.

Then (X, $\overline{\Im}$) is Hausdorff and will be called the Hausdorffification of the second kind for the space (X, $\Im$). Clearly (X, $\overline{\Im}$) is unique.

If instead of $V_{x,y}$ –$W_{y,x}$ and $W_{y,x}$ -$V_{x,y}$ we would have considered $G_{x,y}$ –$H_{y,x}$ and $H_{y,x}$ -$G_{x,y}$, and called then G and H respectively, then the topology generated by $\Im \cup$ G$\cup$ H would have given us a Hausdorff topology on X. But it would not have been unique.

Example 6.1

Let A be an infinite set, and $\Im$ the cofinite topology on X. Then $\overline{\Im}$ is unique and is the discrete topology on X.

Example 6.2

Let X=$\prod_{\alpha} X_\alpha$, where X_α's are the distinct infinite sets with the corresponding cofinite topologies $\Im_\alpha$. Then (X, $\Im$) is locally Hausdorff space of the first kind, where $\Im$ is the product topology on X. Each $(X_\alpha, \overline{\Im}_\alpha)$ is a discrete space for each α , but (X, $\overline{\Im}$), the product space of the collection {$(X_\alpha, \overline{\Im}_\alpha)$}, is not a discrete space.

6. Characterisations of T_2-spaces

We here digress and prove a few characterisations of a Hausdorff space which resemble the definition of a $\Im_1$-space.

We now state and prove below four characterisations of a Hausdorff space.

Theorem 7.1

For a topological space X, the following five statements are equivalent:

(i) X is Hausdorff,

(ii) For each pair of distinct points x, y in X, there exists an open set G_{xy} in X such that $x \in G_{xy}$, $y \notin \overline{G}_{xy}$.

(iii) For each pair of distinct points x, y in X, there exists an open sets G_{xy} and H_{yx} in X such that $x \in G_{xy}$, $y \notin \overline{G}_{xy}$, $y \in H_{yx}$, $x \notin \overline{H}_{yx}$.

(iv) For each compact subset K of X and for each $x \in X$ with $x \notin K$, there exist disjoint open sets G and H in x such that $K \subseteq G$, $y \in H$;

(v) For each pair of disjoint compact subsets K_1 and K_2 of X, there exist disjoint open sets G_1 and G_2 of X such that $K_1 \subseteq G_1$, $K_2 \subseteq G_2$.

Proof: (i) $\Rightarrow$ (ii)

Let X be a Hausdorff space and let x, y in X ($x \neq y$). Then there exist disjoint open sets G and H such that $x \in G$, $y \in H$. It is clear that $y \notin \overline{G}$. (ii) follows by writing $G_{xy}=G$.

(ii) $\Rightarrow$ (i)

Let (ii) hold. Let $x, y \in X$, with $x \neq y$. Then there exists an open set H_{xy} in X such that $y \in H_{xy}$ and $G_{xy} \cap H_{xy} = \Phi$. Hence X is Hausdorff.

(ii) $\Leftrightarrow$ (iii). Obvious.

(iv) $\Rightarrow$ (i). Obvious since every singleton set is compact.

(i) $\Rightarrow$ (iv). Let (i) hold. Let X be Hausdorff and let K be a compact subset of X and $x_0 \in X$ with $x_0 \notin K$. Since X is Hausdorff, for each $x \in K$, there exist open sets G_x and H_x in X such that $x \in G_x$, $x_0 \in H_x$, and $G_x \cap H_x = \Phi$. Since K is compact and $\{G_x\}$ is an open cover of K, $\{G_x\}$ has a finite subcover $\{G_{x_1}, \ldots, G_{x_n}\}$, say. Let $G = G_{x_1} \cup \ldots \cup G_{x_n}$ and $H = H_{x_1} \cap \ldots \cap G_{x_n}$. Then G and H are open and disjoint, and $K \subseteq G$, $x_0 \in H$. Thus (iv) holds.

(v) $\Rightarrow$ (i). since every singleton subset is compact, it is obvious.

(i) $\Rightarrow$ (v). It will be sufficient to prove that (iv) $\Rightarrow$ (v). It is clear that the method of proof of (i) $\Rightarrow$ (iv) can be similarly used to prove that (iv) $\Rightarrow$ (v).

[N.B. Wilansky too proved the equivalence of (i) and (iv) in his book [2]]

As mentioned in the second paragraph of the abstract, the conditions (ii) and (iii) in the statement of the above theorem regarding a T_2-space closely resemble the conditions (i) and (ii) in page 2 for T_1-space

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