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RESEARCH ARTICLE

MAPPINGS BETWEEN C*- ALGEBRAS THAT PRESERVE THE SPECTRUM

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ABSTRACT

In this study, we show that a * - homomorphism $\varphi : \mathrm{A} \to \mathrm{B}$ between unital commutative C*- algebras A and B which verify $r_B\big(\varphi(x)\big) = r_A(x)$ for any $x \in A_+$ satisfies the property to preserve spectrum and hence adjoint mapping $\varphi^* : \Delta(B) \to \Delta(A)$ is surjective, that is, φ^* maps maximal ideal space of B to maximal ideal space of A.

Keywords: C*- algebras, maximal ideal space, positive element, topological divisor of zero.

1. INTRODUCTION

In this paper, the relation between the property preserve spectrum of a homomorphism from one C*- algebra to another, topological divisor of zero and positive elements of these C*- algebras and the mapping of their maximal ideals is examined.

Let A be a unital Banach algebra and $x \in A$. If there exists a sequence (y_n) in A such that $\|y_n\| = 1$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} x y_n = \lim_{n\to\infty} y_n x = 0,$$

then x is called a topological divisor of zero. [5] If A is a unital commutative Banach algebra and x is a topological divisor of zero in A, then x is not invertible in A. [3]

If A is a unital Banach algebra, then the set $\{\lambda \in \mathbb{C} : (x - \lambda 1_A) \notin A^{-1}\}$ is called spectrum of x in A, denoted by $\sigma_A(x)$, where A^{-1} denotes the set of invertible elements of A. $\sigma_A(x)$ is a nonempty compact subset of $\mathbb{C}$ for every x in A. The resolvent set of x is defined by $\rho_A(x) = \mathbb{C} \setminus \sigma_A(x)$. The spectral radius of x is characterized by $r_A(x) = \sup\{\, |\lambda| : \lambda \in \sigma_A(x)\}$. If A is a unital commutative Banach algebra, then for every x in A, the limit

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

exists. [5]

Let A be a complex algebra. An involution on A is a mapping $* : x \to x^*$ from A into A satisfying the following conditions.

i. $(x+y)^* = x^* + y^*$,

ii. $(\lambda x)^* = \bar{\lambda} x^*$,

iii. $(xy)^* = y^* x^*$,

iv. $(x^*)^* = x$

for all $x, y \in A$ and $\lambda \in \mathbb{C}$. Then A is called a * - algebra or an algebra with involution. If * - algebra A is a Banach algebra and involution on it is isometric; that is, $\|x^*\| = \|x\|$ for all $x \in A$, then A is called a Banach * - algebra. If * - algebra A is a Banach algebra and its norm satisfies the equation $\|x^*x\| = \|x\|^2$ for all $x \in A$, then A is said to be a C^*- algebra. [3]

Let A be a * - algebra. An element $x \in A$ is said to be hermitian if $x^* = x$. Let A be a unital C^*-algebra. For each hermitian element x of A, $r_A(x) = \|x\|$. [2]

2. PRELIMINARY NOTES

When A is a commutative complex algebra with unit, every proper ideal of A is contained in a maximal ideal of A and every maximal ideal of A is closed. The set of all maximal ideals in A is denoted by $M(A)$. Let A is a complex algebra and ϕ is a linear functional on A. If $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$, then ϕ is called a complex homomorphism on A. The set of nonzero complex homomorphisms on A is denoted by $\Delta(A)$. For x in commutative Banach algebra A, $\hat{x} : \Delta(A) \to \mathbb{C}$, Gelfand transform of x, is defined by $\hat{x}(h) = h(x)$ for every h in $\Delta(A)$. The set $\hat{A} = \{\hat{a} : a \in A\}$ is called the set of Gelfand transforms on A. [5]

The ε-open neighbourhood $U_\varepsilon(h_0, a_1, \dots, a_n)$ at any $h_0 \in \Delta(A)$ with respect to the Gelfand topology is given

$$\{h \in \Delta(A) : |\hat{a}_i(h_0) - \hat{a}_i(h)| < \varepsilon\}$$

where $\varepsilon > 0$, $n \in \mathbb{N}$ and $a_1, \dots, a_n$ are arbitrary elements of A. [3]

The followings are true when A is a unital commutative Banach algebra.

i. Every maximal ideal of A is the kernel of some $h \in \Delta(A)$.

ii. If $h \in \Delta(A)$, then the kernel of h is a maximal ideal of A.

iii. An element $x \in A$ is invertible in A if and only if $h(x) \neq 0$ for every $h \in \Delta(A)$.

iv. $\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta(A)$. [5]

Theorem 2.1. Let A and B be C^*- algebras with identities such that $A \subset B$. If $x \in A$, then $\sigma_A(x) = \sigma_B(x)$. [1]

Definition 2.2. Let A and B be C^*- algebras, $\varphi : A \to B$ be a homomorphism. If φ satisfies $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$, then this mapping is called a * - homomorphism. [2]

Definition 2.3. Let A be a unital C^*- algebra. A hermitian element x of A is said to be positive if $\sigma_A(x) \subset [0, \infty)$. We write $x \geq 0$ to mean that x is positive, and denote by A_+ the set of all positive elements of A. [1]

Theorem 2.4. If A is a unital C^*- algebra and x is a hermitian element of A, then the following statements are equivalent.

i. $x \geq 0$.

ii. $x = h^2$ for some hermitian h in A.

iii. $x = y^*y$ for some y in A. [1]

Theorem 2.5. Let A and B be C^*- algebras. If $\varphi : A \to B$ is a * - homomorphism, then $\varphi(A)$ is a C^*-subalgebra of B. [4]

Theorem 2.6. Let A and B be unital commutative C^*- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. Then $\sigma_A(x) = \sigma_B(\varphi(x))$ for each $x \in A$ if and only if $A^{-1} = \varphi^{-1}(B^{-1})$. [6]

Theorem 2.7. Let A and B be unital commutative $C*$- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. In that case, if $x = 1_A$ whenever $\varphi(x) = 1_B$ for each $x \in A$, then $\varphi(A^{-1}) = \varphi(A)^{-1}$. [7]

Theorem 2.8. Let A and B be unital commutative $C*$- algebras, $\varphi : A \to B$ be a * - homomorphism. Then $\varphi^*\Delta(B) \subset \Delta(A)$. [6]

Theorem 2.9. Let A and B be unital commutative $C*$- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. In that case, $A^{-1} = \varphi^{-1}(B^{-1})$ if and only if $\varphi^*\Delta(B) = \Delta(A)$. [6]

Theorem 2.10. Let A be unital commutative $C*$- algebra, and x an element of A which does not have a left inverse. Then x^*x is not invertible in the $C*$- subalgebra B generated by 1 and x^*x , so there exist $y_1, y_2, \ldots \in B$ such that $\|y_n\| = 1$ and $\|y_n x^* x\| \to 0$ as $n \to \infty$. Hence if $x \in A$ is not invertible, x is a topological divisor of zero. [2]

3. RESULTS

Theorem 3.1. Let A and B be unital commutative $C*$- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. Then $r_B(\varphi(x)) = r_A(x)$ for each $x \in A_+$ if and only if φ is one - to - one.

Proof. Let $r_B(\varphi(x)) = r_A(x)$ for each $x \in A_+$. Since a^*a is positive for each $a \in A$ by Theorem 2.4,
$\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r_B(\varphi(a^*a)) = r_A(a^*a) = \|a^*a\| = \|a\|^2$.
Hence φ is one - to - one.

Conversely, assume that φ is one - to - one. There exists $y \in A$ such that $(x - \lambda 1_A)y = y(x - \lambda 1_A) = 1_A$ for each $\lambda \notin \sigma_A(x)$. Thus it is obtained that $\varphi(x - \lambda 1_A).\varphi(y) = \varphi(y).\varphi(x - \lambda 1_A) = \varphi(1_A) = 1_B$ and $\lambda \notin \sigma_{\varphi(A)}(\varphi(x))$. Also for each $\lambda \notin \sigma_{\varphi(A)}(\varphi(x))$, there exists $y \in A$ such that $\varphi(x - \lambda 1_A).\varphi(y) = \varphi(y).\varphi(x - \lambda 1_A) = 1_B$. In this case $(x - \lambda 1_A)y = y(x - \lambda 1_A) = 1_A$ by hypothesis and so $\lambda \notin \sigma_A(x)$. Thus, we have seen that $\sigma_{\varphi(A)}(\varphi(x)) = \sigma_A(x)$ for each $x \in A$.

In the other hand, $\sigma_B(y) = \sigma_{\varphi(A)}(y)$ for each $y \in \varphi(A)$ from Theorem 2.1 and Theorem 2.5. Hence $\sigma_B(\varphi(x)) = \sigma_A(x)$ for $x \in A_+$ and this implies $r_B(\varphi(x)) = r_A(x)$ for each $x \in A_+$.

Theorem 3.2. Let A and B be unital commutative $C*$- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. Then the following statements are equivalent.

i. $r_B(\varphi(x)) = r_A(x)$ for each $x \in A$.
ii. $r_B(\varphi(x)) = r_A(x)$ for each $x \in A_+$.
iii. $\|\varphi(x)\| = 1$ whenever $\|x\| = 1$ for each $x \in A$.
iv. φ is one - to - one.
v. $x = 1_A$ whenever $\varphi(x) = 1_B$ for each $x \in A$.
vi. $\sigma_B(\varphi(x)) = \sigma_A(x)$ for each $x \in A$.
vii. $A^{-1} = \varphi^{-1}(B^{-1})$.

Proof.

$(\mathbf{i}) \Rightarrow \mathbf{ii})$) It follows from $A_+ \subset A$.

$(\mathbf{ii}) \Rightarrow \mathbf{iii})$) Let $r_B(\varphi(x)) = r_A(x)$ for each $x \in A_+$. Since a^*a is positive for each $a \in A$ by Theorem 2.4, $\|\varphi(a)\| = \|a\|$. In that case, $\|\varphi(x)\| = 1$ whenever $\|x\| = 1$ for each $x \in A$.

$(\mathbf{iii}) \Rightarrow \mathbf{iv})$) Let $\|\varphi(x)\| = 1$ whenever $\|x\| = 1$ for each $x \in A$. Given any $x \in A$, since $\left\|\frac{x}{a}\right\| = 1$ whenever $\|x\| = a \neq 0$, $\left\|\varphi\left(\frac{x}{a}\right)\right\| = 1$ by hypothesis and so it is obtained that $\|\varphi(x)\| = |\varphi(a)| = a = \|x\|$. Also if $\|x\| = 0$, then $\|\varphi(x)\| = 0$. Therefore it is easily seen $\|\varphi(x)\| = \|x\|$ for each $x \in A$ and φ is one - to - one.

$(\mathbf{iv}) \Rightarrow \mathbf{v})$) Let φ be one - to - one. Then, since $\varphi(x) = \varphi(1_A)$ whenever $\varphi(x) = 1_B$ for each $x \in A$, $x = 1_A$ by hypothesis.

(**v**) $\Longrightarrow$ **vi**)) Let $x = 1_A$ whenever $\varphi(x) = 1_B$ for each $x \in A$.

Since $\varphi(x) - \lambda 1_B = \varphi(x - \lambda 1_A) \notin \varphi(A)^{-1}$ for each $\lambda \in \sigma_{\varphi(A)}(\varphi(x))$, we write $x - \lambda 1_A \notin A^{-1}$ by Theorem 2.7 and so $\lambda \in \sigma_A(x)$. Also we find $y \in A$ such that $\varphi(x - \lambda 1_A).\varphi(y) = \varphi(y).\varphi(x - \lambda 1_A) = 1_B$ for each $\lambda \notin \sigma_{\varphi(A)}(\varphi(x))$. Hence, $(x - \lambda 1_A)y = y(x - \lambda 1_A) = 1_A$ by hypothesis and so $\lambda \notin \sigma_A(x)$. Therefore $\sigma_{\varphi(A)}(\varphi(x)) = \sigma_A(x)$ for each $x \in A$. $\sigma_A(x) = \sigma_B(\varphi(x))$ for each $x \in A$ follows from $\sigma_B(y) = \sigma_{\varphi(A)}(y)$ for each $y \in \varphi(A)$.

(**vi**) $\Longrightarrow$ **vii**)) It follows from Theorem 2.6.

(**vii**) $\Longrightarrow$ **i**)) $\sigma_B(\varphi(x)) = \sigma_A(x)$ for each $x \in A$ whenever $A^{-1} = \varphi^{-1}(B^{-1})$ by Theorem 2.6 and so $r_B(\varphi(x)) = r_A(x)$ for each $x \in A$.

Theorem 3.3. Let A and B be unital commutative C^*- algebras, $\varphi : A \to B$ be a * - homomorphism with $\varphi(1_A) = 1_B$. Then $r_B(\varphi(x)) = r_A(x)$ for each $x \in A$ if and only if $\varphi^*\Delta(B) = \Delta(A)$.

Proof. Let $r_B(\varphi(x)) = r_A(x)$ for each $x \in A$. Then for each $g \in \Delta(A)$, there exists $I \in M(A)$ such that $Kerg = I$. If we denote by J_0 the smallest ideal of B containing $\varphi(I)$, then $J_0 = B$ or $J_0 \neq B$.

If $J_0 = B$, then we can write

$$\sum_{i=1}^{k} v_i.\varphi(u_i) = 1$$

for $v_1, v_2, \dots, v_k \in B$ and $u_1, u_2, \dots, u_k \in I$. Since $u_i \in Kerg$ for each $i = 1,2, \dots, k$, $u_i \notin A^{-1}$. Then each u_i is topological divisor of zero by Theorem 2.10. In this case, there exists a sequence $({y_n}^{(i)})$ in A such that $\|{y_n}^{(i)}\| = 1$ for each $n \in \mathbb{N}$, and $\|{y_n}^{(i)} u_i\| \to 0$ as $n \to \infty$, for each $i = 1,2, \dots, k$. If we assume that $y_n = {y_n}^{(1)} {y_n}^{(2)} \dots {y_n}^{(k)}$ for each $i = 1,2, \dots, k$, it is easily seen that

$$\|y_n u_i\| = \|{y_n}^{(1)} {y_n}^{(2)} \dots {y_n}^{(i-1)} {y_n}^{(i+1)} \dots {y_n}^{(k)} ({y_n}^{(i)} u_i)\| \leq \|{y_n}^{(i)} u_i\|$$

and obtained that $\|y_n u_i\| \to 0$ as $n \to \infty$.

Also there exists $M > 0$ such that $\|\varphi(x)\| \leq M.\|x\|$ for each $x \in A$ since φ is continuous. In the other hand, there is a $n_0 \in \mathbb{N}$ such that given any $\varepsilon > 0$, $\|y_n u_i\| < \frac{\varepsilon}{M}$ for all $n \geq n_0$ since $\|y_n u_i\| \to 0$ as $n \to \infty$ for each $i = 1,2, \dots, k$. Thus it is seen that $\|\varphi(y_n u_i)\| \to 0$ as $n \to \infty$ using $\|\varphi(y_n u_i)\| \leq M.\|y_n u_i\| < \varepsilon$ for each $i = 1,2, \dots, k$ and same $n_0 \in \mathbb{N}$.

Moreover, if we remember

$$\sum_{i=1}^{k} v_i.\varphi(u_i) = 1,$$

then it is clear that $\varphi(y_n) = \sum_{i=1}^{k} v_i.\varphi(y_n u_i)$ and so $\|\varphi(y_n)\| \leq \sum_{i=1}^{k} \|v_i\|.\|\varphi(y_n u_i)\|$. Then $\|\varphi(y_n)\| \to 0$ for same $n_0 \in \mathbb{N}$. Therefore, given any $\varepsilon > 0$, $\|\varphi(y)\| < \varepsilon$ and so $r_B(\varphi(y)) < \varepsilon$ for $y = {y_{n_0}}^{(1)} {y_{n_0}}^{(2)} \dots {y_{n_0}}^{(k)}$.

$({y_{n_0}}^{(i)})^{\wedge} \neq 0$ and $({y_{n_0}}^{(i)})^{\wedge}(g) = g({y_{n_0}}^{(i)}) \neq 0$ for $g \in \Delta(A)$ since ${y_{n_0}}^{(i)} \neq 0$ for each $i = 1,2, \dots, k$. Hence if we say $|g(y)| = s$, then

$$|g(y)| = |g({y_{n_0}}^{(1)})||g({y_{n_0}}^{(2)})| \dots |g({y_{n_0}}^{(k)})| = s > 0.$$

At the same time, since $r_A(y) = \sup_{h \in \Delta(A)} |h(y)|$ and $g \in \Delta(A)$, $r_A(y) \geq s$. In that case, $s \leq r_A(y) = r_B(\varphi(y)) < \varepsilon$ and so $s < \varepsilon$ by hypothesis. Then it is not satisfied that $r_B(\varphi(y)) < s$ and this is a contradiction. This contradiction shows that $J_0 \neq B$.

In this case, there exists $J \in M(B)$ such that $J_0 \subset J$ and also $f \in \Delta(B)$ such that $Kerf = J$. Since $I \in M(A)$ and $A/_I \cong \mathbb{C}$, we find $\lambda \in \mathbb{C}$ and $t \in I$ such that $a = \lambda.1 + t$ for each $a \in A$. Therefore, $(\varphi^* f)(a) = \lambda + f(\varphi(t))$. Again for $t \in I$, $\varphi(t) \in Kerf$ and hence $(\varphi^* f)(a) = \lambda$. Using the fact that

$t \in I = Kerg$, $g(a) = (\varphi^* f)(a)$. Then it is easily seen that $g = \varphi^* f \in \varphi^* \Delta(B)$ and obtained that $\Delta(A) = \varphi^* \Delta(B)$ by Theorem 2.8.

Converse conclusion follows from Theorem 2.9 and Theorem 3.2.

4. DISCUSSION AND CONCLUSION

Corollary 4.1. Let A and B be unital commutative C*- algebras, $\varphi : A \to B$ be a* - homomorphism with $\varphi(1_A) = 1_B$. Then the following statements are equivalent.

i. $r_B(\varphi(x)) = r_A(x)$ for each $x \in A$.

ii. $r_B(\varphi(x)) = r_A(x)$ for each $x \in A_+$.

iii. $\|\varphi(x)\| = 1$ whenever $\|x\| = 1$ for each $x \in A$.

iv. φ is one - to - one.

v. $x = 1_A$ whenever $\varphi(x) = 1_B$ for each $x \in A$.

vi. $\sigma_B(\varphi(x)) = \sigma_A(x)$ for each $x \in A$.

vii. $A^{-1} = \varphi^{-1}(B^{-1})$.

viii. $\varphi^* \Delta(B) = \Delta(A)$, namely φ^* maps the space of complex homomorphisms of B (maximal ideal space of B) to the space of complex homomorphisms of A (maximal ideal space of A).

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