Vol.5.Issue.4.2017 (Oct-Dec)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com
Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

SOME IDENTITIES OF ROGERS-RAMANUJAN TYPE

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ABSTRACT

The following two Identities, namely

for $|q|<1$, $\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 2 \pmod 5$ = $\frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}$

$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 1 \pmod 5$ = $\frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}$

where $(q;q)_n=\prod_{j=1}^{n}(1-q^j)$, $(q;q)_\infty=\prod_{j=1}^{\infty}(1-q^j)$

and$(a_1,a_2,\dots.a_s;q)_\infty=(a_1;q)_\infty\,(a_2;q)_\infty\dots..(a_s;q)_\infty$

which are known as the celebrated original Rogers-Ramanujan Identity. These two identities have motivated extensive research over the past hundred years. They were first proved by L. J. Rogers in 1894 that was completely ignored. They were rediscovered without proof by Ramanujan sometime before 1913. Also in 1917, these identities were rediscovered and proved independently by Issai Schur. In the ensuing decades, numerous identities that are similar to the Rogers-Ramanujan Identities has been discovered by several eminent mathematicians like Jackson, W. N. Bailey, G. E. Andrews, L. J. Slater, A.K. Agarwal, etc.

The Rogers-Ramanujan Identities have two aspects: one analytical and the other is combinatorial. The present paper intends to give a brief discussion on Original Rogers-Ramanujan Identities and to derive some new identities of Rogers-Ramanujan Type analytically by using some general transformation between Basic Hypergeometric Series and with the incorporation of some identities from Lucy Slater's famous list of 130 identities of Rogers-Ramanujan type.

Key words: Rogers-Ramanujan Identity, Slater's Identity, Basic Hypergeometric Series, Jacobi's Triple Product Identity, Bailey Pair etc.

INTRODUCTION

For |q|<1, the q-shifted factorial is defined by

$$(a;q)_0 = 1$$

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ for n≥1}$$

and $(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k)$.

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,....,a_m;q)_n = (a_1;q)_n(a_2;q)_n....(a_m;q)_n$$

$$(a_1,a_2,....,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty....(a_m;q)_\infty.$$

The Basic Hyper geometric Series is

$$_{p+1}\phi_{p+r}\left(\begin{matrix} a_1,a_2,.......,a_{p+1};q;x \\ b_1,b_2,.......,b_{p+r}\end{matrix}\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n....(a_{p+1};q)_n x^n(-1)^{nr}q^{\frac{n(n-1)r}{2}}}{(q;q)_n(b_1:q)_n(b_2;q)_n.....(b_{p+r};q)_n}$$

The series $_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For r=0 it converges only when |x|<1.

Ramanujan's Theta function: Ramanujan's Theta function ([4], P.11, Eq. (1.1.5)) is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \text{ for } |ab|<1.$$

The following special cases of f (a, b) arise so often that they were given their own notation by Ramanujan ([4], P.11):

$$\phi(q) = f(q,q)$$

$$\psi(q) = f(q,q^3)$$

$$f(-q) = f(-q,-q^2)$$

Jacobi's triple product identity :(See [3], P.2, Eq. (1.1.7))

For |ab| < 1, $f(a,b) = (-a,-b,ab;ab)_\infty$

An immediate corollary ([3], P-2, Eq. (1.1.8), (1.1.9), (1.1.10)) of this identity is thus

$$f(-q) = (q;q)_\infty$$

$$\phi(q) = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$$

We now list some general transformations. Most of them can be derived as limiting case of transformations between basic hyper geometric series. Let a, b, c, d, γ and q ∈**C**, |q|<1. Then

$$\sum_{n=0}^{\infty}\frac{q^{n^2}\gamma^n}{(q/b;q)_n(q;q)_n} = (-\gamma q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}\gamma^n(-\frac{q}{b};q)_{2n}}{(q^2;q^2)_n(\frac{q^2}{b^2};q^2)_n(-\gamma q^2;q^2)_n} \quad (1.1)$$

$$\sum_{n=0}^{\infty}\frac{(d;q)_{2n}q^{n^2-n}(-\frac{c^2}{d^2})^n}{(q^2;q^2)_n(c;q)_{2n}} = \frac{(\frac{c^2}{d^2};q^2)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}\gamma^n(-c)^n}{(q;q)_n(-\frac{c}{d};q)_n} \quad (1.2)$$

$$\sum_{n=0}^{\infty}\frac{q^{3n^2-2n}(-a^2)^n}{(q^2;q^2)_n(a;q)_{2n}} = \frac{1}{(a;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(-a)^n}{(q;q)_n} \quad (1.3)$$

$$\sum_{n=0}^{\infty}\frac{(a;q)_n q^{n^2-n}(-b)^n}{(q;q)_n(ab;q^2)_n} = \frac{(b;q^2)_\infty}{(ab;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n q^{n^2-n}(-bq)^n}{(q^2;q^2)_n(b;q^2)_n} \quad (1.4)$$

$$\sum_{n=0}^{\infty}\frac{(a^2;q)_n q^{(3n^2+n)/2}(a)^{2n}}{(q;q)_n} = \frac{(a^2q;q)_\infty}{(-aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-a;q)_n q^{(n^2-n)/2}(aq)^n}{(aq,q;q)_n} \quad (1.5)$$

The transformation (1.1) is appeared as (6.1.12) on page 41 in [3]. The transformation (1.2) follows from (3.5.4) on pages 77-78 in [5], after replacing c with $^{aq}/_{c}$, then letting $a \to 0$ and finally letting$b \to \infty$. It is also appeared as (6.1.17) on page 41 in [3]. The transformation (1.3) follows from (1.2) upon letting$d \to \infty$, and then replacing $c\ with\ a$. This transformation is also appeared as (6.1.18) on page 41 in [3]. The transformation (1.4) follows from a result of Andrews in [6] (see also Corollary 1.2.3 of [7], where it follows after replacing t by $t/_b$, then letting $b \to \infty$ and finally replacing t by b). Finally, the transformation (1.5) is appeared as (6.1.21) on page 42 in [3].

2. We shall now introduce some identities from the Lucy Slater's famous list of Rogers-Ramanujan Type Identities. Each of them below that appears in [3] is designated with a "Slater number" S.n.

$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{f(-q,-q^4)}{f(-q)}$, (see [3], Equation (2.5.1) p.11); ($S_{.14}$) (2.1)

$\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2}}{(-q;q^2)_n(q^4;q^4)_n} = \frac{f(-q^2,-q^3)}{f(-q^2)}$, (see [3], Equation (2.5.7) p.12); ($S_{.19}$) (2.2)

$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q;q)_{2n+1}} = \frac{f(-q^2,-q^{10})}{f(-q)}$(see [3], Equation (2.12.2) p.17); ($S_{.50}$) (2.3)

$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{f(q,q^7)}{f(-q^2)}$, (see [3], Equation (2.8.9) p.15); ($S_{.38}$) (2.4)

$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}} = \frac{f(q^3,q^5)}{f(-q^2)}$, (see [3], Equation (2.8.10) p.15); ($S_{.39}$) (2.5)

$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q,-q^5)}{\psi(-q)}$, (see [3], Equation (2.6.2) p.13) (2.6)

3. Identities related to modulo 8:

Replacing q by q^2 in (1.1), we get

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}\gamma^n}{(q^2/b;q^2)_n(q^2;q^2)_n} = (-\gamma q^4;q^4)_\infty \sum_{n=0}^{\infty}\frac{q^{2n^2}\gamma^n(-q^2/b;q^2)_{2n}}{(q^4;q^4)_n(\frac{q^4}{b^2};q^4)_n(-\gamma q^4;q^4)_n} \tag{3.1}$$

Setting $b = 1/q$ and $\gamma = q^2$ in (3.1), we have

$$(-q^6;q^4)_\infty \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^4)_n\,(q^6;q^4)_n(-q^6;q^4)_n} = \sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^3;q^2)_n(q^2;q^2)_n}$$

which on some reduction, yields

$$\frac{(-q^6;q^4)_\infty}{(1-q)}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^2)_{2n}(-q^6;q^4)_n} = \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}} \tag{3.2}$$

Now using (2.4) in (3.2) we get the following identity

$$\frac{(-q^6;q^4)_\infty}{(1-q)}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^2)_{2n}(-q^6;q^4)_n} = \frac{f(q,q^7)}{f(-q^2)}$$

$$= \frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{3.3}$$

Again, placing $q^{1/2}$ in place of q in transformation (1.1), we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2/2}\gamma^n}{(q^{1/2}/b;q^{1/2})_n(q^{1/2};q^{1/2})_n} = (-\gamma q;q)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2/2}\gamma^n(-q^{1/2}/_b;q^{1/2})_{2n}}{(q;q)_n(\frac{q}{b^2};q)_n(-\gamma q;q)_n} \tag{3.4}$$

which for $b = q^{1/4}$ and $\gamma = 1$ gives

$$\sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q^{1/4};q^{1/2})_n(q^{1/2};q^{1/2})_n} = (-q;q)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2/2}(-q^{1/4};q^{1/2})_{2n}}{(q;q)_n\,(q^{7/8};q)_n(-q;q)_n}$$

Now, taking $q \to q^4$ we get

$$(-q^4;q^4)_\infty \sum_{n=0}^{\infty}\frac{q^{2n^2}(-q;q^2)_{2n}}{(q^8;q^8)_n\,(q^{7/2};q^4)_n} = \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n(q^2;q^2)_n}$$

$$= \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}$$

$$= \frac{f(q^3,q^5)}{f(-q^2)}, \text{ (on using (2.5))}$$

$$=\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{3.5}$$

4. Identities related to modulo 5:

Replacing q by $q^{1/2}$ in (1.3), we get

$$\sum_{n=0}^{\infty}\frac{q^{(3n^2-2n)/2}(-a^2)^n}{(q;q)_n(a;q^{1/2})_{2n}}=\frac{1}{(a;q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{q^{(n^2-n)/2}(-a)^n}{(q^{1/2};q^{1/2})_n} \tag{4.1}$$

Setting $a=-q^{1/2}$ in (4.1), we have, on some simplification, the following:

$$\frac{1}{(-q^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q^{1/2};q^{1/2})_n}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2/2}}{(-q^{1/2};q)_n(q^2;q^2)_n} \tag{4.2}$$

Now taking $q\to q^2$ in (4.2) and then using (2.2), we obtain the following identity:

$$\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(qq)_n}=\frac{f(-q^2,-q^3)}{f(-q^2)}=\frac{(q^2,q^3,q^5;q^5)_\infty}{(q^2;q^2)_\infty}$$

$$=\frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty(q;q)_\infty}$$

Hence it reduces to the original Rogers-Ramanujan Identity, viz,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(qq)_n}=\frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n\not\equiv 0,2,3 \pmod 5 \tag{4.3}$$

Again, Setting $a=-q$ in (4.1), we have

$$\sum_{n=0}^{\infty}\frac{q^{(3n^2+2n)/2}}{(q;q)_n(-q;q^{1/2})_{2n}}=\frac{1}{(-q;q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{q^{(n^2+n)/2}}{(q^{1/2};q^{1/2})_n} \tag{4.4}$$

Now using (2.1) in (4.4) after replacing q by q^2, it yields the following identity

$$(-q^2;q)_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2+2n)}}{(q^2;q^2)_n(-q^2;q)_{2n}}=\frac{f(-q,-q^4)}{f(-q)}=\frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n\not\equiv 0,1,4 \pmod 5 \tag{4.5}$$

5. Identities related to modulo12:

Placing q^2 in place of q in transformation (1.4), we have

$$\sum_{n=0}^{\infty}\frac{(a^2;q^2)_n q^{(3n^2+n)}(a)^{2n}}{(q^2;q^2)_n}=\frac{(a^2q^2;q^2)_\infty}{(-aq^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-a;q^2)_n q^{n^2-n}(aq^2)^n}{(aq^2,q^2;q^2)_n} \tag{5.1}$$

Setting a=q in (5.1), it reduces to:

$$\sum_{n=0}^{\infty}\frac{(q^2;q^2)_n q^{(3n^2+3n)}}{(q^2;q^2)_n}=\frac{(q^4;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^3;q^2)_n(q^2;q^2)_n}$$

which reduces to the following identity after some reduction:

$$\frac{(-q^3;q^2)_\infty}{(q^4;q^2)_\infty(1-q)}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_n q^{(3n^2+3n)}}{(q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q;q)_{2n+1}}$$

$$=\frac{f(-q^2,-q^{10})}{f(-q)},\ \text{(on using (2.3))}$$

$$=\frac{(q^2,q^{10},q^{12};q^{12})_\infty}{(q;q)_\infty}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n\not\equiv 0,2,10 \pmod{12} \tag{5.2}$$

References

[1]. F.H. Jackson, *Transformation of q-series, Messenger of Math* 39 (1910) 145-153.

[2]. G.E. Andrews,*On Basic Hypergeometric Series, mock theta functions, and partitions I, Quart. J. Math.* Oxford Ser. (2) 17 (1966) 64-80.

[3]. J. M. Laughlin, A. V. Sills & P. Zimmer, *The Electronic Journal of Combinatorics* 15(2008), #DS15.

[4]. G. E. Andrews and B. C. Berndt, *Ramanujan Lost Note Book, part 1, Springer* 2005.

[5]. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed.,Cambridge, 2004.

[6]. G.E. Andrews,*On Basic Hypergeometric Series, mock theta functions, and partitions II, Quart. J. Math.* Oxford Ser. (2) 17 (1966) 132-143.

[7]. G. E. Andrews and B. C. Berndt, *Ramanujan Lost Note Book,* part I, Springer. 2005

[8]. G. E. Andrews and B. C. Berndt, *Ramanujan Lost Note Book,* part II, Springer.

[9]. L. J. Slater, *Further Identities of Rogers-Ramanujan Type. Proc.London Math*. Soc.54 (1952) 147-167.