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RESEARCH ARTICLE

ON N-CONE METRIC SPACES

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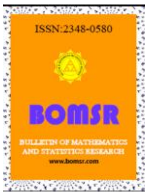

ABSTRACT
In this paper, we prove a fixed point theorem for a contractive mapping on complete N-cone metric spaces illustrating an example and get Banach contraction principal as a consequence.

Keywords: N-cone metric, fixed point, fixed point theorem
AMS(2010) Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 2007, Huang and Zhang [2] defined the notion of cone metric space as a generalization of metric spaces. They replaced the set of real numbers by ordered Banach spaces. After that they proved various fixed point theorems for contractive mapping on this space.

Let E be a real Banach space and P be a subset of E. P is said to be a cone if and only if:

1) P is non-empty, closed and $P \neq \{0\}$,
2) $a,b \in R, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$,
3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to show that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int} P$, $\operatorname{int} P$ denotes the interior of P.

The cone P is said to be normal if and only if there exists a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is said to be the normal constant of P. The cone P is said to be regular, if $\{x_n\}$ is sequence such that

$x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y$

for some $y \in E,$ then there is a $x \in E$ such that $\|x_n - x\| \to 0 \ (n \to \infty).$ Similarly, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Remark 1.1. [3] *If E is a real Banach space with cone P and $\alpha \le \lambda\alpha$ for $\alpha \in P$ and $0 < \lambda < 1,$ then $\alpha = 0.$*

In 2013, the notion of $N-$cone metric space was introduced by Malviya and Fisher [4]. Also, they proved various fixed point theorems for asymptotically regular maps on complete $N-$cone metric space.

Throughout this paper, we assume E is a real Banach space, P is a cone in E where $\operatorname{int} P$ is a non-empty set and $\le$ is a partial ordering with respect to $P.$

Definition 1.1. [4] *Let X be a non-empty set. Suppose the mapping $N : X \times X \times X \to E$ satisfies the following conditions; for all $x, y, z, a \in X,$*

1) $N(x, y, z) \ge 0,$
2) $N(x, y, z) = 0$ if and only if $x = y = z,$
3) $N(x, y, z) \le N(x, x, a) + N(y, y, a) + N(z, z, a).$

Then N is called a $N-$cone metric on X and (X, N) is called a $N-$cone metric space.

Example 1.1. [4] *Let $E = R^3, P = \{(x, y, z) : x, y, z \ge 0\} \subset E,$ $X = R, a * b = a.b$ and $N : X^3 \to E$ is defined by*

$$N(x,y,z) = \left(\alpha\left(|y+z-2x| + |y-z|\right), \beta\left(|y+z-2x| + |y-z|\right), \gamma\left(|y+z-2x| + |y-z|\right)\right)$$

where α, β, γ are positive constants. Then, (X, N) is a $N-$cone metric space.

Lemma 1.1. [4] *Let (X, N) be a $N-$cone metric space. Then, $N(x, x, y) = N(y, y, x)$ for all $x, y \in X$.*

Definition 1.2. [4] *Let (X, N) be a $N-$cone metric space. The open $N-$ball $B_N(x, c)$ is defined as*

$$B_N(x, c) = \{y \in X \ : \ N(y, y, x) << c\}$$

for $c \in E$ with $0 << c$ and for all $x \in X$.

Definition 1.3. [4] *Let (X, N) be a $N-$cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. $\{x_n\}$ is called a convergent sequence if for every $c \in E$ with $0 << c$, there exists a natural number N such that $N(x_n, x_n, x) << c$ for all $n > N.$ Here, x is said to be the limit of sequence $\{x_n\}$ and this is denoted by $\lim x_n = x$ or $x_n \to x$ as $(n \to \infty).$*

Definition 1.4. [4] *Let (X, N) be a $N-$cone metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called a Cauchy sequence in X if for any $c \in E$ with $0 << c$, there exists a natural number N such that $N(x_n, x_n, x_m) << c$ for all $n, m > N.$*

Definition 1.5. [4] *If every Cauchy sequence is convergent in a $N-$cone metric space, then this space is said to be a complete $N-$cone metric space.*

Lemma 1.2. [1] *Let (X, N) be a $N-$cone metric space, P be a normal cone with normal constant $K > 0$ and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$*

.

Lemma 1.3. [1] *Let (X,N) be a $N-$cone metric space, $\{x_n\}$ be a sequence in X and $x\in X$. $\{x_n\}$ is convergent to x if and only if $N(x_n,x_n,x)\to 0$ as $n\to\infty$.*

Lemma 1.4. [1] *Let (X,N) be a $N-$cone metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is a Cauchy sequence if and only if $N(x_n,x_n,x_m)\to 0$ as $n,m\to\infty$.*

2. MAIN RESULTS

In this chapter, we prove a fixed point theorem for a contractive mapping on a complete $N-$cone metric spaces and obtain some results from it.

Theorem 2.1. *Let (X,N) be a complete $N-$cone metric space and P be a normal cone with normal constant $K>0$. Assume that the self mapping $T : X\to X$ satisfying the following contractive condition*

$$N(Tx,Ty,Tz)\le kN(x,y,z)+\lambda N(Tx,Tx,y)+\mu N(x,x,z)\ \text{for all}\ x,y,z\in X$$

for some fixed $k,\lambda,\mu\in[0,1)$ with $k+\lambda+\mu<1$. Then T has a unique fixed point in X and for every $x\in X$, iteration of the sequence $\{T^n x\}$ converges to the fixed point.

Proof: Let $x\in X$ and $x_1=Tx_0$, $x_2=Tx_1=T^2x_0$ for all $x_0\in X$. In general,

$$x_{n+1}=Tx_n=T^{n+1}x_0,\dots$$

for all $n\in N$. We get

$$\begin{aligned} N(x_{n+1},x_{n+1},x_n) &= N(Tx_n,Tx_n,Tx_{n-1}) \\ &\le kN(x_n,x_n,x_{n-1})+\lambda N(Tx_n,Tx_n,x_n)+\mu N(x_n,x_n,x_{n-1}) \\ &\le kN(x_n,x_n,x_{n-1})+\lambda N(x_{n+1},x_{n+1},x_n)+\mu N(x_n,x_n,x_{n-1}) \\ &= (k+\mu)N(x_n,x_n,x_{n-1})+\lambda N(x_{n+1},x_{n+1},x_n). \end{aligned}$$

So,

$$\begin{aligned} (1-\lambda)N(x_{n+1},x_{n+1},x_n) &\le (k+\mu)N(x_n,x_n,x_{n-1}) \\ N(x_{n+1},x_{n+1},x_n) &\le \frac{(k+\mu)}{(1-\lambda)}N(x_n,x_n,x_{n-1}) \\ &= \rho N(x_n,x_n,x_{n-1}),\ \text{where}\ \rho=\frac{(k+\mu)}{(1-\lambda)}<1 \\ &= \rho^2 N(x_{n-1},x_{n-1},x_{n-2}) \\ &\ \ \vdots \\ &= \rho^n N(x_1,x_1,x_0). \end{aligned}$$

Now for any $n>m$, we obtain

$$
\begin{aligned}
N(x_n,x_n,x_m) &\le N(x_n,x_n,x_{n-1}) + N(x_n,x_n,x_{n-1}) + N(x_m,x_m,x_{n-1})\\
&= 2N(x_n,x_n,x_{n-1}) + N(x_m,x_m,x_{n-1})\\
&= 2N(x_n,x_n,x_{n-1}) + N(x_{n-1},x_{n-1},x_m)\\
&\le 2N(x_n,x_n,x_{n-1}) + N(x_{n-1},x_{n-1},x_{n-2}) + N(x_{n-1},x_{n-1},x_{n-2}) + N(x_m,x_m,x_{n-2})\\
&= 2N(x_n,x_n,x_{n-1}) + 2N(x_{n-1},x_{n-1},x_{n-2}) + N(x_m,x_m,x_{n-2})\\
&= 2N(x_n,x_n,x_{n-1}) + 2N(x_{n-1},x_{n-1},x_{n-2}) + N(x_{n-2},x_{n-2},x_m)\\
&\ \ \vdots\\
&\le 2N(x_n,x_n,x_{n-1}) + 2N(x_{n-1},x_{n-1},x_{n-2}) + \dots + 2N(x_{m+2},x_{m+2},x_{m+1})\\
&\quad + N(x_{m+1},x_{m+1},x_m)\\
&\le 2\rho^{n-1}N(x_1,x_1,x_0) + 2\rho^{n-2}N(x_1,x_1,x_0) + \dots + 2\rho^{m+1}N(x_1,x_1,x_0)\\
&\quad + \rho^{m}N(x_1,x_1,x_0)\\
&= 2(\rho^{n-1} + \rho^{n-2} + \dots + \rho^{m+1})N(x_1,x_1,x_0) + \rho^{m}N(x_1,x_1,x_0)\\
&= 2\rho^{m+1}(1 + \rho + \rho^2 + \dots + \rho^{n-m-2})N(x_1,x_1,x_0) + \rho^{m}N(x_1,x_1,x_0)\\
&\le 2\frac{\rho^{m+1}}{1-\rho}N(x_1,x_1,x_0) + \rho^{m}N(x_1,x_1,x_0), \text{as } \rho<1\\
&= \left(2\frac{\rho^{m+1}}{1-\rho} + \rho^{m}\right)N(x_1,x_1,x_0), \text{as } \rho<1\\
&= \frac{\rho^{m}(2\rho+1)}{1-\rho}N(x_1,x_1,x_0), \text{as } \rho<1.
\end{aligned}
$$

From normality of P, for normal constant $K>0$, $\|N(x_n,x_n,x_m)\| \le \frac{\rho^m(2\rho+1)}{1-\rho}K\|N(x_1,x_1,x_0)\|$. Then, $N(x_n,x_n,x_m)\to 0$ as $n,m\to\infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is a complete $N-$cone metric space, there exists a $x\in X$ such that $x_n\to x$ as $n\to\infty$. Then,

$$
\begin{aligned}
N(Tx,Tx,x) &\le N(Tx,Tx,Tx_n) + N(Tx,Tx,Tx_n) + N(x,x,Tx_n)\\
&\le kN(x,x,x_n) + \lambda N(Tx,Tx,x) + \mu N(x,x,x_n) + kN(x,x,x_n) + \lambda N(Tx,Tx,x)\\
&\quad + \mu N(x,x,x_n) + N(x,x,Tx_n)\\
&\le 2(k+\mu)N(x,x,x_n) + 2\lambda N(Tx,Tx,x) + N(x,x,x_{n+1}).
\end{aligned}
$$

This implies

$$(1-2\lambda)N(Tx,Tx,x) \le 2(k+\mu)N(x,x,x_n) + N(x,x,x_{n+1}),$$

$$N(Tx,Tx,x) \le \frac{1}{1-2\lambda}\{2(k+\mu)N(x,x,x_n) + N(x,x,x_{n+1})\}.$$

If we take limit as $n\to\infty$ and by using Lemma 4, then $N(Tx,Tx,x)=0$. From second condition of $N-$cone metric definition, $Tx=x$. Therefore, x is a fixed point of T in X.

Now, we must show uniqueness of fixed point. Suppose that y be another fixed point of T in X. We have

$$\begin{aligned} N(x,x,y) &= N(Tx,Tx,Ty) \\ &\le kN(x,x,y) + \lambda N(Tx,Tx,x) + \mu N(x,x,y) \\ &= kN(x,x,y) + \lambda N(x,x,x) + \mu N(x,x,y) \\ &= (k+\mu)N(x,x,y). \end{aligned}$$

By using Remark 1, we get $N(x,x,y)=0.$ Therefore $x=y.$

If we take $\lambda=\mu=0$ in Theorem 1, then we obtain the usual Banach contraction principal in the setting of a $N-$cone metric space.

Corollary 2.1. *Let (X,N) be a complete $N-$cone metric space and P be a normal cone with normal constant $K>0$. Assume that the self mapping $T: X\to X$ satisfying the following contractive condition*

$$N(Tx,Ty,Tz)\le kN(x,y,z) \text{ for all } x,y,z\in X$$

where $k\in[0,1)$ is a constant. Then, T has a unique fixed point in X and for every $x\in X$, iteration of the sequence $\{T^n x\}$ converges to the fixed point.

Corollary 2.2. *Let (X,N) be a complete $N-$cone metric space, P be a normal cone with normal constant $K>0$ and $B_N(x_0,c)=\{x\in X : N(x,x,x_0)<<c\}$ be an open $N-ball$ for $c\in E$ with $0<<c$ and $x_0\in X$. Assume that the self mapping $T: X\to X$ satisfying the following contractive condition*

$$N(Tx,Ty,Tz)\le kN(x,y,z) \text{ for all } x,y\in B(x_0,c)$$

where $k\in[0,1)$ is a constant. Then, T has a unique fixed point in $B(x_0,c)$.

Proof: Due to Corollary 1, we only must prove that $B_N(x_0,c)$ is complete and $Tx\in B_N(x_0,c)$ for all $x\in B_N(x_0,c)$. Let $\{x_n\}$ is a Cauchy sequence in $B_N(x_0,c)$. Since $B_N(x_0,c)\subset X$, it is also a Cauchy sequence in X. Since X is complete $N-$cone metric space, $\{x_n\}$ is convergent, that is, there exists a $x\in X$ such that $x_n\to x$ as $n\to\infty$. Then we get

$$\begin{aligned} N(x,x,x_0) &\le N(x,x,x_n)+N(x,x,x_n)+N(x_0,x_0,x_n) \\ &= 2N(x,x,x_n)+N(x_n,x_n,x_0) \\ &\le 2N(x,x,x_n)+c. \end{aligned}$$

Since $x_n\to x$, we get $N(x,x,x_0)=0.$ Then, $x\in B_N(x_0,c)$. Thus, $B_N(x_0,c)$ is complete.

Corollary 2.3. *Let (X,N) be a complete $N-$cone metric space and P be a normal cone with normal constant $K>0$. Assume that the self mapping $T: X\to X$ satisfying the following contractive condition*

$$N(T^n x,T^n y,T^n z)\le kN(x,y,z) \text{ for all } x,y,z\in X$$

where $k\in[0,1)$ is a constant and n is a positive integer. Then, T has a unique fixed point in X.

Proof: Due to Corollary 1, T^n has a unique fixed point x. Since $T^n(Tx)=T(T^n x)=Tx,$ Tx is also a fixed point of T^n. Since the fixed point of T^n is unique, $Tx=x$. Then x is also a fixed point of T. As the fixed point of T is also fixed point of T^n, T has a unique fixed point.

Example 2.1. *Let $E=R^3$, $P=\{(x,y,z): x,y,z\ge 0\}\subset E$, $X=R$, $a*b=a.b$ and $N: X^3\to E$ is defined by*

$N(x,y,z)=(\alpha(|x-z|+|y-z|),\beta(|x-z|+|y-z|),\gamma(|x-z|+|y-z|))$

where α,β,γ are positive constants. Then (X,N) is a complete $N-$cone metric space. Define a mapping $T:X\to X$ such that $Tx=\frac{x}{2}$. Then, T satisfies the following condition given in Theorem 1 as follows;

$N(Tx,Ty,Tz)\le kN(x,y,z)+\lambda N(Tx,Tx,y)+\mu N(x,x,z), \text{for all } x,y,z\in X$

with constant $k=\frac{1}{2}$ and $\lambda=\mu=0.$ Then, T has a unique fixed point $0\in X$.

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