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RESEARCH ARTICLE

COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS

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ABSTRACT
The aim of this paper is to prove a common fixed point theorem from the class of compatible continuous mappings to a larger class of mappings having weakly compatible mappings without appeal to continuity which generalizes the results of Jungck [4], Fisher [2], Kang and Kim [9], Jachymski [3] and Rhoades et al. [10].
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1. Introduction

The theory of fixed point has a broad set of applications in various field of mathematics. In 1922, Polish mathematician, Stephen Banach published his famous contraction mapping principle. Since then, this principle has been extended and generalized in several ways either by using the contractive condition or imposing some additional conditions on an ambient spaces. In particular, this principle is to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others.

In 1976, Jungck [5] proved a common fixed point theorem for commuting mappings generalizing the Banach's fixed point theorem. There then follows a flood of paper involving contractive definitions that do not require the continuity of the mapping.

On the other hand Sessa [11] defined weak commutativity and proved common fixed point theorem for weakly commuting mappings. Further, Jungkck [6] introduced more generalized commutativity, the so- called compatibility which is more general than that of weak commutativity. Since then various fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings have been obtained by many authors.

It has been known from the paper of Kannan [8] that there exists maps that have discontinuity in the domain but which have fixed points. In 1998, Jungck and Rhoades [7] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible but converse need not be true.

In this paper, we prove a fixed point theorem foe weakly compatible mappings without appeal to continuity which generalize the result of Fisher [2], Jachymski [3], Kang and Kim [9], and Rhoades et al. [10].

2. Preliminaries

DEFINITION 2.1 [7]. A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

EXAMPLE 2.1. Let $X = [0,3]$ be equipped with the usual metric space $d(x,y) = I\, x - y\, I$.

Define $f, g : [0,3] \to [0,3]$ by $f(x) = \begin{cases} x \; if \; x \in [0,1] \\ 3 \; if \; x \in [0,3] \end{cases}$ and $g(x) \begin{cases} 3 - x \; if \; x \in [0,1] \\ \;\; 3 \; if \; x \in [0,3] \end{cases}$

Then for any $x \in [1,3]$, $fgx = gfx$, showing that f , g are weakly compatible maps on [0,3]

REMARK 2.1. Weakly compatible maps need not be compatible*. Let X = [2,20] and d be the usual metric on X . Define mappings* $B, T : X \to X \; by \; Bx = x$ if $x = 2 \; or > 5, Bx = 6 \; if \; 2 < x \leq 5, Tx = x$ if $x = 2, Tx$ =12 if $2 < x \leq 5$, $Tx = x - 3$ if $x > 5$. The mappings B and T are non-compatible since sequence $\{x_n\}$ defined by $x_n = 5 + \left(\frac{1}{n}\right), n \geq 1$. Then $Tx_n \to 2 \; Bx_n = 2, TBx_n = 2, BTx_n = 6$. But they are weakly compatible since they commute at coincidence point at $x = 2$.

DEFINITION 2.2. Consider the set L of all real continuous functions $g : [(0,\infty)^5 \to (0,\infty)^5]$ satisfying the following properties:

(i) g is non-decreasing in 4th and 5th variable;

(ii) there is an $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\lambda = \lambda_1 \lambda_2 < 1$ and if $u, v \in [0,\infty)$ satisfying

$u \leq g\,(v,v,u,u+v,0)$ or $u \leq g\,(v,u,v,u+v,0)$ then $u \leq \lambda_1 v$ and if $u, v \in [0,\infty)$ satisfying

$u \leq g\,(v,v,u,0,u+v)$ or $u \leq g\,(v,u,v,0,u+v)$ then $u \leq \lambda_2 v$;

(iii) if $u \in [0,\infty)$ is such that $u \leq g\,(u,0,0,u,u)$ or $u \leq g\,(0,u,0,u,u)$ or $u \leq g\,(0,0,u,u,u)$ then $u = 0$.

3. Main Results

Theorem 3.1. Let (A, S) and (B,T) be weakly compatible pairs of self maps of a complete metric space (X ,d) satisfying the following conditions.

a) $A(X) \subset T(X)$ and $B(X) \subset S(X)$ (3.1)

b) $d\,(Ax,By) \leq g\,\big(d\,(Sx,Ty)\,, d\,(Ax,Sx), d\,(By,Ty), d\,(Ax,Ty), d\,(By,Sx)\big)$ (3.2)

for all $x, y \in X$, where $g \in L$.

Then A, B, S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$ be arbitrary. We choose a point x_1 such that $Tx_1 = Ax_0$ and for this point x_1 , there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, $n = 0,1,2,\ldots..$ (3.3)

Now using (3.2), we have the following

$d(y_1, y_2) = d(Ax_2, Bx_1)$

$\leq g(d(Sx_2, Tx_1), d(Ax_2, Sx_2), d(Bx_1, Tx_1), d(Ax_2, Tx_1), d(Bx_1, Sx_2))$

$\leq g(d(y_1, y_0), d(y_2, y_1), d(y_1, y_0), d(y_2, y_0), d(y_1, y_1))$

$\leq g(d(y_0, y_1), d(y_1, y_2), d(y_0, y_1), d(y_0, y_2), 0)$

$\leq g(d(y_0, y_1), d(y_1, y_2), d(y_0, y_1), d(y_0, y_1) + d(y_1, y_2), 0)$

which implies, in view of Definition 2.2, that

$$d(y_1, y_2) \leq \lambda_1 d(y_0, y_1)$$

Again

$d(y_2, y_3) = d(Ax_2, Bx_3)$

$\leq g(d(Sx_2, Tx_3), d(Ax_2, Sx_2), d(Bx_3, Tx_3), d(Ax_2, Tx_3), d(Bx_3, Sx_2))$

$\leq g(d(y_1, y_2), d(y_2, y_1), d(y_3, y_2), d(y_2, y_2), d(y_3, y_1))$

$\leq g(d(y_1, y_2), d(y_1, y_2), d(y_2, y_3), 0, d(y_1, y_3))$

$\leq g(d(y_1, y_2), d(y_1, y_2), d(y_2, y_3), 0, d(y_1, y_2) + d(y_2, y_3))$

which gives, in view of Definition 2.2, that

$$d(y_2, y_3) \leq \lambda_2 d(y_1, y_2)$$
$$\leq \lambda_1 \lambda_2 d(y_0, y_1)$$
$$\leq \lambda d(y_0, y_1)$$

Proceeding in this way, by induction, we have

$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \leq \cdots \leq \lambda^n d(y_0, y_1)$ for all n =1,2,3,…

Next we prove that $\{y_n\}$ is a Cauchy sequence.

If $m > n$, then by Triangular inequality, we have

$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m)$

$\leq [\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}]\, d(y_0, y_1)$

$\leq \frac{\lambda^n}{1-\lambda} d(y_0, y_1)$

Hence $\lim_{m>n\to\infty} d(y_n, y_m) = 0$

As a result, the sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and $\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$

i.e. $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$.

Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Su$. Then using (3.2), we have

$d(Au, z) \leq d(Au, Bx_{2n-1}) + d(Bx_{2n-1}, z)$

$\leq g(d(Su, Tx_{2n-1}), d(Au, Su), d(Bx_{2n-1}, Tx_{2n-1}), d(Au, Tx_{2n-1}), d(Bx_{2n-1}, Su)) + d(Bx_{2n-1}, z)$

Taking the limit as $n \to \infty$ yields

$d(Au, z) \leq g(0, d(Au, Su), 0, d(Au, z), d(z, Su))$

$= g(0, d(Au, z), 0, d(Au, z), 0)$

$\mathbf{\leq 0}$

i.e. $z = Au = Su$.

Again since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $z = Tv$. Then using (3.2), we have

$$d(z, Bv) = d(Au, Bv)$$
$$\leq g\big(d(Su,Tv), d(Au, Su), d(Bv,Tv), d(Au,Tv), d(Bv, Su)\big)$$
$$\leq g\big(0, 0, d(Bv, z), 0, d(Bv, z)\big)$$

which implies, in view of Definition 2.2, that
$d(z, Bv) \leq 0$ i.e. $z = Bv = Tv$.

Thus $Au = Su = Bv = Tv = z$.

Since pair of maps A and S are weakly compatible, then $ASu = SAu$ i.e. $Az =$

Sz. Now we show that z is a fixed point of A.

If $Az \neq z$, then by (3.2), we have

$$d(Az, z) = d(Az, Bv)$$
$$\leq g(d(Sz,Tv), d(Az, Sz), d(Bv,Tv), d(Az,Tv), d(Bv, Sz))$$
$$\leq g(d(Az, z), 0, 0, d(Az, z), d(Az, z))$$

which implies, in view of Definition 2.2, that

$d(Az, z) = 0$ i.e. $Az = z$. Hence $Az = Sz = z$.

Similarly, pair of maps B and T are weakly compatible , we have $Bz = Tz = z$.

Thus $z = Az = Sz = Bz = Tz$ and therefore z is a common fixed point of A, B, S and T.

Finally, to prove the uniqueness, suppose that z and w, $z \neq w$ are common fixed points of A, B, S and T.
Then using (3.2), we obtain

$$d(z, w) = d(Az, Bw)$$
$$\leq g\big(d(Sz,Tw), d(Az, Sz), d(Bw,Tw), d(Az,Tw), d(Bw, Sz)\big)$$
$$\leq g\big(d(z, w), 0, 0, d(z, w), d(z, w)\big)$$

which implies, in view of Definition 2.2, that

$d(z, w) = 0$ implies $z = w$

Corollary 3.1. Let (A, S) and (B,T) are weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (3.1), (3.3) and $d(Ax, By) \leq hM(x, y)$, $0 \leq h < 1, x, y \in X$, where

$$M(x,y) = \max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \left[\frac{d(Ax,Ty)+d(By,Sx)}{2}\right]\right\}$$

then A, B, S and T have a unique common fixed point in X.

Proof. The assertion follows from Theorem 3.1 with $g(u, v, w, x, y) =$

$$h \max\left\{u, v, w, \frac{1}{2}(x+y)\right\}.$$

Indeed $g \in L$ is continuous. First, we have

$$g(v, v, u, u+v, 0) = h \max\left\{v, v, u, \frac{1}{2}(u+v)\right\}.$$

So if

$$u \le g(v, v, u, u+v, 0) = h \max\left\{v, v, u, \frac{1}{2}(u+v)\right\} \text{implies that } u \le hv \text{ or } u \le \frac{h}{2}(u+v).$$

Therefore

$$u \le kv \text{ with } k = \max\left\{h, \frac{h}{2-h}\right\} < 1.$$

Similarly, if

$$u \le g(v, u, v, 0, u+v) = h \max\left\{v, u, v, \frac{1}{2}(u+v)\right\}$$

gives $u \le kv$ with $k = \max\ (h, \frac{h}{2-h}) < 1$ Therefore A, B, S and T satisfies condition (ii).
Moreover, if

$$u \le g(u, 0, 0, u, u) = h \max\left\{u, 0, 0, \frac{1}{2}(u+u)\right\}.$$

Then $u = 0$ with $0 \le h < 1$. Therefore condition (iii) is satisfied

Corollary 3.2. Let (A, S) and (B, T) are weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (3.1), (3.3) and

$$d(Ax, By) \le a_1 d(Sx, Ty) + a_2 d(Ax, Sx) + a_3 d(By, Ty) + a_4 d(Ax, Ty) + a_5 d(By, Sx)$$

for all $x, y \in X$ and for some $a_1, a_2, a_3, a_4, a_5 \ge 0$

with $\max\{a_1 + a_2 + a_3 + 2a_4, a_1 + a_2 + a_3 + 2a_5, a_1 + a_4 + a_5\} < 1$

Then A, B, S and T have a unique common fixed point in X .

Proof. The assertion follows from Theorem 3.1 with $g(u, v, w, x, y) = a_1 u + a_2 v + a_3 w + a_4 x + a_5 y$

Indeed, $g \in L$ is continuous. First, we have $g(v, v, u, u+v, 0) = a_1 v + a_2 v + a_3 u + a_4 (u+v)$

So, if $u \le g(v, v, u, u+v, 0) = a_1 v + a_2 v + a_3 u + a_4 u + a_4 v$ which implies that

$$u \le hv \text{ with } h = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} < 1.$$

Similarly, if $u \le g(v, u, v, 0, u+v) = a_1 v + a_2 u + a_3 v + a_5 u + a_5 v$.

Then

$$u \le kv \text{ with } k = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} < 1$$

i.e. $u \le \lambda v$ where $\lambda = \max\{h, k\} < 1$. Therefore A, B, S and T satisfies condition (ii).

Moreover, if $u \leq g(u,0,0,u,u) = a_1u + a_4u + a_5u$. Then $u = 0$ as $a_1 + a_4 + a_5 < 1$. Therefore condition (iii) is satisfied.

Remark 3.1. Theorem 3.1 generalizes the result of Jungck [4] by using weakly compatible maps without continuity at *S* and *T* . Theorem 3.1 and Corollary 3.1 also generalize the result of Fisher [2] by employing weakly compatible maps instead of commutativity of maps. Further, the results of Jachymski [3], Kang and Kim [9], Rhoades et al. [10] are also generalized by using weakly compatible maps.

References

[1]. Chugh R. and Kumar S., Common fixed points for weakly compatible maps, *Proc. Indian Acad. Sci.,* **111(2)**(2001), 241-247.

[2]. Fisher B., Common fixed points of four mappings, *Bull. Inst. Math. Acad. Sci.,* **11**(1983), 103-113.

[3]. Jachymski J., Common fixed point theorems for some families of maps, *J. Pure Appl. Math.,* **25**(1994), 925-930.

[4]. Jungck G., Compatible mappings and common fixed points (2), *Int. J. Math. Math. Sci.,* **11**(1988), 285-288.

[5]. Jungck G., Commuting maps and fixed points, *Amer. Math. Month.,* **83**(1976), 261-265.

[6]. Jungck G., Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.,* **9**(1986), 771- 779.

[7]. Jungck G. and Rhoades B. E., Fixed points for set valued functions without continuity, *Indian J. Pure Appl. Math.,* **29(3)**(1998), 227-238.

[8]. Kannan R., Some results on fixed points, *Bull. Cal. Math. Soc.,* **60**(1968), 71-76.

[9]. Kang S. M. and Kim Y. P., Common fixed point theorems, *Math. Japonica,* **37(6)**(1992), 1031-1039.

[10]. Rhoades B. E., Park S. and Moon K. B., On generalizations of the Meir-Keeler type contraction maps, *J. Math. Anal. Appl.,* **146**(1990), 482-487.

[11]. Sessa S., On a weak commutativity condition of mappings in fixed point considerations, Pub. Inst. Math., 32(46)(1982), 149-153.