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CONJUGACY CLASS PRIME GRAPHS WITH FEW VERTICES

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ABSTRACT

In this note, we provide a complete classification for the prime graphs regarding conjugacy class sizes of finite groups with vertex number not exceeding 4.

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1. Introduction

For a finite group G, we associate a prime vertex graph $\Delta'(G)$ to G, whose vertices are prime divisors of a nontrivial conjugacy class size of G, and distinct vertices p, q are adjacent just when pq divides some conjugacy class size of G. We also say there is an edge incident to p and q in the situation and there is no edge between p and q otherwise. A question naturally arises: Which graphs can be realized as $\Delta'(G)$ for some group G? It is proved that the graph $\Delta'(G)$ has at most 2 connected components and the graph diameter of $\Delta'(G)$ is at most 3 when $\Delta'(G)$ is connected (for instance, see [11,Theorem 8.3]). Another natural question is: What can be said about the structure of group G when we impose some particular restrictions on the number of vertices and the status of edges of the graph $\Delta'(G)$? It is proved the graph $\Delta'(G)$ has two connected components if and only if G is a quasi-Frobenius group with abelian kernel and complements (for instance, see [11,Theorem 8.1]).

By routine combinatorial calculations, we see the graphs with four vertices have at most eleven possibilities. A series of works in [6,7,8,12] has completed the classification of the character degree vertex graphs $\Gamma(G)$ with at most four vertices. A fairly interesting topic once is the classification of (character degree) prime vertex graphs $\Delta(G)$ with at most four vertices and solvable

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groups, for example, see [9,13]. Dually, in this note, we show there are exactly seven graphs which can occur as (conjugacy size) prime graphs $\Delta'(G)$ for G. Particularly, we prove the following results.

THEOREM *Let G be a finite group such that $\Delta'(G)$ has exactly four vertices. Then $\Delta'(G)$ is just one of the graphs shown in Figure 1. Furthermore, the top six graphs are only realized by solvable groups, the last graph is realized by solvable groups and nonsolvable groups.*

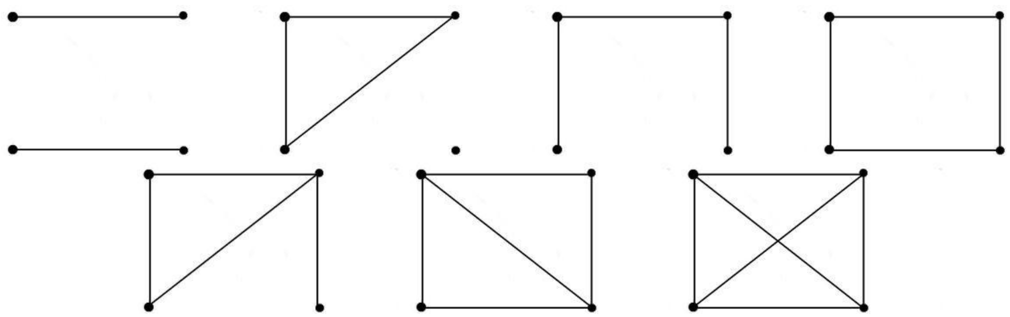

Figure 1: The graphs of THEOREM

We refer to B. Huppet's book [10] for the notation and terminology that we need, but do not explain here.

2. Preliminaries

Lemma 2.1. *Let G be a finite group, then the connected component number of the graph $\Delta'(G)$ is at most 2.*

Proof. For example, see [11].

Lemma 2.2. *For group G, the graph $\Delta'(G)$ is disconnected if and only if G is a quasi-Frobenius group with abelian kernel and complements. In particular, each connected component is a complete graph.*

Proof. See [11] or Theorem 4.1 and the following Remarks in [1].

Lemma 2.3. *Let G be a quasi-Frobenius group with abelian kernel N and complements H. Then the conjugacy class sizes of G are 1, $|N/Z(G)|$ and $|H/Z(G)|$.*

Proof. See Lemma 4.4 of [4].

A vertex p in a graph $\Delta'(G)$ is called a complete vertex if p is adjacent to all other vertices; otherwise, p is called an incomplete vertex, i.e., p is not adjacent to at least one other vertex.

Lemma 2.4. *If $\Delta'(G)$ contains at most two complete vertices for the group G, then G is solvable.*

Proof. See Corollary B of [2].

3. Results

When $\Delta'(G)$ is an empty set, G is apparently an abelian group. If $\Delta'(G)$ has only a vertex for some group G, then since the vertex set of $\Delta'(G)$ is $\pi(G/Z(G))$, we see $G = P \times A$, where P is a nonabelian Sylow p-subgroup and A is abelian. When $\Delta'(G)$ has exactly two vertices, likewise we get that $G = H \times A$, where H is a Hall $\{p, q\}$-subgroup and A is abelian, and so G is solvable by Burnside's theorem. Using GAP[5], we know $\Delta'(S_3)$ exactly contains two isolated vertices 2, 3; and $\Delta'(SL_2(3))$ has just two vertices 2 and 3 with an edge 6. Here S_3 stands for symmetric group of degree 3. By a simple count, we see the graphs $\Delta'(G)$ with three vertices has four possibilities, yet we get the following precise result.

Theorem 3.1. *If the graph $\Delta'(G)$ has exactly three vertices for some finite group G, then $\Delta'(G)$ is one of the following three graphs listed in Figure 2.*

Proof. It is seen that the graphs with three vertices has four possibilities. Lemma 2.1 implies that the graph with three isolated vertices is of nonexistence, thus $\Delta'(G)$ is one of the graphs in Figure 2. Since the first graph has two components, Lemma 2.2 yields that the groups with this graph must be solvable. The second graph has two incomplete vertices, Lemma 2.4 gives the groups with this graph are also solvable. By GAP, we obtain that the graph $\Delta'(D_{30})$ of the dihedral group D_{30} (SmallGroup(30, 3)) has vertices 2, 3, 5 with just one edge 15, and so is the first graph in Figure 2. The graph of the semidirect product $C_{15}:C_4$ (SmallGroup(60, 5)) has vertices 2, 3, 5 with exactly two edges 10, 15, and so its graph is the second one in Figure 2. The graph of the alternating group A_5 possesses vertices 3, 4, 5 with edges 12, 15, 20, and so it is a triangle. The graph $\Delta'(S_3 \times D_{10})$ owns vertices 2, 3, 5 with edges 6, 10, 15 and hence it is also a triangle. Note that A_5 is nonsolvable and $S_3 \times D_{10}$ is solvable.

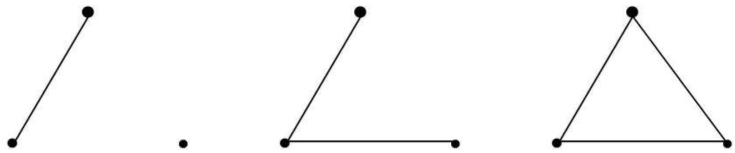

Figure 2: The graphs of three vertices

PROOF OF THEOREM:

The graphs with four vertices has eleven possibilities. Lemma 2.1 implies that there do not exist four-vertex graphs with two or four isolated vertices. Lemma 2.2 shows that there is no four-vertex graph consisting of an angle and an isolated vertex. Theorem A of [3] asserts that at least two are joined among any three vertices for a conjuacy size prime graph with at least three vertices. Thus there is no existence of the four-vertex graph which has a complete vertex and the others have degree one. Theorem 2.4 yields the groups with the top six graphs are all solvable. Thus the four-vertex graph of nonsolvable groups is the last graph in Figure 1.

By using GAP, we obtain for the Frobenius group $C_{91}:C_6$ (SmallGroup(546, 11)) whose $\Delta'(G)$ has vertices 2, 3, 7, 13 with edges 6, 91 and thus it is the first graph of Figure 1. The graph of dihedral group D_{210} (SmallGroup(210, 11)) has vertices 2, 3, 5, 7 with the unique edge 105 and so it is the second one in Figure 1. The graph of group $(C_5 \times (C_7 : C_3)) : C_2$ (SmallGroup(210, 3)) possesses vertices 2, 3, 5, 7 with edges 6, 14, 35, and then it is the third graph in Figure 1. The graph of group $D_{10} \times (C_7 : C_3)$ (SmallGroup(210, 2)) is a square with vertices 2, 3, 5, 7 and edges 6, 14, 15, 35. The graph of semidirect product $C_{105}:C_4$ (SmallGroup(420, 15)) is the fifth one with vertices 2, 3, 5, 7 and edges 10, 105. For the group $(C_{35}:C_4):C_3$ (SmallGroup(420, 15)), its graph is a square with a diagonal line, as the sixth graph. The graph of the solvable group $D_{30} \times D_{14}$ (SmallGroup(420, 14)) is the unique complete graph (of Figure 1) with vertices 2, 3, 5, 7 and edges 14, 15, 30, 105. For the nonsolvable group $A_5 \times D_{10}$ (SmallGroup(840, 137)), its graph is also that unique complete graph with vertices 2, 3, 5, 7 as well as edges 6, 105, 140. The proof is finished.

References

[1]. G. Alfandary, On graphs related to conjugacy classes of groups, Israel J. Math., 1994, 86: 211-220.

[2]. C. Casolo, S. Dolfi, E. Pacifici, L. Sanus, Incomplete vertices in the prime graph on conjugacy class sizes of finite groups, J. Algebra, 2013, 376: 46-57.

[3]. S. Dolfi, On independent sets in the class graph of a finite group, J. Algebra, 2006, 303: 216-224.

[4]. M. Fang, P. Zhang, Finte groups with graphs containing no triangles, J. Algebra, 2003, 264: 613-619.

[5]. The GAP Group, GAP—Groups, algorithms, and programming, version 4.5, http://www.gap-system.org, 2012.

[6]. L. He, G. Zhu, Nonsolvable normal subgroups and irreducible character degrees, J. Algebra, 2012, 372: 68--84.

[7]. L. He, G. Qian, Graphs of nonsolvable groups with four degree-vertices, Sci. China Math., 2015, 58: 1305–1310.

[8]. L. He, M. Lewis, Common divisor character degree graphs of solvable groups with four vertices, Comm. Algebra, 2015, 43(11): 4916–4922.

[9]. B. Huppert, Research in Representation Theory at Mainz (1984-1990), Progress in Mathematics 95, Birkhauser Verlag Basel, 1991, 95: 17-36

[10]. B. Huppert, Character theory of finite groups, Walter deGruyter, Berlin, 1998.

[11]. M. Lewis, An overview of graphs associated with character degrees and conjugacy class sizes in finite groups, Rocky Moutain J. Math., 2008, 38: 175-211.

[12]. G. Qiang, Y. Yang, Nonsolvable groups with no prime dividing three character degrees, J. Algebra, 2015, 436: 145-160.

[13]. J.P. Zhang, On a Problem by Huppert, Acta. Sci. Natur. Univ. Pekinensis, 1998, 34, 143-150.