Vol.6.Issue.1. 2018 (Jan-Mar)
©KY PUBLICATIONS

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com
Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

A NOTE ON RECIPROCITY THEOREM OF RAMANUJAN

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ABSTRACT

In this paper, a simple proof of the two-variable reciprocity theorem of Ramanujan was established using Abel's lemma on summation by parts.

Keywords: Ramanujan's reciprocity theorem, Abel's lemma on summation by parts, difference operators.

2000 Mathematics Subject Classification: 05A30, 33D15.

1 Introduction

In his lost notebook[12] S. Ramanujan has recorded the following beautiful reciprocity theorem.

$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty} \qquad (1.1)$$

where

$$\rho(a,b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-ab)_n}$$

and as usual

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

$$(a)_n := (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad |q|<1$$

The first proof of (1.1) was given by G. E. Andrews[3]. Since then several proofs and applications of (1.1) were independently given by D. D. Somashekara and S. N. Fathima [13], T. Kim, Somashekara and Fathima [11], S. Bhargava, Somashekara and Fathima [7], C. Adiga and N. Anitha[2], B. C. Berndt, S. H. Chan, B. P. Yeap and A. J. Yee [6], and S.-Y. Kang [10], D. D. Somashekara, K. Narasimha Murthy, S. L. Shalini [14], D. D. Somashekara and K. Narasimha Murthy [15]. For more details, refer to the book by Andrews and Berndt [4]. Kang has also obtained a four-variable generalization of (1.1), which is equivalent to the identity of Andrews [Theorem 6, 3] and was further generalized by Z. Zhang [16].

The main objective of this note is to give a simple and elegant proof of (1.1). Using the Heine's transformation[9],

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\beta)_n}{(q)_n(\gamma)_n}\,z^n=\frac{(\gamma/\beta)_\infty\,(\beta z)_\infty}{(\gamma)_\infty(z)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha\beta z/\gamma)_n\,(\beta)_n}{(q)_n(\beta z)_n}\,(\gamma/\beta)^n \quad (1.2)$$

the Jacobi's triple product identity [9]

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}z^n=(qz;q)_\infty(1/z;q)_\infty(q;q)_\infty \quad (1.3)$$

and a reformulated Abel's lemma[1] on summation by parts given by W. Chu [8]. In his paper, Chu defines the backward and forward difference operator ∇ and Δ, respectively, for an arbitrary sequence $\{\tau_k\}$ as

$$\nabla\tau_k = \tau_k - \tau_{k-1} \; and \; \nabla\tau_k = \Delta\tau_k - \tau_{k+1}\,.$$

Then Abel's lemma was reformulated as

$$\sum_{k=-\infty}^{\infty}A_k\nabla B_k=\sum_{k=-\infty}^{\infty}B_k\nabla\,A_k$$

provided that the series on both sides are convergent and $A_kB_{k-1} \rightarrow 0$ as

2. A Proof of (1.1).

Setting $\alpha = -1/a,\ \beta = q,\ z = -bq$ and let $\gamma \rightarrow 0$ in (1.2), we obtain

$$\rho(b,a)=1+\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}b^n a^{-n-1}}{(-bq)_{n+1}}$$

$$=1+\frac{1}{a}\sum_{n=0}^{\infty}-\left(\frac{1}{a}\right)_n(-bq)^n \quad (2.1)$$

Next

$$\rho(a,b)=1+\left(-\frac{1}{a}\right)\sum_{n=1}^{\infty}-\left(\frac{1}{a}\right)_{-n}(-bq)^{-n} \qquad (2.2)$$

on using

$$(a)_{-n}=\frac{(-1)^n q^{n(n+1)/2}}{(a)^n(q/a)_n}$$

On using (2.1) and (2.2) on the left side of (1.1) and after some simplifications we obtain

$$\sum_{n=-\infty}^{\infty}-\left(\frac{1}{a}\right)_n(-bq)^n=\frac{(a/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty} \quad (2.3)$$

Now let

$$A_n = -\left(\frac{1}{a}\right)_n \text{ and } B_n = (-bq)^n$$

Then

$$\nabla A_n = \frac{(-1/aq)_n\, q^n}{1+aq} \text{ and } \Delta B_n = (-bq)^n(1+bq)$$

Hence

$$\sum_{n=-\infty}^{\infty}\left(-\frac{1}{a}\right)_{-n}(-bq)^{-n} = \frac{1}{(1+bq)}\sum_{n=-\infty}^{\infty} A_n\Delta B_n$$

$$= \frac{1}{(1+bq)}\sum_{n=-\infty}^{\infty} B_n\nabla A_n$$

$$= \frac{1}{(1+aq)(1+bq)}\sum_{n=-\infty}^{\infty}\left(-\frac{1}{aq}\right)_{-n}(-bq)^{-n}\; q^n$$

Iterating 'm' times and letting $m \to \infty$ we obtain

$$\sum_{n=-\infty}^{\infty}\left(-\frac{1}{a}\right)_{-n}(-bq)^{-n} = \frac{1}{(-aq)_\infty(-bq)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}(b/a)^n \quad (2.4)$$

Using (1.3) with $z = b/a$, on the right side of (2.4) we obtain (2.3) and hence (1.1).

References

[1]. Abel. N. H , Untersuchungen ¨uber die Reiche $1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\ldots\ldots..$, *J. Reine. Angew. Math* 1 (1826) 311 − 339.

[2]. Adiga. C and Anitha. N, On a reciprocity theorem of Ramanujan, Tamsui *Oxford J. Math. Sci.* 22(1) (2006) 9 − 15.

[3]. Andrews. G. E, Ramanujan's lost notebook, I partial θ functions, *Adv. in. Math*. 41 (1981) 137 − 172.

[4]. Andrews. G. E and Berndt. B. C, Ramanujan's Lost Notebook Part II, Springer, New York 2009.

[5]. Berndt. B. C, Ramanujan's Notebooks, Part III, Springer-Verlag, New York 1991.

[6]. Berndt. B. C, Chan. S. H, Yeap. B. P and Yee. A. J, A reciprocity theorem for certain q - series found in Ramanujan's lost notebook. *Ramanujan J* 13 (2007) 29 − 40.

[7]. Bhargava. S, Somashekara. D. D and Fathima. S. N, Some q - gamma and q - beta function identities deducible from the reciprocity theorem of Ramanujan's, *Adv. Stud. Contemp. Math* 11(2) (2005) 227 − 234.

[8]. Chu. W, Abel's lemma on summation by parts and Ramanujan's1ψ1 summation, *Aeq. Math.* 72 (2006) 172 − 176.

[9]. Gasper. G and Rahman. M, Basic hypergeometric series, Encyclopedia of Mathematics, Cambridge University Press. Cambridge, 2004.

[10]. Kang. S.-Y, Generalisations of Ramanujan's reciprocity theorem and their applications, *J. London. Math. Soc.* 75 (2007) 18 − 34.

[11]. Kim. T, Somashekara. D. D and Fathima. S. N, On generalising Jacobi's triple product identity and its applications. *Adv. Stud. Contemp. Math*.9(2) (2004) 165−174.

[12]. Ramanujan. S, The Lost Notebook and Other Unpublished papers, Narosa New Delhi, 1988.

[13]. Somashekara. D. D and Fathima. S. N, An interesting generalisation of Jacobi's triple product identity, *Far. East. J. Math. Sci*. 12 (2003) 255 − 259.

[14]. Somashekara D. D, Narasimha Murthy.K and Shalini. S. L, On New Forms Of The Reciprocity Theorems Gulf *Journal of Mathematics* Vol 3, 1(2015) 104 − 117

[15]. Somashekara D. D and Narasimha Murthy. K On a new transformation formula for bilateral6ψ6 series and applications *The Ramanujan Journal* 42, 1 − 13 (2017).

[16]. Zhang. Z, A note on an identity of Andrews, *Electron J. Combin*. 12 (2005) N3.