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RESEARCH ARTICLE

A TWO – SMALL – PARAMETER DYNAMIC BUCKLING OF A PRE – STATICALLY LOADED QUADRATIC MODEL STRUCTURE WITH NONLINEAR AND LIGHT LINEAR DAMPING BUT STRUCK BY A STEP LOAD

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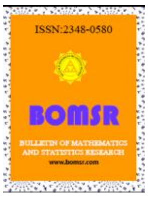

ABSTRACT

This investigation is concerned with analytical investigation of the dynamic buckling of a two – small – parameter pre – statically loaded quadratic elastic model structure that is afterwards struck by a step load. The procedure uses perturbation methods and asymptotic expansions of the relevant variables. The effects of nonlinear damping, imperfection and pre – load are clearly stated both in tabular forms and in graphical plots. All results are of course strictly asymptotic and hence, valid in the limit as the small parameters become increasingly small relative to unity.

Keywords: A two – small – parameter, dynamic buckling, pre-static buckling, nonlinear and light linear damping, step load.

1. Introduction

In this investigation, we beam our search light on the possible stabilizing effects that viscous damping (partly linear and nonlinear) could have on a pre-statically loaded quadratic elastic model structure that is afterwards struck by a step load.

In a series of investigations (Simitses 1983, 1986, 1987a, 1987b),Simitses first propounded and investigated the concept of pre-statically loaded elastic structures where such structures were finally struck by a dynamic load which, in most cases, wereimpulsive in nature. The pre-static load must be applied slowly so as not to generate any inertial response and thereby creating a truly static loading condition after which a dynamic load is further impounded on the structure. In this investigation, we are supplanting a step load of magnitude λ, $0 < \lambda < 1$, on an erstwhile pre-static load of magnitude λ_0 which satisfies the inequality $0 < \lambda < 1$.

The approach adopted here was first developed by Birman (Birman 1989), who investigated the problem of dynamic buckling of anti – symmetric rectangular laminates, while (Tabiei et al. 1999) similarly investigated numerical simulation of cylindrical laminated shells under impulsive lateral pressure.

To date, most investigations on pre-statically loaded structures appear to be approached by way of numerical investigations and such studies include (Tanov et al. 1999, Supratik et al. 2012, Tabiei&Birman1994,Zareiforoush et al. 2010), among others. However, an exception to this rule is the investigation by (Ette&Udo – Akpan2016) which forms the basis on which the current study is an extension.

2. Formulation of the Problem

The elastic model structure under investigation, was first studied, on the dynamic buckling setting by (Budiansky& Hutchinson 1966a, Hutchinson &Budiansky1966b, Ette&Udo – Akpan2016) and has continued to serve as a mathematical generalization of most real – life engineering structures in common use. In its simplest form, the model structure is composed of a two – arm elastic quadratic model structure (that is, with quadratic nonlinearity), with each arm of length L, and carrying mass M at their meeting point below which is hung a spring with restoring force per unit length of $KL(x - \alpha x^2)$, where K and L are constants for K > 0, L > 0, $\alpha > 0$, and x is the additional displacement from the point of equilibrium while $\bar{x}$ is the imperfection measurement (see Fig. 1).

A horizontal force $\bar{F}(T)$ is applied and as in (Budiansky& Hutchinson 1966a, Hutchinson &Budiansky1966b, Ette&Udo – Akpan2016), the nondimensional equation of motion of the undamped structure is given by

$$\ddot{\xi} + (1 - \lambda f(\bar{t}))\xi - a\xi^2 = \lambda\bar{\xi} f(\bar{t}), \quad \bar{t} > 0 \tag{1a}$$

$$\xi(0) = \dot{\xi}(0), \quad (\dot{\ }) = \frac{d(\ )}{d\bar{t}} \tag{1b}$$

where,

$$\xi = \frac{x}{L}, \quad \bar{\xi} = \frac{\bar{x}}{L}, \quad \bar{t} = T\sqrt{\frac{KL}{M}}, \ \lambda f(\bar{t}) = \frac{2F(0)}{KL^2}\left(\frac{F(T)}{F(0)}\right), \ F(0) \neq 0, \tag{1c}$$

where$0 < \bar{\xi} << 1$, $0 < \lambda < 1$ and $\lambda f(\bar{t})$ is the actual loading history. In our investigation, $\lambda f(\bar{t})$ will be a step load superposed on an erstwhile pre-static load.

However, since the structure under investigation is assumed damped, then the required governing equation of motion for any general load $\lambda f(\bar{t})$ is

$$\ddot{\xi} + 2\left(\left(\dot{\xi}\right)^2\theta + \epsilon\dot{\xi}\right) + (1 - \lambda f(\bar{t}))\xi - a\xi^2 = \lambda\bar{\xi} f(\bar{t}), \quad \bar{t} > 0 \tag{2}$$

where, $0 < \epsilon << 1$. Thus, the viscous damping is partly light and linear on one hand and partly heavy and nonlinear on the other hand. The nonlinear damping coefficient θ shall take the value $\theta = 1$ if nonlinear damping is allowed and $\theta = 0$ if nonlinear damping is not allowed. Our aim is to determine the dynamic buckling load λ_D for which the structure buckles dynamically and this is obtained (Budiansky& Hutchinson 1966a, Hutchinson &Budiansky1966b)from the condition

$$\frac{d\lambda}{d\xi_a} = 0 \tag{3}$$

where, ξ_a is the maximum displacement. We define λ_D as the largest load parameter for the solution of the problem to be bounded.

3.1 Static Deformation

Let ξ_0 be the displacement when the structure is pre-statically loaded. Then ξ_0 satisfies the equation (from (2))

$$(1-\lambda_0)\xi_0 - a\xi_0^2 = \lambda_0\bar{\xi} \quad (4)$$

where we have neglected the inertia term as well as other time dependent terms and set $f(\bar{t}) \equiv 1$. The solution of (4) is

$$\xi_0 = \frac{(1-\lambda_0) \pm \sqrt{(1-\lambda_0)^2 - 4\lambda_0\bar{\xi}}}{2a} \quad (5)$$

However, since we are looking for the asymptotic solution, we shall not make use of (5) but now let

$$\xi_0 = \sum_{i=o}^{\infty} \xi_0^{(i)}\,\bar{\xi}^i \quad (6)$$

Thus, we get

$$\boldsymbol{O}(\bar{\xi}): (1-\lambda_0)\xi_0^{(1)} = \lambda_0 \quad (7a)$$

$$\boldsymbol{O}(\bar{\xi}^2): (1-\lambda_0)\xi_0^{(2)} = a\left(\left(\xi_0^{(1)}\right)^2\right) \quad (7b)$$

$$\boldsymbol{O}(\bar{\xi}^3): (1-\lambda_0)\xi_0^{(3)} = 2a\xi_0^{(1)}\xi_0^{(2)} \quad (7c)$$

On solving (7a – c), we get

$$\xi_0^{(1)} = \frac{\lambda_0}{1-\lambda_0} \equiv B_0 \quad (8a)$$

$$\xi_0^{(2)} = \frac{B_0^2}{1-\lambda_0}, \quad \xi_0^{(3)} = \frac{2B_0^3}{(1-\lambda_0)^2} \quad (8b)$$

etc.

Thus, we get

$$\xi_0 = B_0\bar{\xi} + \frac{aB_0^2}{1-\lambda_0}\bar{\xi}^2 + \frac{2aB_0^3}{(1-\lambda_0)^2}\bar{\xi}^3 + \cdots \quad (8c)$$

3.2. Dynamic Deformation at imposition of Step Load

If the structure were trapped by a step load at inception, then the equation of motion would be as in (2) for $f(\bar{t}) = 1$. Let $\zeta(\bar{t})$ be the displacement due to combined static load λ_0 and step load of magnitude λ. Then, the combined displacement is

$$\zeta(\bar{t}) = \xi_0 + \xi(\bar{t}) \quad (9)$$

So that equation of motion at this stage is

$$\ddot{\zeta} + 2\left(\theta\dot{\zeta}^2 + \epsilon\dot{\zeta}\right) + (1-\lambda_0)\,\xi_0 + (1-\lambda)\xi + a\zeta^2 = (\lambda_0+\lambda)\bar{\xi}, \quad \bar{t} > 0 \quad (10a)$$

$$\zeta(0) = \dot{\zeta}(0) = \xi(0) = \dot{\xi}(0) = 0 \quad (10b)$$

where we shall henceforth multiply the nonlinear damping in (10a) namely $\dot{\zeta}^2$, by a constant θ, for $\theta = 0$ or $\theta = 1$. When $\theta = 1$, we get a nonzero nonlinear damping whereas when $\theta = 0$, we get a case of no nonlinear damping.

On substituting for $\zeta(\bar{t})$ in (10a) from (9) and knowing that the required equation of motion is the difference (10a) – (4), we get

$$\ddot{\xi} + 2\left(\theta\dot{\xi}^2 + \epsilon\dot{\xi}\right) + (1-\lambda)\xi - a(2\beta\,\xi\xi_0 + \xi^2) = \lambda\bar{\xi}, \quad \bar{t} > 0 \quad (11a)$$

$$\xi(0) = \dot{\xi}(0) = 0, \quad (11b)$$

for$\beta = 1$ or $\beta = 0$. When $\beta = 1$, we get the required equation of motion at the imposition of step load on the pre-static load, but when $\beta = 0$, we retrieve (2). We specifically note the coupling term (in (11a)) between the displacements produced by the pre-static load λ_0 and that produced by the imposed step load with magnitudeλ. For the solution of (11a, b), we let

$$\tau = \epsilon\bar{t}, \qquad t = \bar{t}(1-\lambda)^{\frac{1}{2}} + \frac{1}{\epsilon}\left(\mu_1\bar{\xi} + \mu_2\bar{\xi}^2 + \cdots\right) \tag{12}$$

$$\mu_i = \mu_i(\tau), \qquad \mu_i(0) = 0, \qquad i = 1, 2, 3, \ldots$$

$$\therefore \dot{\xi} = \frac{\partial\xi}{\partial t}\frac{\partial t}{\partial \bar{t}} + \frac{\partial\xi}{\partial t}\frac{\partial t}{\partial \tau}\frac{d\tau}{d\bar{t}} + \frac{\partial\xi}{\partial \tau}\frac{d\tau}{d\bar{t}} = (1-\lambda)^{\frac{1}{2}}\xi_{,t} + \left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,t} + \epsilon\xi_{,\tau} \tag{13a}$$

where$(\acute{\ }) = \frac{d(\ldots)}{d\tau}$ and where a comma preceding a subscript denotes partial differentiation. We also get

$$\ddot{\xi} = (1-\lambda)\xi_{,tt} + \left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)^2\xi_{,tt} + \epsilon^2\xi_{,\tau\tau} + 2(1-\lambda)^{\frac{1}{2}}\left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,tt}$$
$$+ 2\epsilon\left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,t\tau} + 2\epsilon(1-\lambda)^{\frac{1}{2}}\xi_{,t\tau} + \epsilon\left(\mu_1''\bar{\xi} + \mu_2''\bar{\xi}^2 + \cdots\right)\xi_{,t} \tag{13b}$$

On substituting (13a,b) into (11a, b), we get

$$\Big[(1-\lambda)\xi_{,tt} + \left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)^2\xi_{,tt} + \epsilon^2\xi_{,\tau\tau} + 2(1-\lambda)^{\frac{1}{2}}\left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,tt}$$
$$+ 2\epsilon\left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,t\tau} + 2\epsilon(1-\lambda)^{\frac{1}{2}}\xi_{,t\tau} + \epsilon\left(\mu_1''\bar{\xi} + \mu_2''\bar{\xi}^2 + \cdots\right)\xi_{,t}\Big]$$
$$+ 2\Bigg[\theta\left\{(1-\lambda)^{\frac{1}{2}}\xi_{,t} + \left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,t} + \epsilon\xi_{,\tau}\right\}^2$$
$$+ \epsilon\left\{(1-\lambda)^{\frac{1}{2}}\xi_{,t} + \left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,t} + \epsilon\xi_{,\tau}\right\}\Bigg] + (1-\lambda)\xi - a(2\beta\xi\xi_0 + \xi^2)$$
$$= \lambda\bar{\xi} \tag{14}$$

We let

$$\xi(t,\tau) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\xi^{(ij)}(t,\tau)\bar{\xi}^i\epsilon^j, \tag{15}$$

where the (ij) as in $\xi^{(ij)}$ denote superscripts and not powers.
On equating equations of orders of $\xi\epsilon$ in (14), we get

$$\boldsymbol{O}(\bar{\xi}):\ \xi^{10}_{,tt} + \xi^{10} = \frac{\lambda}{1-\lambda} \equiv B \tag{16}$$

$$\boldsymbol{O}(\bar{\xi}\epsilon):\ \xi^{11}_{,tt} + \xi^{11} = -2(1-\lambda)^{-\frac{1}{2}}\left(\xi^{10}_{,t\tau} + \xi^{10}_{,t}\right) \tag{17}$$

$$\boldsymbol{O}(\bar{\xi}\epsilon^2):\ \xi^{12}_{,tt} + \xi^{12} = -2(1-\lambda)^{-\frac{1}{2}}\left(\xi^{11}_{,t\tau} + \xi^{11}_{,t}\right) - 2(1-\lambda)^{-1}\xi^{10}_{,\tau} - (1-\lambda)^{-1}\xi^{10}_{,\tau\tau} \tag{18}$$

$$\boldsymbol{O}(\bar{\xi}^2):\ \xi^{20}_{,tt} + \xi^{20}$$
$$= -2(1-\lambda)^{-\frac{1}{2}}\mu_1'\xi^{10}_{,tt} - 2\theta\left(\xi^{10}_{,t}\right)^2 + (1-\lambda)^{-1}\left\{2\beta\xi_0^{(1)}\xi_0^{10} + (\xi^{10})^2\right\} \tag{19}$$

$$\boldsymbol{O}(\bar{\xi}^2\epsilon):\ \xi^{21}_{,tt} + \xi^{21}$$
$$= -2(1-\lambda)^{-\frac{1}{2}}\mu_1'\xi^{11}_{,tt} - 2(1-\lambda)^{-1}\mu_1'\xi^{10}_{,t\tau} - 2(1-\lambda)^{-\frac{1}{2}}\xi^{20}_{,t\tau} - (1-\lambda)^{-1}\mu_1''\xi^{10}_{,t}$$
$$- 4\theta(1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,t}\xi^{11}_{,t} - 4\theta\xi^{10}_{,t}\xi^{10}_{,\tau} - 2(1-\lambda)^{-\frac{1}{2}}\xi^{20}_{,t} - 2(1-\lambda)^{-1}\mu_1'\xi^{10}_{,t}$$
$$+ a(1-\lambda)^{-1}\left\{2\beta\xi^{11}\xi_0^{(1)} + 2\xi^{10}\xi^{11}\right\} \tag{20}$$

$$\boldsymbol{O}(\bar{\xi}^2\epsilon^2):\ \xi^{22}_{,tt} + \xi^{22}$$
$$= -2(1-\lambda)^{-\frac{1}{2}}\mu_1'\xi^{12}_{,tt} - 2(1-\lambda)^{-1}\mu_1'\xi^{11}_{,t\tau} - 2(1-\lambda)^{-\frac{1}{2}}\xi^{21}_{,t\tau} - (1-\lambda)^{-1}\mu_1''\xi^{11}_{,t}$$
$$- 4\theta\left\{\xi^{10}_{,t}\xi^{12}_{,t} + \left(\xi^{11}_{,t}\right)^2\right\} - 4\theta(1-\lambda)^{-\frac{1}{2}}\left(\xi^{10}_{,t}\xi^{11}_{,\tau} + \xi^{11}_{,t}\xi^{10}_{,\tau}\right) - 2(1-\lambda)^{-\frac{1}{2}}\xi^{21}_{,t}$$
$$- 2(1-\lambda)^{-1}\mu_1'\xi^{11}_{,t} + a(1-\lambda)^{-1}\left[2\beta\xi_0^{(1)}\xi^{12} + \{(\xi^{11})^2 + 2\xi^{10}\xi^{12}\}\right] - (1-\lambda)^{-1}\xi^{20}_{,\tau\tau}$$
$$- 2(1-\lambda)^{-1}\xi^{20}_{,\tau}$$
$$- 2(1-\lambda)^{-1}\theta\left(\xi^{10}_{,\tau}\right)^2 \tag{21}$$

etc.

The associated initial conditions, evaluated at $(t,\tau) = (0,0)$ are

$$\xi^{ij}(0,0) = 0, \quad i = 1,2,3,\dots, j = 0,1,2,3,\dots \tag{22a}$$

$$\boldsymbol{O}(\bar{\xi}):\ \xi^{10}_{,t} = 0, \quad \boldsymbol{O}(\bar{\xi}\epsilon):\ \xi^{11}_{,t} + \xi^{10}_{,\tau} = 0, \quad \boldsymbol{O}(\bar{\xi}\epsilon^2):\ \xi^{12}_{,t} + \xi^{11}_{,\tau} = 0 \tag{22b}$$

$$\boldsymbol{O}(\bar{\xi}^2):\ \xi^{20}_{,t} + (1-\lambda)^{-\frac{1}{2}}\mu_1'(0)\xi^{10}_{,t} = 0 \tag{22c}$$

$$\boldsymbol{O}(\bar{\xi}^2\epsilon):\ \xi^{21}_{,t} + (1-\lambda)^{-\frac{1}{2}}\{\mu_1'(0)\xi^{11}_{,t} + \xi^{10}_{,\tau}\} = 0 \tag{22d}$$

$$\boldsymbol{O}(\bar{\xi}^2\epsilon^2):\ \xi^{22}_{,t} + (1-\lambda)^{-\frac{1}{2}}\{\mu_1'(0)\xi^{12}_{,t} + \xi^{11}_{,\tau}\} = 0 \tag{22e}$$

etc.

On solving (16) with (22a, b), we get

$$\xi^{10}(t,\tau) = \alpha_{10}(\tau)\cos t + \beta_{10}(\tau)\sin t + B \tag{23a}$$

$$\alpha_{10}(0) = -B, \quad \beta_{10}(0) = 0 \tag{23b}$$

On substituting (23a) into (17) and ensuring a uniformly valid solution in t, we get

$$\alpha_{10}' + \alpha_{10} = 0, \quad \beta_{10}' + \beta_{10} = 0 \tag{24}$$

On solving (24) with (23b), we get

$$\alpha_{10}(\tau) = -Be^{-\tau}, \quad \beta_{10}(\tau) = 0 \tag{25a}$$

Thus, we get

$$\alpha_{10}'(0) = B, \quad \alpha_{10}''(0) = -B \tag{25b}$$

We also get

$$\xi^{10}(t,\tau) = \alpha_{10}(\tau)\cos t + B \tag{26}$$

Solving the remaining equation in the substitution into (17), we get,

$$\xi^{11}(t,\tau) = \alpha_{11}(\tau)\cos t + \beta_{11}(\tau)\sin t \tag{27a}$$

$$\alpha_{11}(0) = 0, \quad \beta_{11}(0) = -\alpha_{10}'(0) = -B \tag{27b}$$

We next substitute for terms, from (26) and (27a) into (18) and get

$$\begin{aligned}\xi^{12}_{,tt} &+ \xi^{12} \\ &= -2(1-\lambda)^{-\frac{1}{2}}\{-\alpha_{11}'\sin t + \beta_{10}'\cos t - \alpha_{11}\sin t + \beta_{11}\cos t\} \\ &- 2(1-\lambda)^{-1}\alpha_{10}'\cos t \\ &- (1-\lambda)^{-1}\alpha_{10}''\cos t\end{aligned} \tag{28a}$$

To ensure a uniformly valid solution in t, we equate to zero the coefficients of cost and sintin (28a) and respectively get

$$\beta_{11}' + \beta_{11} = -\frac{1}{2}(1-\lambda)^{-\frac{1}{2}}(\alpha_{10}' + \alpha_{10}''), \quad and \quad (\alpha_{11}' + \alpha_{10}) = 0 \tag{28b}$$

On solving (28b) using (27b), we get

$$\beta_{11}(\tau) = e^{-\tau}\left[-\frac{1}{2}\int(1-\lambda)^{-\frac{1}{2}}(\alpha_{10}' + \alpha_{10}'')e^{\tau}d\tau + \beta_{11}(0)\right] \tag{28c}$$

$$\alpha_{11}(\tau) = 0, \quad \beta_{11}'(0) = -\beta_{11}(0) - \frac{1}{2}(1-\lambda)^{-\frac{1}{2}}(\alpha_{10}' + \alpha_{10}'') \tag{28d}$$

$$\text{i.e } \beta_{11}'(0) = B \tag{28e}$$

The remaining equation in (28a) is next solved to get

$$\xi^{12}(t,\tau) = \alpha_{12}(\tau)\cos t + \beta_{12}(\tau)\sin t \tag{29a}$$

where,

$$\alpha_{12}(0) = \beta_{12}(0) = 0 \tag{29b}$$

We next substitute into (19) and get

$$\xi^{20}_{,tt} + \xi^{20} = 2\mu_1'(1-\lambda)^{-\frac{1}{2}}\alpha_{10}\cos t + a(1-\lambda)^{-1}\left[\left\{\left(\frac{\alpha_{10}^2}{2} + B^2\right) + 2B\alpha_{10}\cos t + \frac{\alpha_{10}^2}{2}\cos 2t\right\}\right.$$
$$\left. + 2\beta\xi_0^{(1)}(\alpha_{10}\cos t + B)\right] - \theta\alpha_{10}^2(1-\cos 2t) \qquad (30a)$$

To ensure a uniformly valid solution in t, we equate to zero, in (30a) and get

$$\mu_1' = -(1-\lambda)^{\frac{1}{2}}B\left\{a(1-\lambda)^{-1} + \beta\left(\frac{B_0}{B}\right)\right\}$$

We shall however write

$$\mu_1' = -B(1-\lambda)^{\frac{1}{2}}r_0, \quad r_0 = \left\{a(1-\lambda)^{-1} + \beta\left(\frac{B_0}{B}\right)\right\} \qquad (30c)$$

The remaining equation in (30a) is

$$\xi^{20}_{,tt} + \xi^{20} = r_1 + r_2\cos 2t, \quad \xi^{20}(0,0) = 0, \quad \xi^{20}_{,t}(0,0) + \mu_1'(0)(1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,t}(0,0) = 0 \quad (31a)$$

where,

$$r_1 = a(1-\lambda)^{-1}\left\{\left(\frac{\alpha_{10}^2}{2} + B^2\right) + 2\beta B\xi_0^{(1)}\right\} - \theta\alpha_{10}^2 \qquad (31b)$$

$$r_2 = \alpha_{10}^2\left(\theta + \frac{a(1-\lambda)^{-1}}{2}\right) \qquad (31c)$$

and where

$$r_1(0) = \frac{3}{2}(1-\lambda)^{-1}aB^2q_0, \quad q_0 = \left[1 + \frac{4\beta}{3}\left(\frac{B_0}{B}\right) - \frac{2\theta}{3a(1-\lambda)}\right] \qquad (31d)$$

$$r_2(0) = B^2q_1, \quad q_1 = \left(\theta + \frac{a(1-\lambda)^{-1}}{2}\right) \qquad (31e)$$

$$r_1'(0) = B^2q_2, \quad q_2 = (2 - a(1-\lambda)^{-1}) \qquad (31f)$$

$$r_1''(0) = -2B^2q_2, \quad r_2'(0) = -B^2(2\theta + a(1-\lambda)^{-1}) \qquad (31g)$$

$$r_2''(0) = 2B^2(2\theta + a(1-\lambda)^{-1}) \qquad (31h)$$

The solution of (31a) is

$$\xi^{20}(t,\tau) = \alpha_{20}(\tau)\cos t + \beta_{20}(\tau)\sin t + r_1 - \frac{r_2\cos 2t}{3} \qquad (32a)$$

$$\alpha_{20}(0) = \frac{r_2(0)}{3} - r_1(0) = B^2q_{16} \qquad (32b)$$

$$q_{16} = \frac{q_1}{3} - \frac{3}{2}a(1-\lambda)^{-1}q_0, \quad \beta_{20}(0) = 0 \qquad (32c)$$

We next substitute in (20) and get

$$\xi^{21}_{,tt} + \xi^{21} = 2\mu_1'(1-\lambda)^{-\frac{1}{2}}\beta_{11}\sin t + 2\mu_1'(1-\lambda)^{-1}\alpha_{10}'\sin t$$
$$+ 2(1-\lambda)^{-\frac{1}{2}}\left\{\alpha_{20}'\sin t - \beta_{20}\cos t - \frac{2r_2'\sin 2t}{3}\right\}$$
$$+ 2\theta\alpha_{10}\beta_{11}\sin 2t + 2(1-\lambda)^{-\frac{1}{2}}\alpha_{10}\alpha_{10}'\sin 2t$$
$$+ 2(1-\lambda)^{-\frac{1}{2}}\left\{\alpha_{20}\cos t - \beta_{20}\cos t - \frac{2r_2\sin 2t}{3}\right\}$$
$$+ 2\mu_1'(1-\lambda)^{-1}\alpha_{10}\sin t$$
$$+ a(1-\lambda)^{-1}\left\{2\beta\xi_0^{(1)}\beta_{11}\sin t + \alpha_{10}\beta_{11}\sin 2t + 2B\beta_{11}\sin t\right\} \qquad (34a)$$

$$\xi^{21}(0,0) = 0, \quad \xi^{21}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\{\mu_1'(0)\xi^{11}_{,t}(0,0) + \xi^{10}_{,\tau}(0,0)\} = 0 \qquad (34b)$$

To ensure a uniformly valid solution in t, we equate to zero, in (34a) the coefficients of cost and sint and respectively get

$$\beta_{20}' + \beta_{20} = 0 \qquad (34c)$$

$$\alpha_{20}' + \alpha_{20} = h_1(\tau) \tag{34d}$$

where,

$$h_1(\tau) = -(1-\lambda)^{\frac{1}{2}}\Big[\mu_1'(1-\lambda)^{-1}\alpha_{10}' + (1-\lambda)^{-\frac{1}{2}}\mu_1'\beta_{11} + (1-\lambda)^{-1}\mu_1'\alpha_{10} + \alpha(1-\lambda)^{-1}\left\{\beta\xi_0^{(1)}\beta_{11} + B\beta_{11}\right\}\Big] \tag{34e}$$

The solutions of (34c, d), using (32b, c) are

$$\beta_{20} = 0, \quad \alpha_{20} = e^{-\tau}\int h_1(\tau)\, e^{\tau} ds \tag{34f}$$

where,

$$h_1(0) = B^2 q_5, \quad q_5 = r_1 - a(1-\lambda)^{-1}\left(1 + \beta\left(\frac{B_0}{B}\right)\right) \tag{34g}$$

and

$$\alpha_{20}'(0) = h_1(0) - \alpha_{20}(0) = B^2 q_6, \quad q_6 = q_5 - q_{16} \tag{34h}$$

On solving (34c, d), we get

$$\beta_{20}(\tau) = 0, \quad \alpha_{20}(\tau) = e^{-\tau}\left[\alpha_{20}(0) + \int h_1(\tau)\, e^{\tau} d\tau\right] \tag{34i}$$

The remaining equation in (34a) is now written as

$$\xi_{,tt}^{21} + \xi^{21} = r_3 \sin 2t \tag{35a}$$

where,

$$r_3 = -\frac{4}{3}(1-\lambda)^{-\frac{1}{2}}(r_2' + r_2) + 2\theta\alpha_{10}\beta_{11} + 2(1-\lambda)^{-\frac{1}{2}}\alpha_{10}\alpha_{10}' + a(1-\lambda)^{-1}\alpha_{10}\beta_{11} \tag{35b}$$

and where

$$r_3(0) = B^2 q_3, \quad q_3 = -\frac{4}{3}(1-\lambda)^{-\frac{1}{2}}\{q_1 - (2\theta + a(1-\lambda)^{-1})\} + 2\theta - 2(1-\lambda)^{-\frac{1}{2}} + a(1-\lambda)^{-1} \tag{35c}$$

The solution of (35a) together with (34b) is

$$\xi^{21}(t,\tau) = \alpha_{21}(\tau)\cos t + \beta_{21}(\tau)\sin t - \frac{r_3 \sin 2t}{3} \tag{36a}$$

$$\alpha_{21}(0) = 0; \quad \beta_{21}(0) - \frac{2r_3(0)}{3} + (1-\lambda)^{-\frac{1}{2}}\{\mu_1'(0)\beta_{11}(0) + \alpha_{10}'(0)\} = 0 \tag{36b}$$

This yields

$$\beta_{21}(0) = B^2 q_4, \quad q_4 = \left\{\frac{2}{3}q_3 - (1-\lambda)^{-\frac{1}{2}}\left(r_0(1-\lambda)^{-\frac{1}{2}} + \frac{1}{B}\right)\right\} \tag{36c}$$

A detailed evaluation of $r_3'(0)$ (to be used later) from (35b), yields

$$r_3'(0) = B^2 q_9 \tag{36d}$$

$$q_9 = 4(1-\lambda)^{-\frac{1}{2}} - \left(\frac{4(1-\lambda)^{-\frac{1}{2}}}{3} + 2\right)\{2\theta + a(1-\lambda)^{-1}\}, \tag{36e}$$

While from (34e), we have

$$h_1'(0) = B^2(1-\lambda)^{-\frac{1}{2}} q_7, \quad q_7 = r_0 - a(1-\lambda)^{-1}\left(1 + \beta\frac{B_0}{B}\right) \tag{37a}$$

Thus, from (34d), we get

$$\alpha_{20}''(0) = h_1'(0) - \alpha_{20}''(0) = B^2 q_8, \quad q_8 = \left(q_7(1-\lambda)^{-\frac{1}{2}} - q_6\right) \tag{37b}$$

The following terms are worthy of evaluation before substitution into (21).

$$-4\theta\xi_{,t}^{10}\xi_{,t}^{12} = -4\theta\left\{\frac{\alpha_{10}\alpha_{12}}{2}(1-\cos 2t) - \frac{\alpha_{10}\beta_{12}}{2}\sin 2t\right\} \tag{38a}$$

$$\left(\xi_{,t}^{11}\right)^2 = \frac{(\beta_{11})^2}{2}(1-\cos 2t) \tag{38b}$$

$$-4(1-\lambda)^{-\frac{1}{2}}\theta\xi_{,t}^{10}\xi_{,\tau}^{11} = 4\theta(1-\lambda)^{-\frac{1}{2}}\left(\frac{\alpha_{10}\beta_{11}'}{2}(1-\cos 2t)\right) \tag{38c}$$

$$-4(1-\lambda)^{-\frac{1}{2}}\theta\xi_{,t}^{11}\xi_{,\tau}^{10} = -4\theta(1-\lambda)^{-\frac{1}{2}}\left(\frac{\alpha_{10}'\beta_{11}}{2}(1+\cos 2t)\right) \tag{38d}$$

Substituting these and other terms in (21), we get

$$\begin{aligned}
\xi_{,tt}^{22} + \xi^{22} = {} & 2\mu_1'(1-\lambda)^{-\frac{1}{2}}(\alpha_{12}\cos t + \beta_{12}\sin t) \\
& - 2(1-\lambda)^{-1}\mu_1'\beta_{11}'\cos t - 2(1-\lambda)^{-\frac{1}{2}}\left[-\alpha_{21}'\sin t + \beta_{21}'\cos t - \frac{2r_3}{3}\cos 2t\right] \\
& - 4\theta\left[\frac{\alpha_{10}\alpha_{12}}{2}(1-\cos 2t) - \frac{\alpha_{10}\beta_{12}}{2}\sin 2t + \frac{(\beta_{11})^2}{2}(1-\cos 2t)\right] \\
& - 4\theta(1-\lambda)^{-\frac{1}{2}}\left[\frac{\alpha_{10}\beta_{11}'}{2}(1-\cos 2t) + \frac{\alpha_{10}'\beta_{11}}{2}(1+\cos 2t)\right] \\
& - 2(1-\lambda)^{-\frac{1}{2}}\left[\alpha_{21}\sin t + \beta_{21}\cos t - \frac{2r_3}{3}\cos 2t\right] \\
& - 2(1-\lambda)^{-1}\mu_1'\beta_{11}\cos t + a(1-\lambda)^{-1}\Bigg[2\beta\xi_0^{(1)}(\alpha_{12}\cos t + \beta_{12}\sin t) \\
& + \left\{\frac{(\beta_{11})^2}{2}(1-\cos 2t) + \alpha_{10}\alpha_{12}(1\right. \\
& \left. + \cos 2t) + \alpha_{10}\beta_{12}\sin 2t + 2B(\alpha_{12}\cos t + \beta_{12}\sin t)\right\}\Bigg] \\
& - (1-\lambda)^{-1}\left(\alpha_{20}''\cos t - \frac{r_2''}{3}\cos 2t\right) - 2(1-\lambda)^{-1}\left(\alpha_{20}'\cos t - \frac{r_2'}{3}\cos 2t\right) \\
& - \theta(1-\lambda)^{-1}(\alpha_{10}')^2(1+\cos 2t)
\end{aligned} \tag{39a}$$

We ensure a uniformly valid solution in t by equating to zero, in (39a), the coefficients of cost and sint to respectively get

$$\beta_{21}' + \beta_{21} = h_2(\tau) \tag{39b}$$

where,

$$\begin{aligned}
h_2(\tau) = {} & \frac{(1-\lambda)^{-\frac{1}{2}}}{2}\Big[2\left\{(1-\lambda)^{-\frac{1}{2}}\mu_1'\beta_{12} - (1-\lambda)^{-1}\mu_1''\beta_{11}'\right\} - 2(1-\lambda)^{-1}\mu_1'\beta_{11} \\
& - (1-\lambda)^{-1}(\alpha_{20}'' + \alpha_{20}') - (1-\lambda)^{-1}\mu_1''\beta_{11} + 2a(1-\lambda)^{-1}\left(\beta\xi_0^{(1)}\alpha_{12} + B\alpha_{12}\right)\Big]
\end{aligned} \tag{39c}$$

and

$$\alpha_{21}' + \alpha_{21} = h_3(\tau) \tag{39d}$$

$$h_3(\tau) = -\left[(1-\lambda)^{-\frac{1}{2}}\beta_{12}\left(\beta\xi_0^{(1)} + B\right) + \mu_1'\alpha_{12}\right] \tag{39e}$$

We now solve (39b – e) to obtain

$$\beta_{21}(\tau) = e^{-\tau}\left[\beta_{21}(0) + \int h_2(\tau)e^{\tau}d\tau\right] \tag{39f}$$

$$\alpha_{21}(\tau) = e^{-\tau}\int h_3(\tau)e^{\tau}d\tau \tag{39g}$$

The remaining equation in (39a) is now given as

$$\xi_{,tt}^{22} + \xi^{22} = r_4 + r_5\cos 2t + r_6\sin 2t \tag{40a}$$

$$\xi^{22}(0,0) = 0, \quad \xi_{,t}^{22}(0,0) + (1-\lambda)^{-\frac{1}{2}}\{\mu_1'(0)\xi_{,t}^{12}(0,0) + \xi_{,\tau}^{11}(0,0)\} = 0 \tag{40b}$$

where,

$$r_4 = 4\theta(\beta_{11})^2 - 2\alpha_{10}\alpha_{12} + 2(1-\lambda)^{-\frac{1}{2}}\beta_{11}\alpha_{10}' + 2(1-\lambda)^{-\frac{1}{2}}\alpha_{10}\beta_{11}' + a(1-\lambda)^{-1}\left(\frac{(\beta_{11})^2}{2} + 2\alpha_{10}\alpha_{12}\right) - (1-\lambda)^{-1}(r_1'' + 2r_1') - \theta(\alpha_{10}')^2(1-\lambda)^{-1} \quad (40c)$$

$$r_5 = 2\alpha_{10}\alpha_{12} + 4\theta(\beta_{11})^2 + 2(1-\lambda)^{-\frac{1}{2}}\beta_{11}\alpha_{10}' - 2(1-\lambda)^{-\frac{1}{2}}\alpha_{10}\beta_{11}' - (\alpha_{10}')^2(1-\lambda)^{-1} - a(1-\lambda)^{-1}\left(\frac{(\beta_{11})^2}{2} - 2\alpha_{10}\alpha_{12}\right) + \frac{(1-\lambda)^{-1}}{3}(r_2'' + 2r_2') \quad (40d)$$

$$r_6 = \frac{4(1-\lambda)^{-\frac{1}{2}}}{3}r_3' + 2\alpha_{10}\beta_{11} + \frac{4(1-\lambda)^{-\frac{1}{2}}}{3}r_3 + \frac{a(1-\lambda)^{-1}}{2}\alpha_{10}\beta_{12} \quad (40e)$$

We note the following values

$$r_4(0) = B^2 q_{10}, \quad q_{10} = 4\theta + \frac{(1-\lambda)^{-1}}{2} - \theta(1-\lambda)^{-1} \quad (40f)$$

$$r_5(0) = B^2 q_{11}, \quad q_{11} = \left(4\theta + \frac{a}{2}(1-\lambda)^{-1} - \theta(1-\lambda)^{-1}\right) \quad (40g)$$

$$r_6(0) = B^2 q_{12}, \quad q_{12} = \left(\frac{4}{3}(1-\lambda)^{-\frac{1}{2}}(q_3 + q_9) + 2\right) \quad (40h)$$

The solution of (40a, b), using (40c – h) is

$$\xi^{22}(t,\tau) = \alpha_{22}(\tau)\cos t + \beta_{22}\sin t + r_4 - \frac{r_5\cos 2t}{3} - \frac{r_6\sin 2t}{3} \quad (41a)$$

$$\alpha_{22}(0) = r_4(0) - \frac{r_5(0)}{3} = B^2 q_{13}, \quad q_{13} = \left(\frac{q_{11}}{3} - q_{10}\right) \quad (41b)$$

$$\beta_{22}(0) = 0 \quad (41c)$$

So far, we write

$$\xi(t,\tau) = \bar{\xi}(\xi^{10} + \epsilon\xi^{11} + \epsilon^2\xi^{12} + \cdots) + \bar{\xi}^2(\xi^{20} + \epsilon\xi^{21} + \epsilon^2\xi^{22} + \cdots) \quad (42)$$

3.2: Critical values of independent values at maximum displacement

Following (13a), the condition for maximum displacement is

$$\xi_{,t} + (1-\lambda)^{-\frac{1}{2}}\left[\left(\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right)\xi_{,t} + \epsilon\xi_{,\tau}\right] = 0 \quad (43a)$$

Let t_a, $\bar{t}_a$, τ_a be the values of t, $\bar{t}$ and τ respectively at maximum displacement and let these be expanded asymptotically as

$$t_a = t_0 + \epsilon t_{01} + \epsilon^2 t_{02} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \epsilon^2 t_{12} + \cdots) + \bar{\xi}^2(t_{20} + \epsilon t_{21} + \epsilon^2 t_{22} + \cdots) \quad (43b)$$

$$\bar{t}_a = \bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \epsilon^2\bar{t}_{12} + \cdots) + \bar{\xi}^2(\bar{t}_{20} + \epsilon\bar{t}_{21} + \epsilon^2\bar{t}_{22} + \cdots) \quad (43c)$$

$$\tau_a = \epsilon\left[\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \epsilon^2\bar{t}_{12} + \cdots) + \bar{\xi}^2(\bar{t}_{20} + \epsilon\bar{t}_{21} + \epsilon^2\bar{t}_{22} + \cdots)\right] \quad (43d)$$

Each term of (43a) is now expanded as follows

$$\bar{\xi}\xi_{,t}^{10} = \bar{\xi}\Big[\xi_{,t}^{10} + \{\epsilon t_{01} + \epsilon^2 t_{02} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \epsilon^2 t_{12} + \cdots) + \bar{\xi}^2(t_{20} + \epsilon t_{21} + \epsilon^2 t_{22} + \cdots)\}\xi_{,tt}^{10} + \epsilon\,\xi_{,t\tau}^{10}\{\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \cdots) + \bar{\xi}^2(\bar{t}_{20} + \epsilon\bar{t}_{21} + \cdots)\} + \frac{1}{2}\Big\{\Big\{\xi_{,tt\tau}^{10}\{\epsilon t_{01} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \cdots) + \bar{\xi}^2(t_{20} + \epsilon t_{21} + \cdots)\}^2 + 2\epsilon\{\epsilon t_{01} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \cdots) + \bar{\xi}^2(t_{20} + \epsilon t_{21} + \cdots)\} \times \{\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \cdots) + \bar{\xi}^2(\bar{t}_{20} + \epsilon\bar{t}_{21} + \cdots)\}\xi_{,tt\tau}^{10} + \epsilon^2\xi_{,t\tau\tau}^{10}\{\bar{t}_0 + \cdots\}^2 + \cdots\Big\}\Big\} + \cdots\Big]\Big|_{(t_0,0)} \quad (44a)$$

$$\bar{\xi}\epsilon\xi^{11}_{,t} = \bar{\xi}\epsilon\big[\xi^{11}_{,t} + \xi^{11}_{,tt}\{\epsilon t_{01} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \cdots) + \bar{\xi}^2(t_{20} + \epsilon t_{21} + \cdots)\} + \epsilon\xi^{11}_{,t\tau}\{\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \cdots) + \bar{\xi}^2(\bar{t}_{20} + \epsilon\bar{t}_{21} + \cdots)\} + \cdots\big]\Big|_{(t_0,0)} \quad (44b)$$

$$\bar{\xi}\epsilon^2\xi^{12}_{,t} = \bar{\xi}\epsilon^2\big[\xi^{12}_{,t} + \xi^{12}_{,tt}\{\epsilon t_{01} + \cdots\}\big]\Big|_{(t_0,0)} \quad (44c)$$

$$\bar{\xi}^2\xi^{20}_{,t} = \bar{\xi}^2\big[\xi^{20}_{,t} + \xi^{20}_{,tt}\{\epsilon t_{01} + \cdots\} + \epsilon\xi^{20}_{,t\tau}\{\bar{t}_0 + \cdots\}\big]\Big|_{(t_0,0)} \quad (44d)$$

$$\bar{\xi}^2\epsilon\xi^{21}_{,t} = \bar{\xi}^2\epsilon\big[\xi^{21}_{,t} + \xi^{21}_{,tt}\{\epsilon t_{01} + \cdots\} + \epsilon\xi^{21}_{,t\tau}\{\bar{t}_0 + \cdots\}\big]\Big|_{(t_0,0)} \quad (44e)$$

$$\bar{\xi}^2\epsilon^2\xi^{22}_{,t} = \bar{\xi}^2\epsilon^2\xi^{22}_{,t}(t_0,0) + \cdots \quad (44f)$$

$$(1-\lambda)^{-\frac{1}{2}}\mu_1'\bar{\xi}^2\xi^{10}_{,t} = (1-\lambda)^{-\frac{1}{2}}\bar{\xi}^2\Big[\mu_1'\xi^{10}_{,t} + \mu_1'\xi^{10}_{,tt}\{\epsilon t_{01} + \epsilon^2 t_{02} + \cdots\} + \epsilon\big(\xi^{10}_{,t}\mu_1'\big)_{,\tau}\{\bar{t}_0 + \epsilon\bar{t}_{01} + \cdots\} + \cdots\Big]\Big|_{(t_0,0)} \quad (44g)$$

$$(1-\lambda)^{-\frac{1}{2}}\mu_1'\bar{\xi}^2\epsilon\xi^{11}_{,t} = (1-\lambda)^{-\frac{1}{2}}\bar{\xi}^2\epsilon\Big[\mu_1'\xi^{11}_{,t} + \mu_1'\xi^{11}_{,tt}\{\epsilon t_{01} + \cdots\} + \epsilon\big(\xi^{11}_{,t}\mu_1'\big)_{,\tau}\{\bar{t}_0 + \epsilon\bar{t}_{01} + \cdots\} + \cdots\Big]\Big|_{(t_0,0)} \quad (44h)$$

$$(1-\lambda)^{-\frac{1}{2}}\mu_1'\bar{\xi}^2\epsilon^2\xi^{12}_{,t} = (1-\lambda)^{-\frac{1}{2}}\bar{\xi}^2\epsilon^2\big[\mu_1'\xi^{12}_{,t} + \cdots\big]\Big|_{(t_0,0)} \quad (44i)$$

$$(1-\lambda)^{-\frac{1}{2}}\bar{\xi}\epsilon\xi^{10}_{,\tau} = (1-\lambda)^{-\frac{1}{2}}\bar{\xi}\epsilon\Bigg[\xi^{10}_{,\tau} + \xi^{10}_{,t\tau}\{\epsilon t_{01} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \cdots) + \bar{\xi}^2(t_{20} + \epsilon t_{21} + \cdots)\} + \epsilon\xi^{10}_{,\tau\tau}\{\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \cdots) + \bar{\xi}^2(\bar{t}_{20} + \epsilon\bar{t}_{21} + \cdots)\} + \frac{1}{2}\Bigg\{\Big\{\xi^{10}_{,tt\tau} + \{\epsilon t_{01} + \cdots + \bar{\xi}(t_{10} + \cdots) + \bar{\xi}^2(t_{20} + \cdots)\}^2 + 2\epsilon\xi^{10}_{,t\tau\tau}\{\epsilon t_{01} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \cdots) + \bar{\xi}^2(t_{20} + \epsilon t_{21} + \cdots)\} \times \{\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \cdots) + \bar{\xi}^2(\bar{t}_{20} + \epsilon\bar{t}_{21} + \cdots)\}\Big\}\Bigg\} + \epsilon^2\xi^{10}_{,\tau\tau\tau}\{\bar{t}_0 + \cdots\}^2 + \ldots\Bigg]\Bigg|_{(t_0,0)} \quad (44j)$$

$$(1-\lambda)^{-\frac{1}{2}}\bar{\xi}\epsilon^2\xi^{11}_{,\tau} = (1-\lambda)^{-\frac{1}{2}}\bar{\xi}\epsilon^2\big[\xi^{11}_{,\tau} + \xi^{11}_{,t\tau}\{\epsilon t_{01} + \cdots\} + \epsilon\xi^{11}_{,t\tau}\{\bar{t}_0 + \epsilon\bar{t}_{01} + \cdots\} + \cdots\big]\Big|_{(t_0,0)} \quad (44k)$$

$$(1-\lambda)^{-\frac{1}{2}}\bar{\xi}^2\epsilon\xi^{20}_{,\tau} = (1-\lambda)^{-\frac{1}{2}}\bar{\xi}^2\epsilon\big[\xi^{20}_{,\tau} + \xi^{20}_{,t\tau}\{\epsilon t_{01} + \cdots\} + \epsilon\xi^{20}_{,\tau\tau}\{\bar{t}_0 + \cdots\}\big]\Big|_{(t_0,0)} + \cdots \quad (44l)$$

$$(1-\lambda)^{-\frac{1}{2}}\bar{\xi}^2\epsilon^2\xi^{21}_{,\tau} = (1-\lambda)^{-\frac{1}{2}}\bar{\xi}^2\epsilon^2\xi^{21}_{,\tau}(t_0,0) + \cdots \quad (44m)$$

By substituting (44a – m) into (43a) and equating the coefficients of $\bar{\xi}^i\epsilon^j$, we get the following equations evaluated at $(t_0, 0)$.

$$\boldsymbol{O}(\bar{\xi}):\ \xi^{10}_{,t} = 0 \quad (45a)$$

$$\boldsymbol{O}(\bar{\xi}\epsilon):\ t_{01}\xi^{10}_{,tt} + \bar{t}_0\xi^{10}_{,t\tau} + \xi^{11}_{,t} + (1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,\tau}\ 0 \quad (45b)$$

$$\boldsymbol{O}(\bar{\xi}\epsilon^2):\; t_{02}\xi^{10}_{,tt} + \bar{t}_{01}\xi^{10}_{,t\tau} + \frac{1}{2}(t_{01})^2\xi^{10}_{,ttt} + t_{01}\bar{t}_0\xi^{10}_{,tt\tau} + \frac{1}{2}(\bar{t}_{01})^2\xi^{10}_{,t\tau\tau} + t_{01}\xi^{11}_{,tt} + \xi^{12}_{,t}$$
$$+ \bar{t}_0\xi^{11}_{,t\tau} + t_{01}(1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,t\tau} + \bar{t}_0(1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,\tau\tau} + (1-\lambda)^{-\frac{1}{2}}\xi^{11}_{,\tau} = 0 \quad (45c)$$

$$\boldsymbol{O}(\bar{\xi}^2):\; t_{10}\xi^{10}_{,tt} + (1-\lambda)^{-\frac{1}{2}}\mu_1'(0)\xi^{10}_{,t} = 0 \quad (45d)$$

$$\boldsymbol{O}(\bar{\xi}^2\epsilon):\; t_{11}\xi^{10}_{,tt} + \bar{t}_{10}\xi^{10}_{,t\tau} + t_{01}t_{10}\xi^{10}_{,ttt} + t_{10}\bar{t}_0\xi^{10}_{,tt\tau} + t_{10}\xi^{11}_{,tt} + (1-\lambda)^{-\frac{1}{2}}\mu_1'(0)t_{01}\xi^{10}_{,tt}$$
$$+ (1-\lambda)^{-\frac{1}{2}}\bar{t}_0\left(\mu_1'\xi^{10}_{,t}\right)_{,\tau} + (1-\lambda)^{-\frac{1}{2}}\mu_1'(0)\xi^{11}_{,t} + t_{10}(1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,t\tau} \quad (45e)$$

etc.

From (45a), we get

$$t_0 = \pi \quad (46a)$$

From (45b), we get

$$t_{01} = \frac{-\left(\bar{t}_0\xi^{10}_{,t\tau} + \xi^{11}_{,t} + (1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,\tau}\right)}{\xi^{10}_{,tt}}\Big|_{(t_0,0)} = \left(1-(1-\lambda)^{-\frac{1}{2}}\right) \quad (46b)$$

From (45d), we get

$$t_{10} = -\frac{1}{\xi^{10}_{,tt}}\left((1-\lambda)^{-\frac{1}{2}}\mu_1'(0)\xi^{10}_{,t}\right)\Big|_{(t_0,0)} = 0 \quad (46c)$$

From (45e), we get (after cancelling terms that vanish)

$$t_{11} = -\frac{1}{\xi^{10}_{,tt}}\left[(1-\lambda)^{-\frac{1}{2}}\mu_1'(0)t_{01}\xi^{10}_{,tt} + (1-\lambda)^{-\frac{1}{2}}\mu_1'(0)B\right] = -Br_0(1-\lambda)^{-\frac{1}{2}} \quad (46d)$$

We may not need t_{02} and so (45c) is here given for completeness and is not simplified. We shall also need $\bar{t}_0$, $\bar{t}_{01}$, $\bar{t}_{10}$, $and\ \bar{t}_{11}$ which are obtained by evaluating the second equation of (12) at the critical values using (43b – d). Thus, from (12) we get

$$t_a = \bar{t}_a(1-\lambda)^{\frac{1}{2}} + \frac{1}{\epsilon}\left(\mu_1(\tau_a) + \mu_2(\tau_a)\bar{\xi}^2 + \cdots\right) \quad (47a)$$

On expanding (47a) using the fact that $\mu_i(0) = 0,\ i = 1,2,3,\dots$, we get

$$t_0 + \epsilon t_{01} + \epsilon^2 t_{02} + \cdots + \bar{\xi}(t_{10} + \epsilon t_{11} + \epsilon^2 t_{12} + \cdots) + \cdots$$
$$= (1-\lambda)^{\frac{1}{2}}\left[\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \cdots) + \cdots\right]$$
$$+ \mu_1'(0)\bar{\xi}\left[\bar{t}_0 + \epsilon\bar{t}_{01} + \epsilon^2\bar{t}_{02} + \cdots + \bar{\xi}(\bar{t}_{10} + \epsilon\bar{t}_{11} + \cdots) + \cdots\right] \quad (47b)$$

On equating the coefficients of $\bar{\xi}^i\epsilon^j$ in (47b), we get

$$\boldsymbol{O}(1):\; t_0 = \pi = (1-\lambda)^{\frac{1}{2}}\bar{t}_0 \quad \therefore\ \bar{t}_0 = \pi(1-\lambda)^{-\frac{1}{2}} \quad (48a)$$

$$\boldsymbol{O}(\epsilon):\; t_{01} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{01} \quad \therefore\ \bar{t}_{01} = t_{01}(1-\lambda)^{-\frac{1}{2}}$$

i.e $\bar{t}_{01} = (1-\lambda)^{-\frac{1}{2}}\left[1-(1-\lambda)^{-\frac{1}{2}}\right] \quad (48b)$

$$\boldsymbol{O}(\bar{\xi}):\; t_{10} = 0 = (1-\lambda)^{\frac{1}{2}}\bar{t}_{10} + \mu_1'(0)\bar{t}_0$$
$$\therefore\ \bar{t}_{10} = \mu_1'(0)\bar{t}_0(1-\lambda)^{-\frac{1}{2}} = -Br_0\pi(1-\lambda)^{-\frac{1}{2}} \quad (48c)$$

$$\boldsymbol{O}(\bar{\xi}\epsilon):\; t_{11} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{11} + \mu_1'(0)\bar{t}_{01}$$
$$\therefore\ \bar{t}_{11} = (t_{11} - \mu_1'(0)\bar{t}_{01})(1-\lambda)^{-\frac{1}{2}} = Br_0 (48d)$$

3.3 Maximum Displacement, ξ_a

The maximum displacement ξ_a is now obtained by evaluating (42) at the critical values of the independent variables, using (46a – d), (47a, b) and (48a – d). Thus, by evaluating (42) at these critical values and carrying out a Taylor series expansion of each $\xi^{ij}(t_a,\tau_a)$ in a manner similar to the expansions in (44a – m), we observe that most terms in the expansion vanish while the non – vanishing ones are the following:

$$\xi_a = \xi(t_a, \tau_a)$$
$$= \bar{\xi}\left[\xi^{10} + \epsilon \bar{t}_0 \xi^{10}_{,\tau} + \epsilon^2\left(\bar{t}_{01}\xi^{10}_{,\tau} + \frac{\bar{t}_0{}^2}{2}\xi^{10}_{,\tau\tau} + t_{01}\xi^{11}_{,t}\right) + \cdots\right]\Big|_{(t_0,0)}$$
$$+ \bar{\xi}^2\left[\xi^{20} + \epsilon\left(\bar{t}_{01}\xi^{10}_{,\tau} + \bar{t}_0\xi^{20}_{,\tau}\right)\right.$$
$$+ \epsilon^2\left(\bar{t}_{11}\xi^{10}_{,\tau} + t_{01}\, t_{11}\xi^{10}_{,tt} + \bar{t}_0\bar{t}_{10}\xi^{10}_{,\tau\tau} + t_{11}\xi^{11}_{,t} + \bar{t}_{01}\xi^{20}_{,\tau} + \frac{1}{2}(t_{01}{}^2)\xi^{20}_{,tt} + \frac{\bar{t}_0{}^2}{2}\xi^{20}_{,\tau\tau} + \xi^{22}\right)$$
$$\left. + \cdots\right]\Big|_{(t_0,0)} + \cdots \quad (49)$$

After simplifying (49), we get

$$\xi_a = 2B\bar{\xi}(1 + \epsilon A_{11} + \epsilon^2 A_{12}) + \frac{8aB^2\bar{\xi}^2}{3(1-\lambda)}\left\{1 + \frac{3\beta}{2}\left(\frac{B_0}{B}\right) - \theta\frac{(1-\lambda)}{a}\right\}(1 + \epsilon A_{21} + \epsilon^2 A_{22}) \quad (50a)$$

where,

$$A_{11} = -\frac{\pi}{2}(1-\lambda)^{-\frac{1}{2}}, \quad A_{12} = \frac{1}{2}\left\{\frac{\pi}{2}(1-\lambda)^{-1} + \left(1 - (1-\lambda)^{-\frac{1}{2}}\right)^2\right\} \quad (50b)$$

$$A_{21} = \frac{3\pi(1-\lambda)^{\frac{1}{2}}}{8aq_{18}}(q_4 - r_0) \quad (50c)$$

$$q_{18} = \left\{1 + \frac{3\beta}{2}\left(\frac{B_0}{B}\right) - \theta\frac{(1-\lambda)}{a}\right\} \quad (50d)$$

$$A_{22} = \frac{3(1-\lambda)}{8aq_{18}}\left[r_0(1-\lambda)^{-1}(\pi - 1) - r_0 + q_4(1-\lambda)^{-\frac{1}{2}}\left(1 - (1-\lambda)^{-\frac{1}{2}}\right) + \frac{q_{15}}{2}\left(1 - (1-\lambda)^{-\frac{1}{2}}\right)^2 + \frac{\pi^2 q_{17}}{2} + \frac{2q_{19}}{3}\right] \quad (50e)$$

where,

$$q_{19} = q_{10} - q_{11} \quad (50f)$$

and

$$\xi^{20}_{,\tau}(t_0,0) = B^2 q_{14}, \quad q_{14} = q_6 + q_2 + \frac{1}{3}(2 + a(1-\lambda)^{-1}) \quad (50g)$$

$$\xi^{20}_{,\tau\tau}(t_0,0) = B^2 q_{15}, \quad q_{15} = 2q_{21} - q_2 - \frac{2}{3}(2 + a(1-\lambda)^{-1}) \quad (50h)$$

$$\xi^{20}_{,tt}(t_0,0) = B^2 q_{17}, \quad q_{17} = 2(1-\lambda)^{-1}q_1 + \frac{4}{3}q_{16} \quad (50i)$$

3.4 Dynamic Buckling Load, λ_D

We can recast (50a – e) simply as

$$\xi_a = C_1\bar{\xi} + C_2\bar{\xi}^2 + \cdots \quad (51a)$$

where,

$$C_1 = 2B(1 + \epsilon A_{11} + \epsilon^2 A_{12}), \quad C_2 = \frac{8aB^2}{3(1-\lambda)}q_{18}(1 + \epsilon A_{21} + \epsilon^2 A_{22}) \quad (51b)$$

As in Ette [14], the determination of λ_D is preceded by first reversing the series (51a) in the form

$$\bar{\xi} = d_1\xi_a + d_2\xi_a^2 + \cdots \quad (52a)$$

By substituting for ξ_a in (52a) from (51a), and equating the resultant coefficients of $\bar{\xi}$, we get

$$d_1 = \frac{1}{C_1}, \quad d_2 = \frac{-C_2}{C_1^3} \quad (52b)$$

The maximization (3) is now easily executed through (52a) (noting that d_1 , d_2 and ξ_a depend on λ) to get

$$d_1 + 2d_2\xi_{aD} = 0 \qquad (52c)$$

whereξ_{aD} is the value of ξ_a at dynamic buckling. This gives

$$\xi_{aD} = -\frac{d_1}{2d_2} = \frac{C_1^2}{2C_2} \qquad (52d)$$

If we next determine (52a) at buckling we get

$$\bar{\xi} = \left(\frac{C_1}{4C_2}\right) \qquad (52e)$$

On substituting for C_1 and C_2 in (52e), we get the dynamic buckling loadλ_D evaluated from

$$(1 - \lambda_D)^2 = \frac{16a\bar{\xi}\lambda_D}{3}\left\{1 + \frac{3\beta}{2}\left(\frac{B_0}{B}\right) - \theta\frac{(1 - \lambda_D)}{a}\right\}\left[\frac{1 + \epsilon A_{21} + \epsilon^2 A_{22}}{1 + \epsilon A_{11} + \epsilon^2 A_{12}}\right] \qquad (53a)$$

The static buckling load λ_S of the structure [11 – 14] is obtained from

$$(1 - \lambda_S)^2 = 4a\bar{\xi}\lambda_S \qquad (53b)$$

We now eliminate the imperfection parameter $\bar{\xi}$ from (53a) using (53b) to get

$$\left(\frac{1 - \lambda_D}{1 - \lambda_S}\right)^2 = \frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right)\left\{1 + \frac{3\beta}{2}\left(\frac{B_0}{B}\right) - \theta\frac{(1 - \lambda_D)}{a}\right\}\left[\frac{1 + \epsilon A_{21} + \epsilon^2 A_{22}}{1 + \epsilon A_{11} + \epsilon^2 A_{12}}\right] \qquad (53c)$$

where (53a, c) are determined at $\lambda = \lambda_D$ and all results are strictly asymptotic.

4.1 Results and Discussion

(a) From Table 1, Fig. 2, we clearly observe that for the case of no pre – load, the nonlinear damping increases the dynamic buckling load against the case of no nonlinear damping. However, as Table 2, Fig. 3 show, the increase in the dynamic buckling load in the case of the same nonlinear damping, is only marginal when there is a pre – load compared to the case of no pre – load.

(b) As Table 3 and Fig. 4 show, the dynamic buckling load increases sharply when there is no pre – load in the presence of nonlinear damping.

(c) As observed from Table 4 and Fig. 5, the dynamic buckling load seems to be decreasing with increased pre – load for a fixed imperfection and also seems to be decreasing with increased imperfection. However, all these changes are only marginal so that the graph plots do not actually show the differences.

Other relevant results are easily obtainable from Tables 5, 6, 7 and Figs 6, 7, 8.

(d) When there is no pre – load and no damping, equations (53a, c) give the following results:

$$(1 - \lambda_D)^2 = \frac{16a\bar{\xi}\lambda_D}{3} \qquad (4.1)$$

$$\left(\frac{1 - \lambda_D}{1 - \lambda_S}\right)^2 = \frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right) \qquad (4.2)$$

Table 1: Relationship between dynamic buckling load, λ_D and imperfection parameter, $\bar{\xi}$ for linearand nonlinear damping, in the presence of a pre – load.

DYNAMIC BUCKLING LOADS FOR λ_0 = 0.4, a = 1 and DAMPING FACTOR ϵ = 0.01		
$\bar{\xi}$ (IMPERFECTION)	λ_D for β = 1 and θ = 1	λ_D for β = 1 and θ = 0
0.01	0.807545	0.788997
0.02	0.749449	0.717822
0.03	0.710265	0.668014
0.04	0.680295	0.629046
0.05	0.655923	0.596872
0.06	0.635357	0.569429
0.07	0.617559	0.545518

0.08	0.601875	0.524339
0.09	0.587861	0.505359
0.1	0.575199	0.488186
0.2	0.490224	0.373871

Table 2: Relationship between dynamic buckling load, $\lambda_D$and imperfection parameter, $\bar{\bar{\xi}}$ for nonlinear damping, in the presence and absence of a pre – load.

DYNAMIC BUCKLING LOADS FOR λ_0 = 0.4, a = 1 and DAMPING FACTOR ϵ = 0.01		
$\bar{\bar{\xi}}$ (IMPERFECTION)	λ_D for β = 1 and θ = 1	λ_D for β = 0 and θ = 1
0.01	0.807545	0.807898
0.02	0.749449	0.749844
0.03	0.710265	0.710658
0.04	0.680295	0.680669
0.05	0.655923	0.656269
0.06	0.635357	0.635668
0.07	0.617559	0.617833
0.08	0.601875	0.602108
0.09	0.587861	0.588051
0.1	0.575199	0.575345
0.2	0.490224	0.489903
0.3	0.440565	0.439785

Table 3: Relationship between dynamic buckling load, $\lambda_D$and imperfection parameter, $\bar{\xi}$ for linear and nonlinear damping, in the absence of a pre – load.

DYNAMIC BUCKLING LOADS for λ_0 = 0.4, a = 1 and DAMPING FACTOR ϵ = 0.01		
$\bar{\bar{\xi}}$ (IMPERFECTION)	λ_D for β = 0 and θ = 1	λ_D for β = 0 and θ = 0
0.01	0.807898	0.789323
0.02	0.749844	0.718109
0.03	0.710658	0.668191
0.04	0.680669	0.629074
0.05	0.656269	0.596723
0.06	0.635668	0.569087
0.07	0.617833	0.544957
0.08	0.602108	0.523551
0.09	0.588051	0.504332
0.1	0.575345	0.486909
0.2	0.489903	0.369771
0.3	0.439785	0.302819

Table 4: Relationship between dynamic buckling load, λ_D and imperfection parameter, $\bar{\xi}$ for nonlinear damping, in the presence of a pre – load, for some fixed values of λ_0.

DYNAMIC BUCKLING LOADS for β = 1 and θ = 1, ϵ = 0.01, a = 1				
$\bar{\xi}$ (IMPERFECTION)	λ_D for $\lambda_0 = 0.4$	λ_D for $\lambda_0 = 0.5$	λ_D for $\lambda_0 = 0.6$	λ_D for $\lambda_0 = 0.7$
0.01	0.807545	0.807374	0.807122	0.806719
0.02	0.749449	0.749267	0.749006	0.748614
0.03	0.710265	0.710091	0.709859	0.709546
0.04	0.680295	0.680141	0.679954	0.679753
0.05	0.655923	0.655797	0.655664	0.655594
0.06	0.635357	0.635259	0.635189	0.635266
0.07	0.617559	0.617496	0.617491	0.617721
0.08	0.601875	0.601847	0.601912	0.602304
0.09	0.587861	0.587868	0.588006	0.588564
0.1	0.575199	0.575245	0.575456	0.576184
0.2	0.490224	0.490661	0.491649	0.494146
0.3	0.440565	0.441394	0.443159	0.447391

Table 5: Relationship between dynamic buckling load, λ_D and imperfection parameter, $\bar{\xi}$ for linear damping, in the presence of a pre – load, for some fixed values of λ_0.

DYNAMIC BUCKLING LOADS for β = 1 and θ = 0, ϵ = 0.01, a = 1				
(IMPERFECTION) $\bar{\xi}$	λ_D for $\lambda_0 = 0.4$	λ_D for $\lambda_0 = 0.5$	λ_D for $\lambda_0 = 0.6$	λ_D for $\lambda_0 = 0.7$
0.01	0.788997	0.788843	0.788619	0.788269
0.02	0.717822	0.717701	0.717549	0.717373
0.03	0.668014	0.667972	0.667967	0.668111
0.04	0.629046	0.629109	0.629297	0.629862
0.05	0.596872	0.597059	0.597481	0.598539
0.06	0.569429	0.569761	0.570444	0.572059
0.07	0.545518	0.545999	0.546966	0.549188
0.08	0.524339	0.524987	0.526259	0.529126
0.09	0.505359	0.506501	0.507777	0.511322
0.1	0.488186	0.489189	0.491125	0.495377
0.2	0.373871	0.376991	0.382776	0.394623
0.3	0.310034	0.315471	0.325251	0.344128

Table 6: Relationship between the dynamic buckling load, λ_D and the static buckling load, λ_S for some fixed values of B_0, in the presence of a pre – load and nonlinear damping.

DYNAMIC BUCKLING LOAD for β = 1and θ = 1, ϵ = 0.01, a = 1				
λ_S	λ_D for $B_0 = 0.4$	λ_D for $B_0 = 0.5$	λ_D for $B_0 = 0.6$	λ_D for $B_0 = 0.7$
0.1	0.235572	0.236736	0.238053	0.239516
0.15	0.283924	0.284669	0.285519	0.286469
0.2	0.326342	0.326847	0.327423	0.328069
0.25	0.365549	0.365894	0.366289	0.366736
0.3	0.402948	0.403179	0.403449	0.403754
0.35	0.439388	0.439536	0.439714	0.439915

0.4	0.475444	0.475533	0.475639	0.475766
0.45	0.511547	0.511589	0.511644	0.511712
0.5	0.548038	0.548044	0.548059	0.548084
0.55	0.585208	0.585189	0.585174	0.585168
0.6	0.623323	0.623284	0.623251	0.623221
0.65	0.662632	0.662581	0.662534	0.662489
0.7	0.703383	0.703326	0.703271	0.703217
0.75	0.745828	0.745769	0.745713	0.745656

Table 7: Relationship between the dynamic buckling load, λ_D and the static buckling load, λ_S for some fixed values of B_0, in the presence of a pre – load and linear damping.

DYNAMIC BUCKLING LOAD for $\beta = 1$ and $\theta = 0$, $\epsilon = 0.01$, a = 1				
λ_S	λ_D for $B_0 = 0.4$	λ_D for $B_0 = 0.5$	λ_D for $B_0 = 0.6$	λ_D for $B_0 = 0.7$
0.1	0.106248	0.113431	0.120634	0.127804
0.15	0.139059	0.144399	0.149939	0.155609
0.2	0.176416	0.180221	0.184268	0.188506
0.25	0.217108	0.219761	0.222628	0.225678
0.3	0.260332	0.262159	0.264149	0.266288
0.35	0.305574	0.306816	0.308173	0.309639
0.4	0.352493	0.353319	0.354225	0.355205
0.45	0.400845	0.401379	0.401965	0.402599
0.5	0.450453	0.450781	0.451141	0.451533
0.55	0.501175	0.501359	0.501563	0.501786
0.6	0.552898	0.552984	0.553081	0.553191
0.65	0.605526	0.605548	0.605574	0.605609
0.7	0.658979	0.658961	0.658945	0.658933
0.75	0.713187	0.713146	0.713108	0.713071

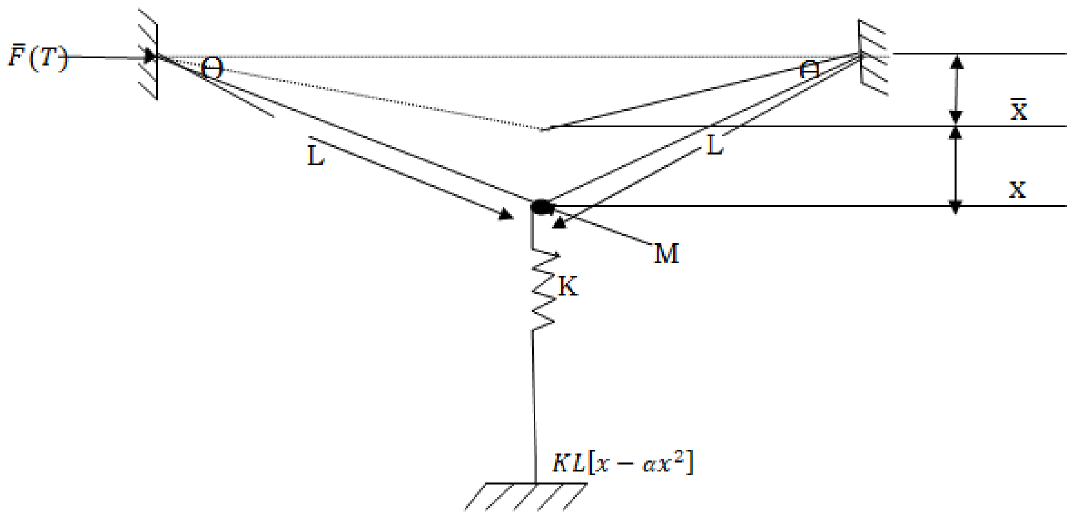

Fig. 1: A simple Quadratic – Elastic Model Structure
(Budiansky and Hutchinson's Model)

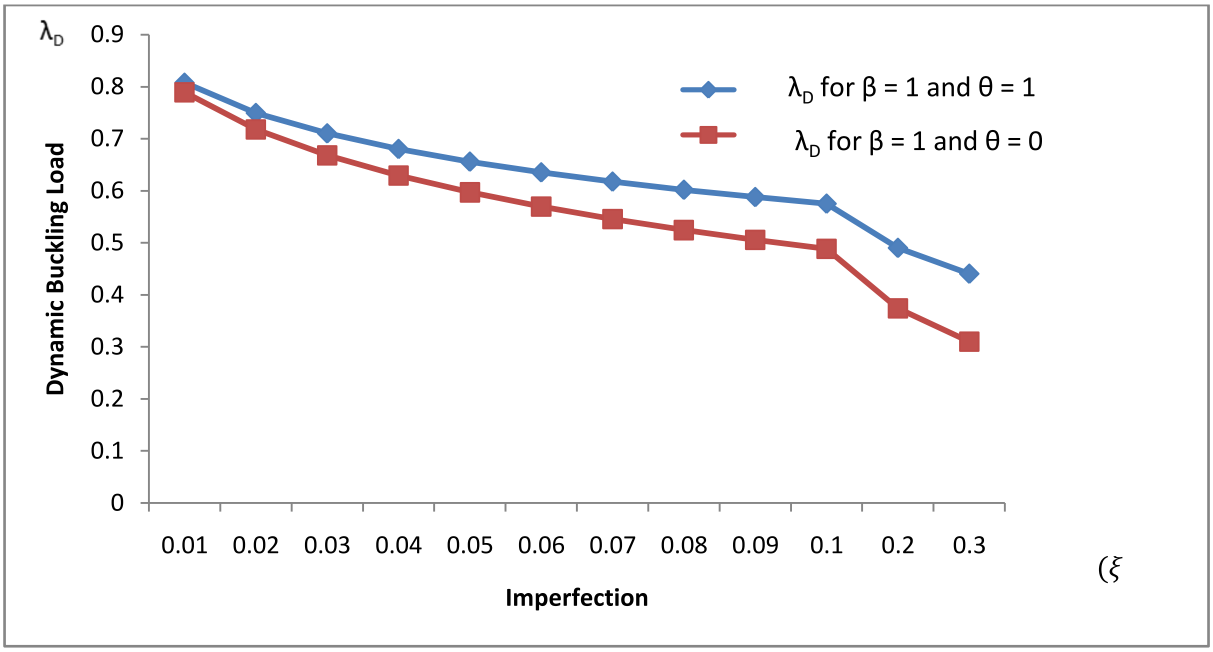

Fig. 2: Graph of Table 1

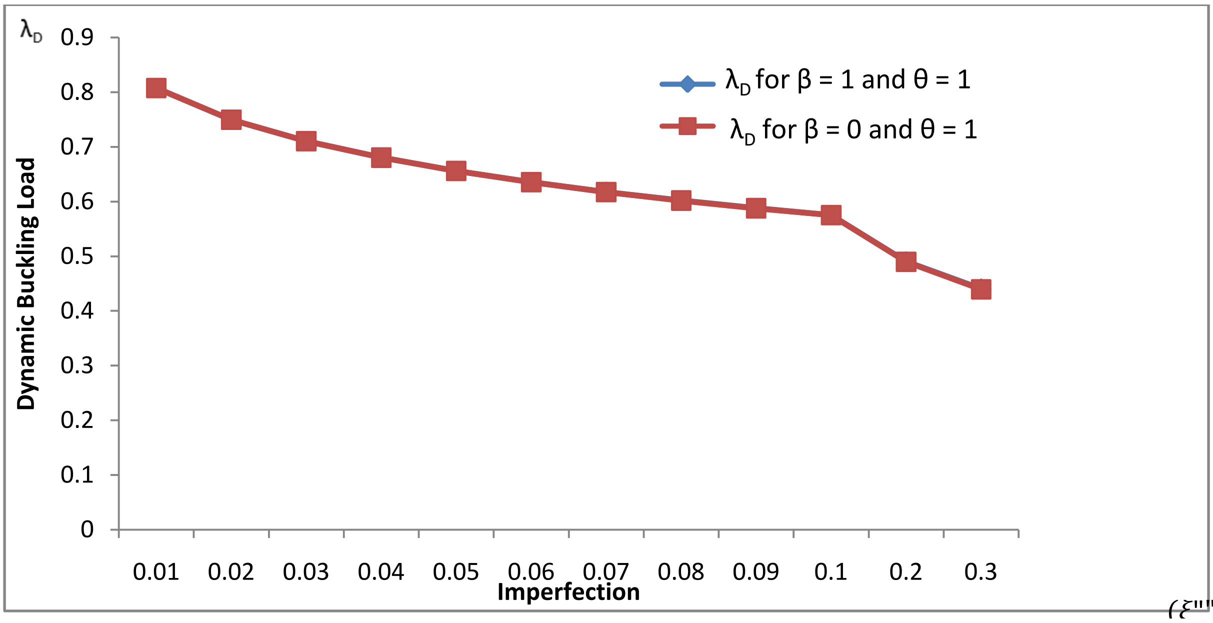

Fig. 3: Graph of Table 2

In this case, there is no appreciable difference in the dynamicbuckling load.

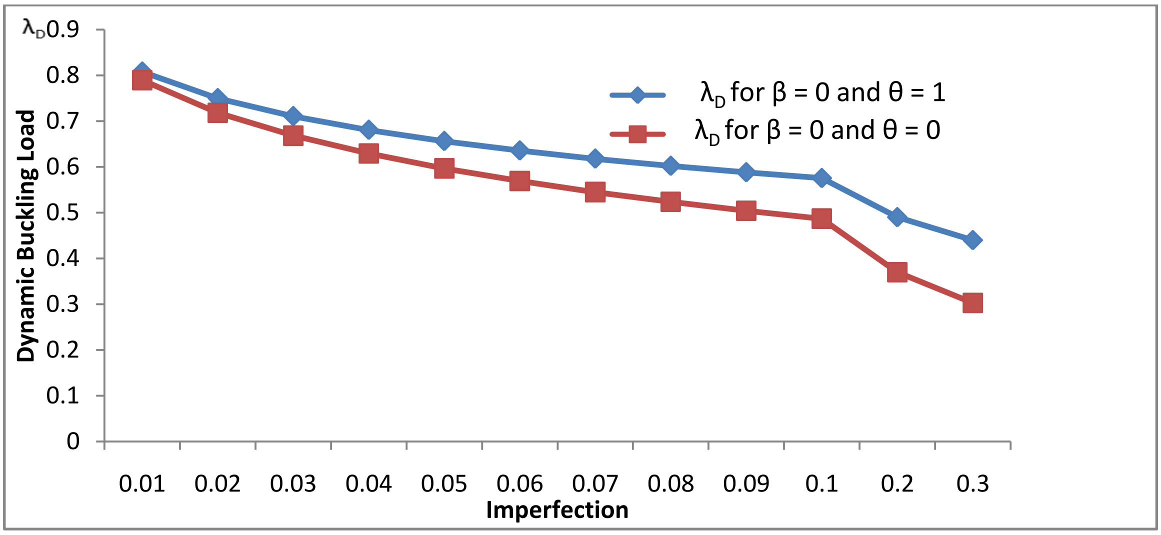

Fig. 4: Graph of Table 3.

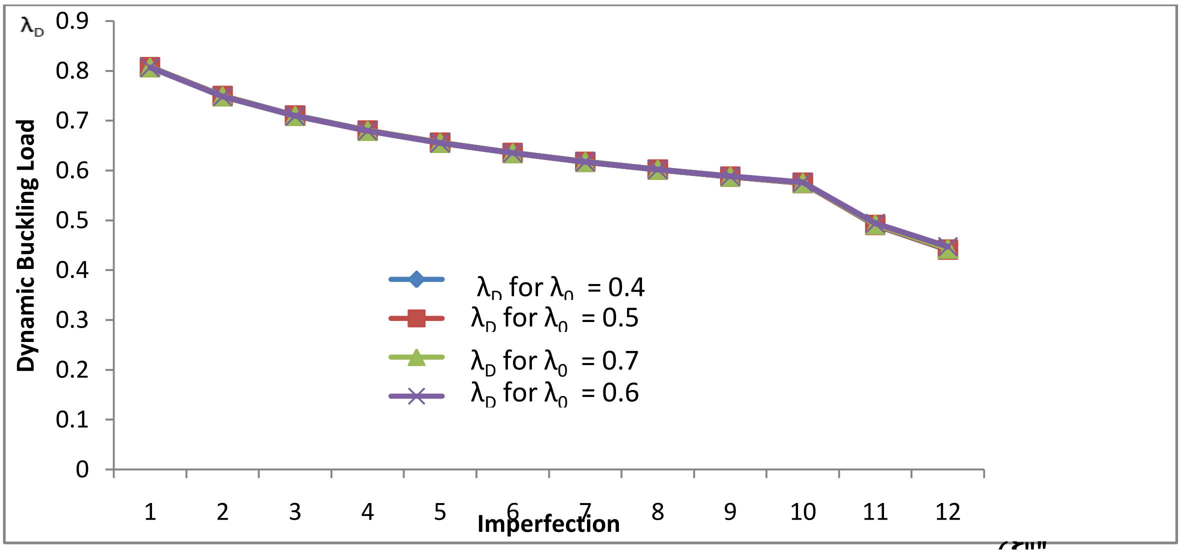

Fig. 5: Graph of Table 4

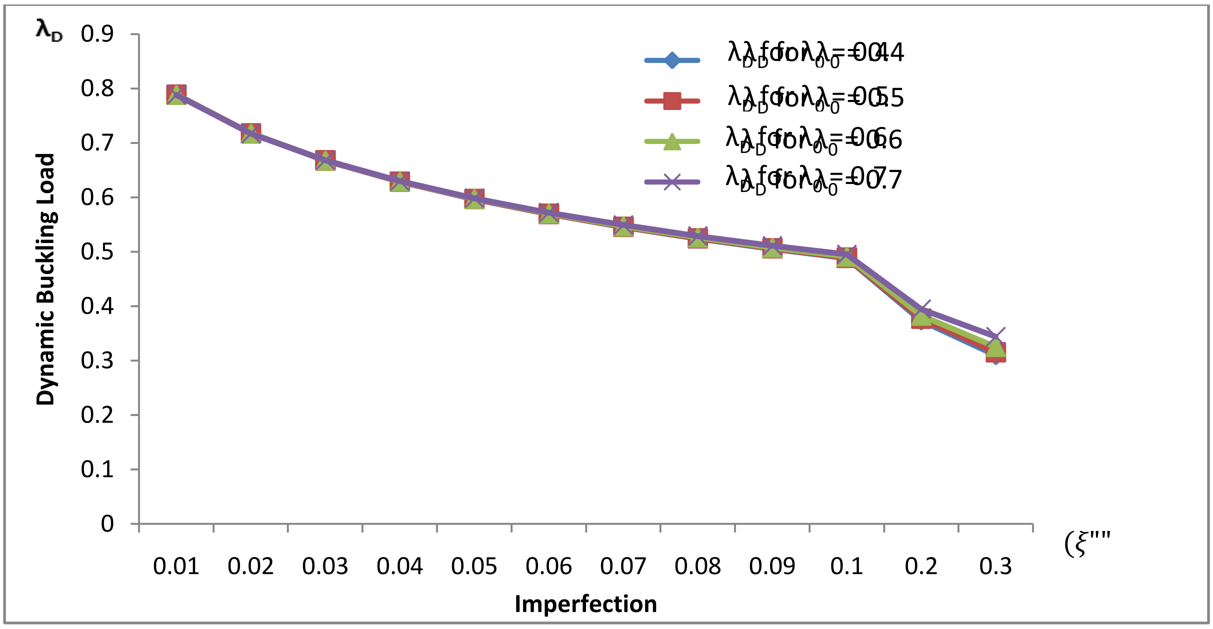

Fig. 6: Graph of Table 5.*No appreciable difference in dynamic buckling load*.

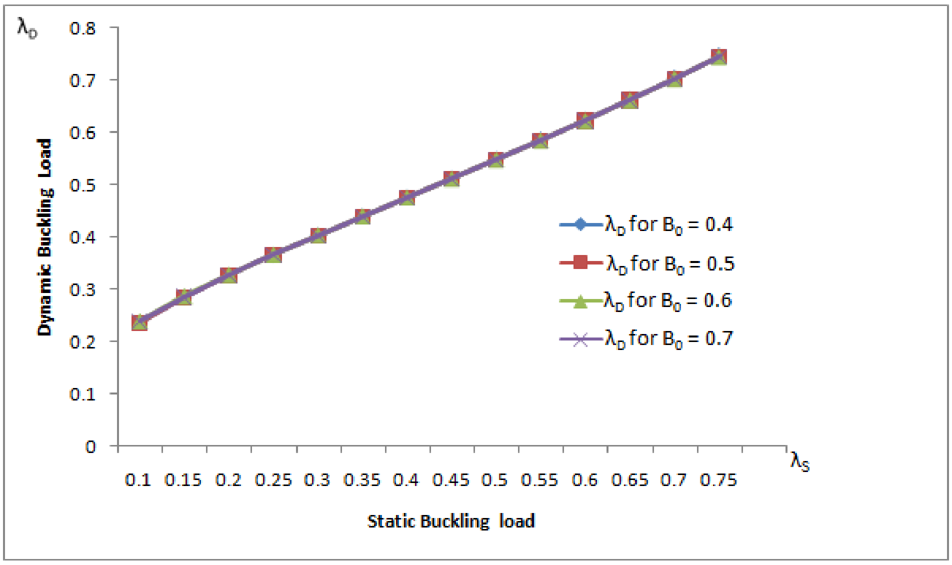

Fig. 7: Graph of Table 6. ***No appreciable changesin the Dynamic Buckling Loads.***

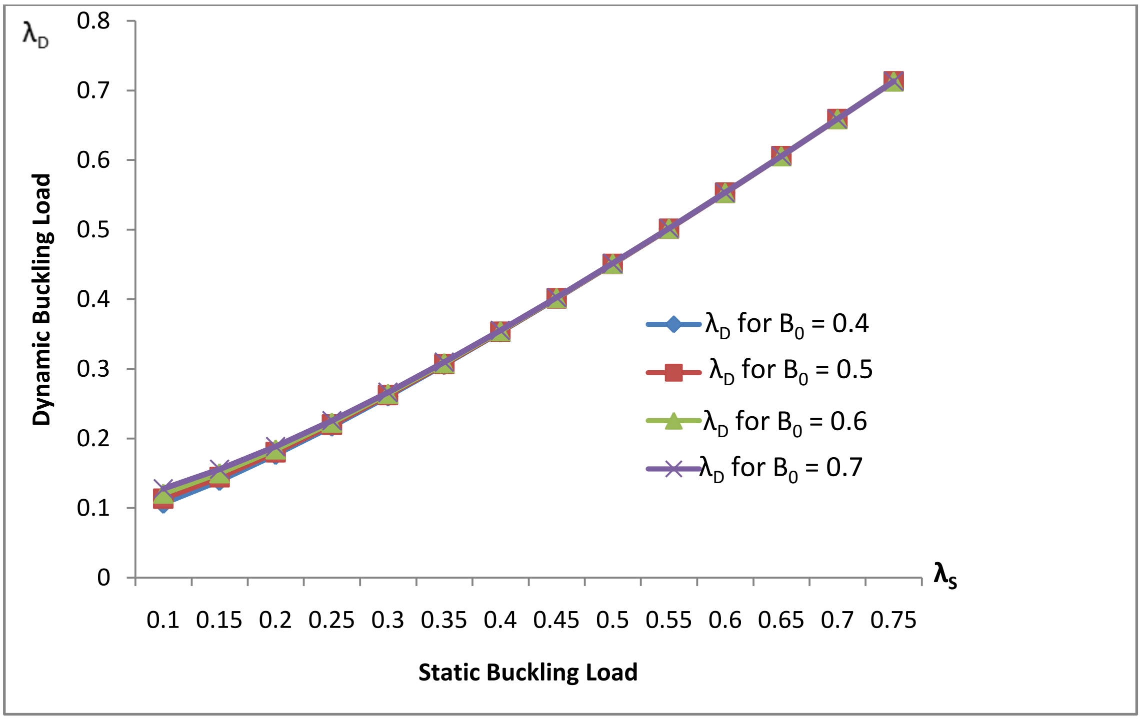

Fig. 8: Graph of Table 7

5. Conclusion

We have judiciously carried out an analytical investigation into the dynamic buckling of a two – small – parameter dynamic buckling of a pre – statically loaded quadratic elastic model structure that has both nonlinear and light linear damping components but is otherwise struck by a step load. The effects of nonlinear damping pre – load and imperfections have been clearly highlighted and discussed. Graphical plots that help to elucidate the results have been given and all results are, of course asymptotic.

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