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ON INTUITIONISTIC MENGER SPACES

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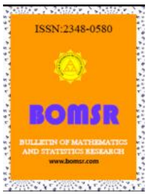

ABSTRACT

In this paper, we obtain the class of topological intuitionistic Menger spaces coincides with semi-metrizable topological spaces.

Key words and phrases. t-norm; t-conorm; semi-metric; intuitionistic probabilistic metric; intuitionistic Menger space, topology.

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Introduction

In 2007, the authors [3] defined non-distance distribution function, using that they introduced intuitionistic Menger spaces as a generalization of Menger spaces, discussed to what topological spaces are intuitionistic Menger metrizable and whether this can be discerned by the t-norm and t-conorm on the space in Menger triangle relations. Building of that work, we prove the class of topological intuitionistic Menger spaces coincides with semi-metrizable topological spaces and no conditions weaker than $\sup\{T(t,t): t < 1\} = 1$ and $\inf\{S(t,t): t > 0\} = 0$ can guarantee that an intuitionistic Menger space is topological. Let us call same basic definitions,

Definition 1 *A binary operation* $T: [0,1] \times 0,1] \to 0,1]$ *is a t-norm if* T *is satisfying the following conditions:*

(a) T is commutative and associative,
(b) $T(a, 1) = a$ for all $a \in 0,1]$,
(c) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in 0,1]$.

Definition 2 *A binary operation* $S: [0,1] \times 0,1] \to 0,1]$ *is a t-conorm if* S *is satisfying the following conditions:*

(a) S is commutative and associative,
(b) $S(a, 0) = a$ for all $a \in 0,1]$,
(c) $S(a, b) \leq S(c, d)$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in 0,1]$.

Remark 1 *The concepts of t-norms and t-conorms are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively [2]. These concepts were originally introduced by Menger [5] in his study of statistical metric spaces developed by Schweizer and Sklar [7,8], and Morrel and Nagata [6]. In literature, several properties and examples for these concepts were proposed.*

Definition 3 ([8]) *A distance distribution function is a function* $F:[-\infty,\infty]\to 0,1]$ *which is left continuous on* $\mathbb{R}$*, non-decreasing and* $F(-\infty)=0$*,* $F(\infty)=1$*.*

We denote by $\triangle$ the family of all distance distribution functions on $[-\infty,\infty]$ and by D the subsets of $\triangle$ containing functions F such that $\lim_{t\to\infty}F(t)=1$. H is a special element of D defined by

$$H(t)=\begin{cases}0, & \text{if} t\le 0\\ 1, & \text{if} t>0\end{cases}$$

If X is a nonempty set, $F: X\times X\to\triangle$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}.

Definition 4 ([3]) *A non-distance distribution function is a function* $L:[-\infty,\infty]\to 0,1]$ *which is right continuous on* $\mathbb{R}$*, non-increasing and* $L(-\infty)=1$*,* $L(\infty)=0$*.*

We will denote by $\triangledown$ the family of all non-distance distribution functions on $[-\infty,\infty]$ and denote by E the subsets of $\triangledown$ containing functions L such that $\lim_{t\to\infty}L(t)=0$. G is a special element of E defined by

$$G(t)=\begin{cases}1, & \text{if} t\le 0\\ 0, & \text{if} t>0\end{cases}$$

If X is a nonempty set, $L: X\times X\to\triangledown$ is said to be a probabilistic non-distance on X and $L(x,y)$ will be denoted by L_{xy}.

Definition 5 ([3]) *A triple* (X,F,L) *is said to be an intuitionistic probabilistic metric space (IPM-space) if* X *is a nonempty set,* F *is a probabilistic distance and* L *is a probabilistic non-distance on* X *satisfying the following conditions: for all* $x,y,z\in X$ *and* $t,s>0$

1. $F_{xy}(t)+L_{xy}(t)\le 1$;
2. $F_{xy}=H$ if and only if $x=y$;
3. $F_{xy}=F_{yx}$;
4. If $F_{xz}(t)=1$ and $F_{zy}(s)=1$, then $F_{xy}(t+s)=1$;
5. $L_{xy}=G$ if and only if $x=y$;
6. $L_{xy}=L_{yx}$;
7. If $L_{xz}(t)=0$ and $L_{zy}(s)=0$, then $L_{xy}(t+s)=0$.

In addition, if Menger triangle inequalities

1. $F_{xy}(t+s)\ge T(F_{xz}(t),F_{zy}(s))$;
2. $L_{xy}(t+s)\le S(L_{xz}(t),L_{zy}(s))$,

where T is a t-norm and S is a t-conorm are verified, then (X,F,L,T,S) is called an intuitionistic Menger space. The functions $F_{xy}(t)$ and $L_{xy}(t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2 *Every Menger space* (X,F,T) *is an intuitionistic Menger space of the form* $(X,F,1-F,T,S)$ *such that t-norm* T *and t-conorm* S *are associated [3], i.e.* $S(x,y)=1-T(1-x,1-y)$ *for any* $x,y\in X$*.*

Example 1 ([3]) *Let* (X,d) *be a metric space. Then the metric* d *induces a distance distribution function* F *defined by* $F_{xy}(t)=H(t-d(x,y))$ *and a non-distance distribution function* L *defined by* $L_{xy}(t)=G(t-d(x,y))$ *for all* $x,y\in X$ *and* $t>0$*. If t-norm* T *is* $T(a,b)=min\{a,b\}$ *and t-conorm* S *is* $S(a,b)=min\{1,a+b\}$ *for all* $a,b\in 0,1]$ *then* (X,F,L,T,S) *is an intuitionistic Menger space.*

We call this intuitionistic Menger space induced by a metric d the induced intuitionistic Menger space.

Remark 3 *Note the above example holds even with the t-norm $T(a,b) = min\{a,b\}$ and t-conorm $S(a,b) = max\{a,b\}$, and hence (X,F,L,T,S) is an intuitionistic Menger space with respect to any t-norm and t-conorm. Also note that, in the above example, t-norm T and t-conorm S are not associated.*

1 Some Properties of Intuitionistic Menger Spaces

In [3], we proved given a t-norm T and a t-conorm S of an intuitionistic Menger space (X,F,L,T,S) satisfying $\sup\{T(t,t): t < 1\} = 1$ and $\inf\{S(t,t): t > 0\} = 0$, the family $\{N_x(\epsilon,\lambda): \epsilon > 0, \lambda \in (0,1)\}$ taken as a neighborhood base at x gives rise to a metrizable topology. In this section, we shall prove that the class of topological intuitionistic Menger spaces coincides with semi-metrizable topological spaces, and no conditions on T and S weaker than $\sup\{T(t,t): t < 1\} = 1$ and $\inf\{S(t,t): t > 0\} = 0$, respectively, can guarantee that an intuitionistic Menger space is topological.

Definition 6 ([4]) *A topological space (X,d,τ) is a semi-metric space if d is a semi-metric, i.e. $d: X \times X \to \mathbb{R}$ is a binary function satisfying $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x = y$, and $d(x,y) = d(y,x)$ such that for each x in X the balls $B_\epsilon(x) = \{y \in X: d(x,y) < \epsilon, \epsilon > 0\}$ form a neighborhood base at x with regard to the topology τ.*

Theorem 1 *If (X,d,τ) is a semi-metric space, then there exist F and L such that (X,F,L,T,S) is an intuitionistic Menger space and such that the family of neighborhoods $\{N_x(\epsilon,\lambda): \epsilon > 0, \lambda \in (0,1)\}$ induces the topology $\tau_{(F,L)}$.*

Proof. For each $x, y \in X$, define

$$F_{xy}(t) = \begin{cases} 1 - \dfrac{d(x,y)}{t + d(x,y)}, & \text{if} t > 0 \\ 0, & \text{if} t \leq 0 \end{cases}$$

$$L_{xy}(t) = \begin{cases} 1 - \dfrac{t}{t + d(x,y)}, & \text{if} t > 0 \\ 1, & \text{if} t \leq 0 \end{cases}$$

$\{F_{xy}: x, y \in X\}$ and $\{L_{xy}: x, y \in X\}$ satisfies (a)-(g) of Definition 5, and (h) and (i) are satisfied for the t-norm T:

$$T(a,b) = \begin{cases} a, & b = 1 \\ b, & a = 1 \\ 0, & a \neq 1 \text{and} b \neq 1 \end{cases}$$

and for the t-conorm S:

$$S(a,b) = \begin{cases} a, & b = 0 \\ b, & a = 0 \\ 1, & a \neq 0 \text{and} b \neq 0 \end{cases}$$

Moreover, $N_x(\epsilon,\lambda) = B_{\lambda\epsilon/1-\lambda}(x)$ for $\epsilon > 0$ and $\lambda \in (0,1)$. Thus, the resulting intuitionistic Menger space is topological with regard to the neighborhoods $N_x(\epsilon,\lambda)$ and this topology is precisely $\tau_{(F,L)}$.

Theorem 2 *If (X,F,L,T,S) is an intuitionistic Menger space such that there exists a topology $\tau_{(F,L)}$ on X for which $\{N_x(\epsilon,\lambda): \epsilon > 0, \lambda \in (0,1)\}$ is a neighborhood base at x for each x in X, then $(X,\tau_{(F,L)})$ is semi-metrizable.*

Proof. Let $x, y \in X$. We define a semi-metric d as follows

$$d(x,y) = \min\{1 + \epsilon - F_{xy}(\epsilon), \epsilon + L_{xy}(\epsilon)\}$$

for all $\epsilon > 0$. Then $d(x,x) = 0$ and if $d(x,y) = 0$ then for each $\delta > 0$ there exists $\epsilon > 0$ such that $1 + \epsilon - F_{xy}(\epsilon) < \delta$ and $\epsilon + L_{xy}(\epsilon) < \delta$. Thus $\epsilon < \delta$ and $y \in N_x(\epsilon,\delta) \subseteq N_x(\delta,\delta)$. So, by (b) and (e) in Definition 5,

$$\bigcap_{\delta>0} N_x(\delta,\delta) = \{x\}$$

and therefore $x = y$. And d is symmetric. Thus, d is a semi-metric on X.
$N_x(\frac{\delta}{2},\frac{\delta}{2}) \subset B_\delta(x)$ because $F_{xy}(\frac{\delta}{2}) > 1 - \frac{\delta}{2}$ and $L_{xy}(\frac{\delta}{2}) < \frac{\delta}{2}$ imply $1 + \frac{\delta}{2} - F_{xy}(\frac{\delta}{2}) < \delta$ and $\frac{\delta}{2} + L_{xy}(\frac{\delta}{2}) < \delta$, and $B_\delta(x) \subset N_x(\epsilon,\epsilon)$ because $d(x,y) < \epsilon$ implies $1 + \epsilon^| - F_{xy}(\epsilon^|) < \epsilon$ and $\epsilon^| + L_{xy}(\epsilon^|) < \epsilon$ for some $\epsilon^| > 0$. Thus, $\epsilon^| < \epsilon$ and we have

$$F_{xy}(\epsilon) \geq F_{xy}(\epsilon^|) > 1 + \epsilon^| - \epsilon > 1 - \epsilon,$$
$$L_{xy}(\epsilon) \leq L_{xy}(\epsilon^|) < \epsilon - \epsilon^| < \epsilon$$

which implies $y \in N_x(\epsilon,\epsilon)$. If $\{N_x(\epsilon,\lambda): \epsilon > 0, \lambda \in (0,1)\}$ is a neighborhood base at x for each x for a topology, then so is the family $\{B_\epsilon(x): \epsilon > 0\}$, proving our assertion.

Remark 4 *Theorems 1 and 2 result in the fact that those intuitionistic Menger spaces which are topological in the aforementioned sense are precisely those semi-metrizable topological spaces.*

As mentioned, there is a sufficient condition on a t-norm T and a t-conorm S such that an intuitionistic Menger space (X,F,L,T,S) with these norms be metrizable, no weaker condition on T and S can guarantee the space be topological:

Theorem 3 *Let T and S be a t-norm and t-conorm, respectively. If $sup_{t<1}T(t,t) < 1$ and $inf_{t>0}S(t,t) > 0$ then there exists an intuitionistic Menger space (X,F,L,T,S) such that the family $\{N_x(\epsilon,\lambda): \epsilon > 0, \lambda \in (0,1)\}$ can not be a neighborhood base at x for any topology.*

Proof. Choose $\alpha > 0$ such that $\sup_{t<1}T(t,t) \leq \alpha < 1$ and $\inf_{t>0}S(t,t) \geq 1 - \alpha > 0$. Let

$$X := \{(p,q) \in \mathbb{R}^2 : 0 \leq p,q < 1 - \alpha\}.$$

For given $x = (p_1,q_1)$ and $y = (p_2,q_2)$ define F_{xy} and L_{xy} as follows:
If $p_1 = p_2$ or $q_1 = q_2$,

$$F_{xy}(\epsilon) = \begin{cases} 0, & \text{if} \epsilon \leq 0 \\ 1 - \max\{|p_1 - p_2|, |q_1 - q_2|\} & \text{if} 0 < \epsilon \leq 1 \\ 1, & \text{if} \epsilon > 1 \end{cases}$$

$$L_{xy}(\epsilon) = \begin{cases} 1, & \text{if} \epsilon \leq 0 \\ \max\{|p_1 - p_2|, |q_1 - q_2|\} & \text{if} 0 < \epsilon \leq 1 \\ 0, & \text{if} \epsilon > 1 \end{cases}$$

If $p_1 \neq p_2$ and $q_1 \neq q_2$,

$$F_{xy}(\epsilon) = \begin{cases} 0, & \text{if} \epsilon \leq 0 \\ \alpha & \text{if} 0 < \epsilon \leq 1 \\ 1, & \text{if} \epsilon > 1 \end{cases}$$

$$L_{xy}(\epsilon) = \begin{cases} 1, & \text{if} \epsilon \leq 0 \\ 1 - \alpha & \text{if} 0 < \epsilon \leq 1 \\ 0, & \text{if} \epsilon > 1 \end{cases}$$

Since the conditions of Definition 5 are satisfied, (X,F,L,T,S) is an intuitionistic Menger space with T and S as a t-norm and a t-conorm, respectively.
Now consider $x,y,z \in X$ and $t,s \geq 0$. If $F_{xy}(t+s) = 1$ and $L_{xy}(t+s) = 0$, then $F_{xy}(t+s) \geq T(F_{xz}(t), F_{zy}(s))$ and $L_{xy}(t+s) \leq S(L_{xz}(t), L_{zy}(s))$. Also, if $F_{xy}(t+s) = 0$ and $L_{xy}(t+s) = 1$,

then $t+s=0$ thus $t=s=0$, and $F_{xz}(t)=0=F_{zy}(s)$ and $L_{xz}(t)=1=L_{zy}(s)$ so that $F_{xy}(t+s)\geq T(F_{xz}(t),F_{zy}(s))$ and $L_{xy}(t+s)\leq S(L_{xz}(t),L_{zy}(s))$. If $0\neq F_{xy}(t+s)\neq 1$ and $0\neq L_{xy}(t+s)\neq 1$ then $F_{xy}(t+s)\geq \alpha$ and $L_{xy}(t+s)\leq \alpha$. If $x=z$ then

$$F_{xy}(t+s)\geq F_{xy}(s)=T(1,F_{xy}(s))\geq T(F_{xz}(t),F_{zy}(s))$$

$$L_{xy}(t+s)\leq L_{xy}(s)=S(0,L_{xy}(s))\leq S(L_{xz}(t),L_{zy}(s))$$

likewise if $y=z$. In the case $x\neq z$ and $y\neq z$ since $\sup_{t<1}T(t,t)\leq \alpha$, $\inf_{t>0}S(t,t)\geq 1-\alpha$ and monotonicity holds for T and S, it suffices to show that $F_{xz}(t)<1, F_{zy}(s)<1, L_{xz}(t)>0$ and $L_{zy}(s)>0$; but we have $F_{xy}(t+s)<1$ and $L_{xy}(t+s)>0$, thus $t\leq 1$ and $s\leq 1$ which implies $F_{xz}(t)<1, F_{zy}(s)<1, L_{xz}(t)>0$ and $L_{zy}(s)>0$ since x,y,z are distinct. Thus triangle inequalities in Definition 5 are satisfied. We omit rest of the proof because, it is the same as the classical case.

Remark 5 *Theorems 1 and 2 can be generalized for the neighborhood systems of ϵ-spheres of a semi-metric and the neighborhoods of an IPM-space regardless of whether they are compatible with a topology in the classical sense, since a neighborhood system, in the terminology of Mamuzic [4], generates a g-topology. A further generalization of Theorem 6 is possible by considering the neighborhoods*

$$N_x^{\varphi}(\epsilon,\lambda)=\{y\in X: F_{xy}(\epsilon)>\varphi(\epsilon)-\lambda, L_{xy}(\epsilon)<\lambda\}$$

for a profile function φ on $\mathbb{R}^+$, as defined by Fritsche [1]. The proof is analogous to that of Theorem 6.

Conclusion 1: *Is it possible to apply Theorem 3 to other intuitionictic spaces built on triangular norm system?*

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