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RESEARCH ARTICLE

A LOWER CONFIDENCE LIMIT FOR RELIABILITY OF A COHERENT SYSTEM WITH COMPONENT RELIABILITIES ESTIMATED FROM CENSORED DATA

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ABSTRACT

We consider the problem of finding an interval estimate for system reliability via the CHA algorithm when the component reliabilities are estimated from arbitrarily right censored data. The system is assumed to be composed of independent components. A closed form expression for the standard error of the system reliability, for a given mission of duration, is obtained. Some well known systems like series, parallel and 2-out-of-3 are considered to illustrate the method of calculating the required lower confidence limit.

Key Words: Structure function, Coherent Structure, the CHA algorithm, Confidence Interval, System Reliability, Kaplan Meier Estimate.

1. Introduction

In this paper, we consider a coherent structure composed of independent components. The problem is to find a lower confidence limit for the system reliability. We confine our attention to the lower confidence limit of system reliability, since it is of most interest to the reliability practitioners in the context of interval estimation of system reliability.

An excellent account on the topic of interval estimation of system reliability is available in [5]. The first step in obtaining the lower confidence limit of a coherent system is to get a point estimate of the system reliability R(t). The basis of estimation of system reliability is the following model (under the assumption of independence of components, see [4]:

$$R(t) = \sum_{j=1}^{2^m-1} 1(j).\prod_{i=1}^{n} r_i(t)^{D(i,j)}, \tag{1.1}$$

which connects the system reliability R(t) with the component reliabilities $r_i(t)$, $i=1,\ldots,n,$ for a mission of duration t. If $\hat{r}_i(t)$ denotes the MLE of ith component reliability, then the MLE of R(t) is given by:

$$\hat{R}(t) = \sum_{j=1}^{2^m-1} 1(j) \prod_{i=1}^{n} \hat{r}_i(t)^{D(i,j)} \tag{1.2}$$

In principle, one can obtain $Var\left(\hat{R}\right)$ from (1.2), and thus, the lower 100(1-α)% confidence limit is calculated as

$$\hat{R}(t) - z_\alpha \sqrt{Var\left(\hat{R}\right)} \tag{1.3}$$

where z_α is the α-fractile of the standard normal distribution (see [6]). Unfortunately, this does not work well, because the distribution of $\hat{R}$ is basically skewed and a skewed distribution is approximated by a symmetric distribution that is normal here. As a result the confidence intervals may be outside the interval [0,1] as noted by Easterling [6]. Dissatisfaction with this approximation led Easterling to consider the use of a binomial distribution with parameters $\hat{n}$ and $\hat{R}$ where $\hat{n}$ = $\frac{\hat{R}(1-\hat{R})}{Var(\hat{R})}$. This is justified in the light of the following observation: since $\hat{R}$ is a probability and it is obtained via binomial sampling, it is logical to treat $\hat{R}$ as a binomial estimate based on $\hat{n}$ trials. Thus, the component test results can be thought of as being equivalent to system results of $\hat{n}$ tests with $\hat{n}\,\hat{R}$ successes. From this consideration one can now easily obtain a lower confidence limit for R(t) as follows.

Consider a hypothetical single binomial experiment (see [5]) where Y denotes the number of system survivals for mission time t such that Y follows a binomial distribution with parameters $\hat{n}$ and $\hat{R}$ where

$$\hat{n} = \frac{\hat{R}(1-\hat{R})}{Var(\hat{R})}. \tag{1.4}$$

Thus, we can determine a 100(1-α)% lower confidence interval limit for R that satisfies

$$\sum_{i=0}^{x} \binom{\hat{n}}{i} R^i (1-R)^{\hat{n}-i} = 1-\alpha \tag{1.5}$$

where $x = \hat{n}\hat{R}$. Using a relationship between the binomial CDF and the incomplete beta function one can solve the equation (4.2). Hence, the required lower confidence limit is obtained from:

$$R_L = B\left(1-\alpha; x, \hat{n} - x + 1\right) \tag{1.6}$$

where $B(\gamma; a, b)$ is the γ-fractile of the beta distribution with parameters a and b not being necessarily integers.

The component reliabilities can be estimated from two types of failure data viz. (i) Binomial Test Data and (ii) arbitrarily right censored data.

Suppose we have a binomial test data where n_i number of items of component i tested for t hours. Let f_i denote the number of failures of component i during the test. If we record the status (failed/ not failed) of an item during test, then we have a sequence of n_i Bernoulli trials with success probability $p_i = r_i(t), i = 1,\ldots, n.$ Thus,

$$\hat{r}_i(t) = 1 - \hat{q}_i(t), \tag{1.7}$$

$$\hat{q}_i \equiv \frac{f_i}{n_i}, \tag{1.8}$$

$$\hat{Var}(\hat{r}_i(t)) = \frac{\hat{r}_i(t)(1-\hat{r}_i(t))}{n_i}, \tag{1.9}$$

$$\hat{E}\left(\hat{r}_i(t)\right) = \hat{r}_i(t). \tag{1.10}$$

In the two papers in ([1] and [2]) , we have treated the same problem as we are discussing here for the cases (i) components with exponential distribution and (ii) component reliabilities are estimated from binomial test data. In the present paper, we use Kaplan-Meier estimation procedure to get estimates of the component reliabilities.

The paper is organized as follows. In section 2, we define the Kaplan-Meier estimate for arbitrarily right censored data. Section 3 contains the analytical expression for the standard error of the estimate of reliability of a coherent system and the corresponding formulas for series, parallel and 2-out-of-3 systems are worked out. Three examples are given in section 4 to illustrate the technique involved in calculating the lower confidence limit using the CHA algorithm [4].

2. Kaplan-Meier Estimate

We follow the treatment given in Hoyland and Rausand (1994). In an incomplete data set, we have the data of the following kind (Y_i, δ_i), i=1,…,n, where n items are put on test at time 0 and the censoring time for item i is S_i which is stochastic. Here $Y_i = \min(T_i, S_i)$, where T_i is the lifetime of component i if it is not exposed to censoring and $\delta_i = 1$ if $T_i \le S_i$ and $\delta_i = 0$ otherwise.

In this scenario, the Kaplan-Meier estimate of the component reliability r(t) is given by:

$$\hat{r}(t) = \prod_{\{j: t_{(j)} < t\}} \frac{n_j - d_j}{n_j} \tag{2.1}$$

where d_j represents the number of items fail at time $t_{(j)}$ and n_j is the number of items at risk immediately before time $t_{(j)}$. An estimate of variance of $\hat{r}$ (t) is given by:

$$\hat{Var}(\hat{r}(t)) = \hat{r}(t)^2 \sum_{\{j: t_{(j)}<t\}} \frac{d_j}{n_j(n_j-d_j)} \tag{2.2}$$

3. **Variance of** $\hat{R}$:

If $\hat{r}_i$ is an estimate of ith component reliability, then an obvious estimate of R is given by:

$$\hat{R} = \sum_{j=1}^{2^m-1} 1(j) \prod_{i=1}^{n} \hat{r}_i^{\,D(i,j)} \tag{3.1}$$

Note
$$\hat{R}^2 = \sum_{j=1}^{2^m-1} \prod_{i=1}^{n} \hat{r}_i^{\,2D(i,j)} + 2\sum_{j<k} 1(j)1(k) \prod_{i=1}^{n} \hat{r}_i^{\,D(i,j)+D(i,k)} \tag{3.2}$$

Hence,

$$E(\hat{R}^2) = \sum_{j=1}^{2^m-1} \prod_{i=1}^{n} E(\hat{r}_i^{\,2D(i,j)}) + 2\sum_{j<k} 1(j)1(k) \prod_{i=1}^{n} E(\hat{r}_i^{\,D(i,j)+D(i,k)}) \tag{3.3}$$

and we have
$$E\hat{R} = \sum_{j=1}^{2^m-1} 1(j) \prod_{i=1}^{n} E\left(\hat{r}_i^{\,D(i,j)}\right) \tag{3.4}$$

Since
$$Var\left(\hat{R}\right) = E\hat{R}^2 - \left(E\hat{R}\right)^2, \tag{3.5}$$

$Var\left(\hat{R}\right)$ can be calculated from (3.3) and (3.4).

The computations involved in (3.5) are straightforward in the light of the CHA algorithm described in [4]. Few examples are given below:

Examples:

(i) Series system:

Here $D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underset{\sim}{1} = (1)$, m=1, n=2 .

Hence,
$$E\hat{R}^2 = E\hat{r}_1^2 E\hat{r}_2^2 \quad \text{and} \quad E\hat{R} = E\hat{r}_1 E\hat{r}_2. \tag{3.6}$$

(ii) Parallel system:

Here $D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\underset{\sim}{1} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, m=n=2.

Hence,
$$E\hat{R}^2 = E\hat{r}_1^2 + E\hat{r}_2^2 + E\hat{r}_1^2 E\hat{r}_2^2 + 2[E\hat{r}_1 E\hat{r}_2 - E\hat{r}_1^2 E\hat{r}_2 - E\hat{r}_1 E\hat{r}_2^2] \tag{3.7}$$

$$E(\hat{R}) = E\hat{r}_1 + E\hat{r}_2 - E\hat{r}_1 E\hat{r}_2 \tag{3.8}$$

(iii) 2-out-of-3 system:

Here $D=\begin{pmatrix}1&1&0&1&1&1&1\\1&0&1&1&1&1&1\\0&0&1&1&1&1&1\end{pmatrix}$, $\underset{\sim}{1}=\begin{pmatrix}1\\1\\1\\-1\\-1\\-1\\1\end{pmatrix}$, m=n=3.

$$\begin{aligned}E\left(\hat{R}^2\right) &= E\hat{r}_1^2 E\hat{r}_2^2 + E\hat{r}_1^2 + E\hat{r}_2^2 E\hat{r}_3^2 + 4E\hat{r}_1^2 E\hat{r}_2^2 E\hat{r}_3^2 \\ &+ 2E\hat{r}_1^2 E\hat{r}_2 + 2E\hat{r}_1 E\hat{r}_2^2 E\hat{r}_3 - 4E\hat{r}_1^2 E\hat{r}_2^2 E\hat{r}_3 \\ &+ 2E\hat{r}_1 E\hat{r}_2 E\hat{r}_3 - 4E\hat{r}_1^2 E\hat{r}_2 E\hat{r}_3 - 4E\hat{r}_1 E\hat{r}_2^2 E\hat{r}_3^2 \end{aligned} \tag{3.9}$$

$$E\left(\hat{R}\right) = E\hat{r}_1 E\hat{r}_2 + E\hat{r}_1 + E\hat{r}_2 E\hat{r}_3 - 2E\hat{r}_1 E\hat{r}_2 E\hat{r}_3 \tag{3.10}$$

4. Illustrative Examples:

Example 4.1: Series system with two identical components

Let r denote the reliability of each component. Then,

$$E\hat{R}^2 = \left(E\hat{r}^2\right)^2 \qquad E\hat{R} = \left(E\hat{r}\right)^2 \tag{4.1}$$

One can easily show that

$$Var\left(\hat{R}\right) = Var\left(\hat{r}\right)\left[Var\left(\hat{r}\right) + 2\left(E\hat{r}\right)^2\right] \tag{4.2}$$

We use the component failure data in Table 9.3, ([7], p.397) to get a Kaplan-Meier estimate of r, the component reliability. The SPSS survival analysis of the data is given below:

Table 4.1: SPSS analysis of the failure data in Hayland and Rausand ([7])

Time	Cumulative survival	Standard error
31.70	.9375	.0605
39.20	.8750	.0827
57.20	.8125	.0976
65.00*	.8125	.0976
65.80	.7445	.1105
70.00	.6771	.1194
75.00*	.6771	.1194
75.20*	.6771	.1194
87.50*	.6771	.1194
88.30*	.6771	.1194
94.20*	.6771	.1194
101.70*	.6771	.1194
105.80	.5078	.1718
109.20*	.5078	.1718
110.00	.2539	.1990
130.00*	.2539	.1990

* indicates the censoring observation

Thus, $$Var\left(\hat{R}(t)\right)=\begin{cases} .0101 & 31.70 \le t < 39.2 \\ .0173 & 39.20 \le t < 57.5 \\ .0221 & 57.5 \le t < 65.8 \\ .0258 & 65.8 \le t < 70.0 \\ .0273 & 70.0 \le t < 105.8 \\ .0447 & 105.8 \le t < 110.0 \\ .0447 & 110.0 \le t \end{cases}$$

Table 4.2:Calculations of the lower confidence limit $\hat{R}_L = B(0.05; x, n-x+1)$

$Var\hat{R}$	$\hat{R}$	$\hat{n}$	x	R_L
.0101	.8798	10.4659	9.2084	.5906
.0173	.7656	10.3652	7.9359	.4639
.0221	.6609	10.1314	6.6965	.3593
.0257	.5550	9.5858	5.3203	.2600
.0273	.4583	9.0857	4.1642	.1796
.0447	.2580	4.2786	1.1041	.0168
.0447	.0645	1.3498	.0871	3.78E-16

Example 4.2: Consider a system with a single component. Let the component failure data be the same as remission times of leukemia patients under drug Placebo given in ([8],p.81). This example generates more confidence in the Easterling's approach ([6]). The results are given in the following table.

Table 4.2: 95% lower confidence limit for R (10+) and R (20+).

Method	R(10+)	R(20+)
Easterling Approach	.20	.02
Normal approximation	.17	0.0
Log(-log) transformation	.18	.016
Likelihood ratio	.20	.016

The values for R(10+) and R(20+) correspond to Normal approximation, Log(-log) transformation and Likelihood ratio method respectively. The details of the methods are available in ([8], p.90).

Example 4.3: Consider now a parallel system with two components. Let the failure times of the components be the same as that of the remission times for two groups of leukemia patients subjected to two kinds of drugs---- one under 6-MP and the other under Placebo (see [8], for details), so that

For component 1: $$\hat{r}_1(10+) = .753 \quad Var\left(\hat{r}_1\right) = .0092 \qquad \textbf{(4.4)}$$

For component 2: $$\hat{r}_2(10+) = .381 \quad Var\left(\hat{r}_2\right) = .0112 \qquad (4.5)$$

where $\hat{r}(10+) = \lim_{h \to 0+} \hat{r}(10+h)$. The time point 10 is taken for sake of illustration only.

Then, the variance of the system reliability can be calculated to be using the CHA algorithm:

$$Var\left(\hat{R}(10+)\right) = .0282 \qquad (4.6)$$

Thus, the 95% lower confidence limit $\hat{R}_L = B(0.05; x, n - x + 1)$ is given in the following table:

Table 4.3: The 95% lower confidence limit for the parallel system considered in Example 4.3:

$Var\left(\hat{R}(10+)\right)$	$\hat{R}(10+)$	$\hat{n}$	x	$\hat{R}_L = B(0.05; x, n - x + 1)$
0.0282	0.847107	4.592792	3.890586	0.362789

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