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RESEARCH ARTICLE

Recurrence Relations of Moments of the Extended Power Lindley Distribution Based on General Progressively Type-II Right Censored Order Statistics and Characterization

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ABSTRACT

In this article, we establish recurrence relations single and product moments based ongeneral progressively Type-II right censored order statistics (GPTIIRCOS). Characterization for extended power Lindley distribution using relation between probability density function and distribution function is obtained. Moreover recurrence relations of single and product moments based on GPTIIRCOS are also used to characterize the distribution.

Keywords: Characterization; Extended Power Lindley Distribution; GeneralProgressively Type-II Right Censored Order Statistics.

1 Introduction

This paper considers a general progressively Type-II censoring scheme, this scheme is conducted as follows: The times of failure for the first k failures are not observed, and the $(k+1)^{th}$,..., $(m)^{th}$failures are observed at which times $R_{k+1}, R_{k+2}, \dots, R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units are censored. The resulting (m − k) ordered failure times are referred to as general progressively Type-II censored scheme. See Davis and Feldstein (1979) and Balakrishnan and Sandhu (1996).

If the failure times are based on anabsolutely continuous distribution function F with probability density function(pdf) f, the joint probability density function of the progressively Type II censored failuretimes $X_{k+1:m:n}, X_{k+2:m:n}, \dots, X_{m:m:n}$, is given by

$$f_{X_{k+1:m:n},\dots,X_{m:m:n}}(x_{k+1},\dots,x_m) = A_{(n,m-1)}[F(x_{k+1},\theta)]^k \prod_{i=k+1}^{m} f(x_i,\theta)[1-F(x_i,\theta)]^{R_i},$$

$$x_{k+1} < x_{k+2} < \cdots < x_m, \tag{1.1}$$

where,

$$A_{(n,m-1)} = \frac{n!}{k!\,(n-k-1)!}\left(\prod_{j=k+2}^{m} n - \sum_{i=k+1}^{j-1} R_i - j + 1\right).$$

Aggrawala and Balakrishnan (1996) derived recurrence relations for single and product moments of progressively Type-II right censored order statistics from exponential, Pareto and power function distributions and their truncated forms. Abd El-Aty and Mohie El-Din (2009) derived recurrence relations for single and double moments of generalized order statisticsfrom the inverted linear exponential distribution and any continuous function. MohieEl-Din, and Kotb (2011) derived recurrence relations for single and product moments of generalized order statistics for modified Burr XII-Geometric distribution and characterization. Salah (2012) also derived moments from progressively Type-II censored data of Marshall-Olkin exponential distribution.Kotb et al. (2013) derivedrecurrence relations for single and product moments of lower generalized order statistics from a general form of distributions and its characterizations. Mohie El-Dinet al. (2013a) derived emperical Bays estimators of reliability performances using progressively Type-II censoring from lomax model. Mohie El-Dinet al.(2013b) derived estimation for parameters of Feller-Pareto distribution from progressively Type-II censoring and some characterizations.Athar et al.(2014) derived some new moments of progressively Type-II right censored order statistics from Lindley distribution.Alkarni(2015) discussed extended power lindley distribution (A new statistical model for non-monotone survival data).Mohie El-Dinet al.(2017a,b) derived characterization for Gompertz and linear failure rate distributions using recurrence relations of single and product moments based on GPTIIRCOS.

The extended power Lindley distribution has been made toward the generalization of some well-known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others.

In this paper, we shall introducerecurrence relations single and product moments order statistics GPTIIRCOS.Characterization for extended power Lindley distribution using relation between probability density function and distribution function is obtained. Moreover recurrence relations of single and product moments based on GPTIIRCOSare also used to characterize the distribution.

Let $X_{k+1:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)} < X_{k+2:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)} < \cdots < X_{m:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)}$ be the $m$ordered observed failure times in a sample of size $n-k$ under general progressively Type-II right censoring scheme from theextended power Lindley distribution with probability density function (pdf) is given by

$$f(x,\theta,\alpha,\beta) = \frac{\alpha\theta^2}{\theta+\beta}[1+\beta x^\alpha]x^{\alpha-1}e^{-\theta x^\alpha}, \qquad \theta,\alpha,\beta > 0\,,\ \ x \geq 0, \tag{1.2}$$

andthe cumulative distribution function(cdf) is given by

$$F(x,\theta,\alpha,\beta) = 1 - \left[1 + \frac{\theta\beta}{\theta+\beta}x^\alpha\right]e^{-\theta x^\alpha}. \tag{1.3}$$

It may be noted that from (1.2) and (1.3) the relation between pdf and cdf is given by,

$$[\theta+\beta+\theta\beta x^\alpha]f(x) = \alpha\theta^2[1+\beta x^\alpha]x^{\alpha-1}[1-F(x)]. \tag{1.4}$$

Note (1)

We can obtain some distributions as particular cases extended power Lindley distributionwith pdf given by (1.2) as follows:

1- For$\beta = 1$, Eq. (1.2) reduces to the case of Power Lindley distribution.

2- For$\alpha = 1 and \beta = 1$, Eq. (1.2) reduces to the case of Lindley distribution.

3- For$\beta = 0$, Eq. (1.2) reduces to the case of Weibull distribution.
4- For$\alpha = 2 and \beta = 0$, Eq. (1.2) reduces to the case of Rayleigh distribution.
5- For$\alpha = 1 and \beta = 0$, Eq. (1.2) reduces to the case of exponentialdistribution.

For any continuous distribution, we shall denote the i^{th}single moment of theGPTIIRCOS in view of Eq. (1.1) as

$$\mu_{q:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i)}} = E\left[X_{q:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)}\right]^i$$

$$= K_{(n,m-1)} \iint \dots \int_{0<x_{k+1}<\dots<x_m<\infty} x_q^i [F(x_{k+1})]^k f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \times$$
$$f(x_{k+2})[1-F(x_{k+2})]^{R_{k+2}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} \dots dx_m, \quad (1.5)$$

and the i^{th} and j^{th} product moments as

$$\mu_{q,s:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i,j)}} = E\left[X_{q:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^i} X_{s:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^j}\right] =$$

$$= K_{(n,m-1)} \iint \dots \int_{0<x_{k+1}<\dots<x_m<\infty} x_q^i x_s^j [F(x_{k+1})]^k f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \times$$
$$f(x_{k+2})[1-F(x_{k+2})]^{R_{k+2}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} \dots dx_m. \quad (1.6)$$

2 Recurrence Relations of Single and Product Moments

In this section we introduce the recurrence relations for single and product momentsbased onGPTIIRCOS.

In the next theorem we introduce the recurrence relations for singlemomentsbased onGPTIIRCOS.

Theorem2. 1

If $X_{k+1:n} \leq X_{k+2:n} \leq \dots \leq X_{n:n}$ be the order stasistics of a random sample of size $(n-k)$ following extended power Lindley distribution, for$k+2 \leq r \leq m-1$, $m \leq n and i \geq 0$, then

$$\mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i+2\alpha)}} = \frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)} \mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i)}} + \frac{(i+2\alpha)}{\alpha\theta(R_r+1)} \mu_{r:m:n}^{(R_{k+1},\dots,R_m)^{(i+\alpha)}}$$
$$+ \frac{(n-R_{k+1}-\dots-R_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1:m-1:n}^{(R_{k+1},R_{k+2},\dots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\dots,R_m)^{(i+2\alpha)}}\right.$$
$$\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r-1:m-1:n}^{(R_{k+1},R_{k+2},\dots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\dots,R_m)^{(i+\alpha)}}\right] - \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i+\alpha)}}$$
$$- \frac{(n-R_{k+1}-\dots-R_r-r)}{(R_r+1)} \left[\mu_{r:m-1:n}^{(R_{k+1},R_{k+2},\dots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\dots,R_m)^{(i+2\alpha)}}\right.$$
$$\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r:m-1:n}^{(R_{k+1},R_{k+2},\dots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\dots,R_m)^{(i+\alpha)}}\right]. \quad (2.1)$$

Proof

From Eq. (1.4) and Eq. (1.5), we get

$$(\theta+\beta)\mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i)}} + \theta\beta\mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i+\alpha)}} =$$
$$C_{(n,m-1)} \iint \dots \int_{0<x_{k+1}<\dots<x_{r-1}<x_{r+1}<\dots<x_m<\infty} [F(x_{k+1})]^k Y_1(x_{r-1},x_{r+1}) \dots \times$$
$$f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \dots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots \times$$
$$f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} dx_{k+2} \dots dx_{r-1} dx_{r+1} \dots dx_m, \quad (2.2)$$

where

$$Y_1(x_{r-1},x_{r+1}) = \alpha\theta^2 \int_{x_{r-1}}^{x_{r+1}} x_r^i (1+\beta x_r^\alpha) x_r^{\alpha-1} [1-F(x_r)]^{R_r+1} dx_r. \quad (2.3)$$

Now, integrating by parts gives

$$Y_1(x_{r-1},x_{r+1}) = \alpha\theta^2 \left\{ \frac{x_{r+1}^{i+\alpha}[1-F(x_{r+1})]^{R_r+1} - x_{r-1}^{i+\alpha}[1-F(x_{r-1})]^{R_r+1}}{i+\alpha} \right.$$
$$+ \frac{\beta x_{r+1}^{i+2\alpha}[1-F(x_{r+1})]^{R_r+1} - \beta x_{r-1}^{i+2\alpha}[1-F(x_{r-1})]^{R_r+1}}{i+2\alpha}$$
$$+ \left(\frac{R_r+1}{i+\alpha}\right) \int_{x_{r-1}}^{x_{r+1}} x_r^{i+\alpha} f(x_r)[1-F(x_r)]^{R_r} dx_r$$
$$\left. + \frac{\beta(R_r+1)}{i+2\alpha} \int_{x_{r-1}}^{x_{r+1}} x_r^{i+2\alpha} f(x_r)[1-F(x_r)]^{R_r} dx_r \right\}. \qquad (2.4)$$

Now substituting for the resultant expression of $Y_1(x_{r-1},x_{r+1})$from Eq. (2.4) in Eq. (2.2) and simplifying, yields Eq. (2.1).This completes the proof.

In the next two theorems we introducerecurrence relations for product momentsbased onGPTIIRCOS.

Theorem 2.2

If$X_{k+1:n} \le X_{k+2:n} \le \ldots \le X_{n:n}$ be the order stasistics of arandom sample of size $(n-k)$ following extended power Lindley distribution, for$k+1 \le r < s \le m-1, m \le n$ thenand$i,j \ge 0$,

$$\mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i+2\alpha,j)}} = \frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)} \mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i,j)}} + \frac{(i+2\alpha)}{\alpha\theta(R_r+1)} \mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha,j)}}$$
$$+ \frac{(n-R_{k+1}-\cdots-R_{r-1}-r+1)}{(R_r+1)} \left[\mu_{r-1,s-1:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\ldots,R_m)^{(i+2\alpha,j)}} \right.$$
$$\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r-1,s-1:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\ldots,R_m)^{(i+\alpha,j)}} \right] - \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i+\alpha,j)}}$$
$$- \frac{(n-R_{k+1}-\cdots-R_r-r)}{(R_r+1)} \left[\mu_{r,s-1:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+2\alpha,j)}} \right.$$
$$\left. + \frac{i+2\alpha}{\beta(i+\alpha)} \mu_{r,s-1:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+\alpha,j)}} \right]. \qquad (2.5)$$

Proof

From Eq. (1.4) and Eq. (1.6), we get

$$(\theta+\beta)\mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i,j)}} + \theta\beta\mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i+\alpha,j)}} =$$
$$C_{(n,m-1)} \iint \ldots \int_{0<x_{k+1}<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty} x_s^j\, [F(x_{k+1})]^k Y_1(x_{r-1},x_{r+1}) f(x_{k+1}) \times$$
$$[1-F(x_{k+1})]^{R_{k+1}} \ldots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \ldots f(x_m) \times$$
$$[1-F(x_m)]^{R_m} dx_{k+1} dx_{k+2} \ldots dx_{r-1} dx_{r+1} \ldots dx_m, \qquad (2.6)$$

Substituting the resultant expression of $Y_1(x_{r-1},x_{r+1})$ from Eq. (2.4) in Eq. (2.6) and simplifying, yields Eq. (2.5).This completes the proof.

Theorem2. 3

If$X_{k+1:n} \le \ldots \le X_{n:n}$ be the order stasistics of a random sample of size $(n-k)$ following extended power Lindley distribution, for $k+1 \le r < s \le m-1$, $m \le n$and$i,j \ge 0$,then

$$\mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i,j+2\alpha)}} = \frac{(j+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_s+1)} \mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i,j)}} + \frac{(j+2\alpha)}{\alpha\theta(R_s+1)} \mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i,j+\alpha)}}$$
$$+ \frac{(n-R_{k+1}-\cdots-R_{s-1}-s+1)}{(R_s+1)} \left[\mu_{r,s-1:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+2\alpha)}} \right.$$
$$\left. + \frac{j+2\alpha}{\beta(j+\alpha)} \mu_{r,s-1:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+\alpha)}} \right] - \frac{j+2\alpha}{\beta(j+\alpha)} \mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i,j+\alpha)}}$$

$$-\frac{(n-R_{k+1}-\cdots-R_s-s)}{(R_s+1)}\Big[\mu_{r,s:m-1:n}^{(R_{k+1},\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+2\alpha)}}$$

$$+\frac{j+2\alpha}{\beta(j+\alpha)}\mu_{r,s:m-1:n}^{(R_{k+1},\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+\alpha)}}\Big]. \quad (2.7)$$

Proof

From Eq. (1.4) and Eq. (1.6), we get

$$(\theta+\beta)\mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i,j)}}+\theta\beta\mu_{r,s:m:n}^{(R_{k+1},R_{k+2},\ldots,R_m)^{(i,j+\alpha)}}=$$

$$C_{(n,m-1)}\iint\cdots\int_{0<x_{k+1}<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty} x_r^i[F(x_{k+1})]^k\,Y_2(x_{s-1},x_{s+1})f(x_{k+1})\times$$
$$[1-F(x_{k+1})]^{R_{k+1}}\ldots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\ldots f(x_m)\times$$
$$[1-F(x_m)]^{R_m}dx_{k+1}dx_{k+2}\ldots dx_{s-1}dx_{s+1}\ldots dx_m, \quad (2.8)$$

where

$$Y_2(x_{s-1},x_{s+1})=\alpha\theta^2\int_{x_{s-1}}^{x_{s+1}}x_s^j(1+\beta x_s^\alpha)x_s^{\alpha-1}[1-F(x_s)]^{R_s+1}dx_s. \quad (2.9)$$

Now, integrating by parts gives

$$Y_2(x_{s-1},x_{s+1})=\alpha\theta^2\Bigg\{\frac{x_{s+1}^{j+\alpha}[1-F(x_{s+1})]^{R_s+1}-x_{s-1}^{j+\alpha}[1-F(x_{s-1})]^{R_s+1}}{j+\alpha}$$
$$+\frac{\beta x_{s+1}^{j+2\alpha}[1-F(x_{s+1})]^{R_s+1}-\beta x_{s-1}^{j+2\alpha}[1-F(x_{s-1})]^{R_s+1}}{j+2\alpha}$$
$$+\left(\frac{R_s+1}{j+\alpha}\right)\int_{x_{s-1}}^{x_{s+1}}x_s^{j+\alpha}f(x_s)[1-F(x_s)]^{R_s}dx_s$$
$$+\frac{\beta(R_s+1)}{j+2\alpha}\int_{x_{s-1}}^{x_{s+1}}x_s^{j+2\alpha}f(x_s)[1-F(x_s)]^{R_s}dx_s\Bigg\}. \quad (2.10)$$

Now substituting for the resultant expression of $Y_2(x_{s-1},x_{s+1})$ from Eq. (2.10) in Eq. (2.8) and simplifying, yields Eq. (2.7).This completes the proof.

3The Characterizations

In this section we introduce the characterizations of the extended power Lindley distribution using the relation between pdf and cdfand using recurrence relations for single and product momentsbased onGPTIIRCOS.

3.1Characterizationvia differential equation for the extendedpowerLindley distribution

In the next theorem we introduce the characterization of the extended powerLindley distribution using relation between pdf and cdf.

Theorem 3.1

Let X be a continuous random variable with pdf $f(\cdot)$, cdf $F(\cdot)$and survival function $[1-F(\cdot)]$. Then X has extendedpower Lindley distribution iff

$$[\theta+\beta+\theta\beta x^\alpha]f(x)=\alpha\theta^2[1+\beta x^\alpha]x^{\alpha-1}[1-F(x)]. \qquad x\geq 0 \qquad (3.1)$$

Proof

Necessity:

From Eq. (1.2) and Eq. (1.3) we can easily obtain Eq. (3.1).

Sufficiency:

Suppose that X is a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. Suppose, also, that Eq. (3.1) is true. Then we have:

$$\frac{-d[1-F(x)]}{1-F(x)}=\frac{\alpha\theta[\theta+\theta\beta x^{\alpha}]x^{\alpha-1}}{\theta+\beta+\theta\beta x^{\alpha}}dx$$
$$=\frac{\alpha\theta[\theta+\beta+\theta\beta x^{\alpha}-\beta]x^{\alpha-1}}{\theta+\beta+\theta\beta x^{\alpha}}dx$$
$$=\alpha\theta x^{\alpha-1}dx-\frac{\alpha\beta\theta x^{\alpha-1}}{\theta+\beta+\theta\beta x^{\alpha}}dx.$$

On integrating, we get

$$-\ln|1-F(x)|=\theta x^{\alpha}-\ln|\theta+\beta+\theta\beta x^{\alpha}|+C,$$

where C is an arbitrary constant.

Now, since$[1-F(0)]=1$, then putting $x=0$ in this equation, we get $C=\ln|\theta+\beta|$.

Therefore,

$$\ln\left|\frac{[\theta+\beta+\theta\beta x^{\alpha}]}{[\theta+\beta][1-F(x)]}\right|=\theta x^{\alpha},$$

or,

$$\frac{[\theta+\beta+\theta\beta x^{\alpha}]}{[\theta+\beta][1-F(x)]}=e^{\theta x^{\alpha}}.$$

Hence,

$$F(x)=1-\left[1+\frac{\theta\beta}{\theta+\beta}x^{\alpha}\right]e^{-\theta x^{\alpha}}.$$

That is the distribution function of extended power Lindley distribution.

This completes the proof.

3.2Characterizationvia recurrence relations for single moments

In the next theorem we introduce thecharacterization of the extended power Lindley distribution using recurrence relations for single momentsbased onGPTIIRCOShas introduced in the following theorems.

Theorem3.2

Let $X_{k+1:n}\leq X_{k+2:n}\leq\ldots\leq X_{n:n}$ be the order stasistics of a random sample of size $(n-k)$. Then X has extended power Lindley distribution iff, for$k+2\leq r\leq m-1, m\leq nandi\geq 0$,

$$\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+2\alpha)}}=\frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)}\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i)}}+\frac{(i+2\alpha)}{\alpha\theta(R_r+1)}\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha)}}$$
$$-\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha)}}+\frac{(n-R_{k+1}-\cdots-R_{r-1}-r+1)}{(R_r+1)}\times$$
$$\left[\mu_{r-1:m-1:n}^{(R_{k+1},\ldots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\ldots,R_m)^{(i+2\alpha)}}+\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r-1:m-1:n}^{(R_{k+1},\ldots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\ldots,R_m)^{(i+\alpha)}}\right]$$
$$-\frac{(n-R_{k+1}-\cdots-R_r-r)}{(R_r+1)}\left[\mu_{r:m-1:n}^{(R_{k+1},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+2\alpha)}}\right.$$
$$\left.+\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+\alpha)}}\right]. \quad (3.2)$$

Proof

Necessity:

Theorem 2.1proved the necessary part of this theorem

Sufficiency:

Suppose that X is a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. Assuming that equation (3.2) holds, then we have:

$$\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+2\alpha)}}=\frac{(i+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)}\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i)}}+\frac{(i+2\alpha)}{\alpha\theta(R_r+1)}\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha)}}$$

$$-\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha)}}+\frac{(n-R_{k+1}-\cdots-R_{r-1}-r+1)}{(R_r+1)}\times$$
$$\left[\mu_{r-1:m-1:n}^{(R_{k+1},\ldots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\ldots,R_m)^{(i+2\alpha)}}+\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r-1:m-1:n}^{(R_{k+1},\ldots,R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\ldots,R_m)^{(i+\alpha)}}\right]$$
$$-\frac{(n-R_{k+1}-\cdots-R_r-r)}{(R_r+1)}\left[\mu_{r:m-1:n}^{(R_{k+1},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+2\alpha)}}\right.$$
$$\left.+\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r:m-1:n}^{(R_{k+1},R_{k+2},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+\alpha)}}\right], \quad (3.3)$$

where,

$$\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha)}}=C_{(n,m-1)}\iint\cdots\int_{0<x_{k+1}<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty}[F(x_{k+1})]^k Y_3(x_{r-1},x_{r+1})\times$$
$$f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}}\ldots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}}f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots\times$$
$$f(x_m)[1-F(x_m)]^{R_m}dx_{k+1}\ldots dx_{r-1}dx_{r+1}\ldots dx_m, \quad (3.4)$$

where

$$Y_3(x_{r-1},x_{r+1})=\int_{x_{r-1}}^{x_{r+1}}x_r^{i+\alpha}f(x_r)[1-F(x_r)]^{R_r}dx_r. \quad (3.5)$$

Now, integrating by parts gives

$$Y_3(x_{r-1},x_{r+1})=\frac{-1}{R_r+1}x_{r+1}^{i+\alpha}[1-F(x_{r+1})]^{1+R_r}+\frac{1}{1+R_r}x_{r-1}^{i+\alpha}[1-F(x_{r-1})]^{1+R_r}$$
$$+\frac{i+\alpha}{R_r+1}\int_{x_{r-1}}^{x_{r+1}}x_r^{i+\alpha-1}[1-F(x_r)]^{1+R_r}dx_r. \quad (3.6)$$

Now by substituting in Eq. (3.4), we get

$$\mu_{r:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha)}}=\frac{i+\alpha}{R_r+1}C_{(n,m-1)}\iint\cdots\int_{0<x_{k+1}<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty}[F(x_{k+1})]^k\times$$
$$f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}}\ldots\int_{x_{r-1}}^{x_{r+1}}x_r^{i+\alpha-1}[1-F(x_r)]^{1+R_r}dx_r\,f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}}$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f(x_m)[1-F(x_m)]^{R_m}dx_{k+1}\ldots dx_{r-1}dx_{r+1}\ldots dx_m$$
$$+\frac{C_{(n,m-1)}}{R_r+1}\iint\cdots\int_{0<x_{k+1}<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty}x_{r-1}^{i+\alpha}[F(x_{k+1})]^k f(x_{k+1})\times$$
$$[1-F(x_{k+1})]^{R_{k+1}}\ldots f(x_{r-1})[1-F(x_{r-1})]^{1+R_r+R_{r-1}}f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots\times$$
$$f(x_m)[1-F(x_m)]^{R_m}dx_{k+1}\ldots dx_{r-1}dx_{r+1}\ldots dx_m$$
$$-\frac{C_{(n,m-1)}}{R_r+1}\iint\cdots\int_{0<x_{k+1}<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty}x_{r+1}^{i+\alpha}[F(x_{k+1})]^k f(x_{k+1})\times$$
$$[1-F(x_{k+1})]^{R_{k+1}}\ldots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}}f(x_{r+1})[1-F(x_{r+1})]^{1+R_r+R_{r+1}}\ldots\times$$
$$f(x_m)[1-F(x_m)]^{R_m}dx_{k+1}\ldots dx_{r-1}dx_{r+1}\ldots dx_m$$
$$=C_{(n,m-1)}\frac{i+\alpha}{R_r+1}\iint\cdots\int_{0<x_{k+1}<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty}[F(x_{k+1})]^k f(x_{k+1})\times$$
$$[1-F(x_{k+1})]^{R_{k+1}}\ldots\int_{x_{r-1}}^{x_{r+1}}x_r^{i+\alpha-1}[1-F(x_r)]^{1+R_r}dx_r\,f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}}$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f(x_m)[1-F(x_m)]^{R_m}dx_{k+1}\ldots dx_{r-1}dx_{r+1}\ldots dx_m+$$
$$\frac{(n-R_{k+1}-\cdots-R_r-r)}{R_r+1}\mu_{r:m-1:n}^{(R_{k+1},\ldots R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+\alpha)}}$$
$$-\frac{(n-R_{k+1}-\cdots-R_{r-1}-r+1)}{R_r+1}\mu_{r-1:m-1:n}^{(R_{k+1},\ldots R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\ldots,R_m)^{(i+\alpha)}}, \quad (3.7)$$

and

$$\mu_{r:m:n}^{(R_{k+1},\dots,R_m)^{(i+2\alpha)}} = C_{(n,m-1)} \frac{i+2\alpha}{R_r+1} \iint \dots \int_{0<x_{k+1}<\dots<x_{r-1}<x_{r+1}<\dots<x_m<\infty} [F(x_{k+1})]^k \times$$
$$f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \dots \int_{x_{r-1}}^{x_{r+1}} x_r^{i+2\alpha-1}[1-F(x_r)]^{1+R_r} dx_r\, f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}}$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} \dots dx_{r-1} dx_{r+1} \dots dx_m$$
$$+\frac{(n-R_{k+1}-\dots-R_r-r)}{R_r+1} \mu_{r:m-1:n}^{(R_{k+1},\dots R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\dots,R_m)^{(i+2\alpha)}}$$
$$-\frac{(n-R_{k+1}-\dots-R_{r-1}-r+1)}{R_r+1} \mu_{r-1:m-1:n}^{(R_{k+1},\dots R_{r-2},(R_{r-1}+R_r+1),R_{r+1},\dots,R_m)^{(i+2\alpha)}}. \tag{3.8}$$

Now by substituting for $\mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i+\alpha)}}$ and $\mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i+2\alpha)}}$ from Eq. (3.7) and Eq. (3.8) in Eq. (3.3), we get

$$(\theta+\beta)\mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i)}} + \theta\beta\mu_{r:m:n}^{(R_{k+1},R_{k+2},\dots,R_m)^{(i+\alpha)}} =$$
$$C_{(n,m-1)}\alpha\theta^2 \iint \dots \int_{0<x_{k+1}<\dots<x_m<\infty} x_r^i(x_r^{\alpha-1}+\beta x_r^{2\alpha-1})[F(x_{k+1})]^k[1-F(x_r)]^{R_r+1}$$
$$f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \dots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots \times$$
$$f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} \dots dx_m. \tag{3.9}$$

We get

$$C_{(n,m-1)} \iint \dots \int_{0<x_{k+1}<\dots<x_m<\infty} x_r^i(\theta+\beta+\theta\beta x_r^\alpha) f(x_r)[1-F(x_r)]^{R_r}[F(x_{k+1})]^k \times$$
$$f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \dots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}$$
$$\times \dots f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} \dots dx_m$$
$$= C_{(n,m-1)}\alpha\theta^2 \iint \dots \int_{0<x_{k+1}<\dots<x_m<\infty} x_r^i(x_r^{\alpha-1}+\beta x_r^{2\alpha-1})[1-F(x_r)]^{R_r+1}[F(x_{k+1})]^k \times$$
$$f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \dots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots \times$$
$$f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} \dots dx_m.$$

We get

$$C_{(n,m-1)} \iint \dots \int_{0<x_{k+1}<\dots<x_m<\infty} \left[(\theta+\beta+\theta\beta x_r^\alpha) f(x_r) - \alpha\theta^2(x_r^{\alpha-1}+\beta x_r^{2\alpha-1})[1-F(x_r)]\right] \times$$
$$x_r^i[1-F(x_r)]^{R_r}[F(x_{k+1})]^k f(x_{k+1})[1-F(x_{k+1})]^{R_{k+1}} \dots f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}} \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{k+1} \dots dx_m = 0.$$

Using Muntz-Szasz theorem, [See, Hwang and Lin (1984)], we get

$$(\theta+\beta+\theta\beta x_r^\alpha) f(x_r) = \alpha\theta^2(x_r^{\alpha-1}+\beta x_r^{2\alpha-1})[1-F(x_r)].$$

Using Theorem 3.1, we get the distribution function of $m \leq nandi \geq 0$,power Lindley distribution.

$$F(x) = 1 - \left[1 + \frac{\theta\beta}{\theta+\beta} x^\alpha\right] e^{-\theta x^\alpha}.$$

This completes the proof.

3.3Characterizationvia recurrence relation for product moments

In this section we characterize the extended power Lindley distribution using recurrence relations for product momentsbased onGPTIIRCOS

Theorem3.3

Let $X_{k+1:n} \leq \dots \leq X_{n:n}$ be the order stasistics of a random sample of size $(n-k)$. Then X has extended power Lindley distribution iff, for $k+1 \leq r < s \leq m-1, m \leq n \text{and} i, j \geq 0$,

$$\mu_{r,s:m:n}^{(R_{k+1},\dots,R_m)^{(i+2\alpha,j)}} = \frac{(j+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_r+1)} \mu_{r,s:m:n}^{(R_{k+1},\dots,R_m)^{(i,j)}} + \frac{(j+2\alpha)}{\alpha\theta(R_r+1)} \mu_{r,s:m:n}^{(R_{k+1},\dots,R_m)^{(i+\alpha,j)}}$$

$$-\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i+\alpha,j)}}+\frac{(n-R_{k+1}-\cdots-R_{r-1}-r+1)}{R_r+1}\times$$
$$\left[\mu_{r-1,s-1:m-1:n}^{(R_{k+1},\ldots,R_{r-2},(R_{r-1}+R_r+1),\ldots,R_m)^{(i+2\alpha,j)}}+\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r-1,s-1:m-1:n}^{(R_{k+1},\ldots,R_{r-2},(R_{r-1}+R_r+1),\ldots,R_m)^{(i+\alpha,j)}}\right]$$
$$-\frac{(n-R_{k+1}-\cdots-R_r-r)}{R_r+1}\left[\mu_{r,s-1:m-1:n}^{(R_{k+1},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+2\alpha,j)}}\right.$$
$$\left.+\frac{i+2\alpha}{\beta(i+\alpha)}\mu_{r,s-1:m-1:n}^{(R_{k+1},\ldots,R_{r-1},(R_r+R_{r+1}+1),R_{r+2},\ldots,R_m)^{(i+\alpha,j)}}\right]. \quad (3.10)$$

Proof

Necessity:

Theorem 2.2 proved the necessary part of this theorem

Sufficiency:

Similarly as proved in theorem 3.2 we obtain the distribution function of extended power Lindley distribution given by

$$F(x)=1-\left[1+\frac{\theta\beta}{\theta+\beta}x^{\alpha}\right]e^{-\theta x^{\alpha}}.$$

This completes the proof.

Theorem3.4

Let $X_{k+1:n}\leq X_{k+2:n}\leq\ldots\leq X_{n:n}$ be the order stasistics of a random sample of size $(n-k)$. Then X has extendedpower Lindley distribution iff, for$k+1\leq r<s\leq m-1$, $m\leq n$and$i,j\geq 0$,

$$\mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i,j+2\alpha)}}=\frac{(j+2\alpha)(\theta+\beta)}{\beta\alpha\theta^2(R_s+1)}\mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i,j)}}+\frac{(j+2\alpha)}{\alpha\theta(R_s+1)}\mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i,j+\alpha)}}$$
$$+\left[\frac{(n-R_{k+1}-\cdots-R_{s-1}-s+1)}{(R_s+1)}\right]\left[\mu_{r,s-1:m-1:n}^{(R_{k+1},\ldots,(R_{s-2},R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+2\alpha)}}\right.$$
$$\left.+\frac{j+2\alpha}{\beta(j+\alpha)}\mu_{r,s-1:m-1:n}^{(R_{k+1},\ldots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+\alpha)}}\right]$$
$$-\frac{(n-R_{k+1}-\cdots-R_s-s)}{(R_s+1)}\left[\mu_{r,s:m-1:n}^{(R_{k+1},\ldots R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+2\alpha)}}\right.$$
$$\left.+\frac{j+2\alpha}{\beta(j+\alpha)}\mu_{r,s:m-1:n}^{(R_{k+1},\ldots R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+\alpha)}}\right]-\frac{j+2\alpha}{\beta(j+\alpha)}\mu_{r,s:m:n}^{(R_{k+1},\ldots,R_m)^{(i,j+\alpha)}}. \quad (3.11)$$

Proof

Necessity:

Theorem 2.3 proved the necessary part of this theorem

Sufficiency:

Similarly as proved in theorem 3.2 we obtain the distribution function of extended power Lindley distribution given by

$$F(x)=1-\left[1+\frac{\theta\beta}{\theta+\beta}x^{\alpha}\right]e^{-\theta x^{\alpha}}.$$

This completes the proof.

Note (2)

The same theorems can be obtained as a special cases for some distributions presented in Note (1).

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