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RESEARCH ARTICLE

EXPLICIT SOLUTIONS OF THE OPERATOR EQUATION $A^* XB + B^*X^*A = C$

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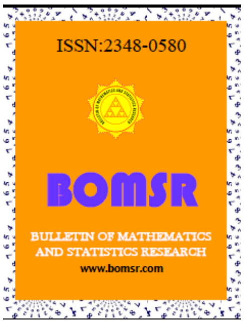

ABSTRACT

In this paper, we study the explicit solutions of the equation $A^*XB+B^*X^*A=C$ for linear bounded operators on Hilbert spaces, where X is the unknown operator. Based on the block operator matrix technique and the Moore-Penrose inverse, the sufficient and necessary conditions for the existence of solutions to the equation are respectively obtained under one of the four cases: $R(B^*)\subseteq R(A^*)$, $R(A^*)\subseteq R(B^*)$, $R(B^*)\subseteq N(A)$ or $R(A^*)\subseteq N(B)$ .Moreover, the general solutions of the considered equation are expressed in terms of the Moore-Penrose inverses of A and B.

AMS classification : 47A62, 47A52

Keywords: Operator equation; Moore-Penrose inverse; Solution

1 Introduction

In this paper H and K denote arbitrary complex Hilbert spaces. We use $B(H,K)$ to denote the set of all linear bounded operators from H to K. Also, $B(H)=B(H,H)$. For $A\in B(H,K)$, the null space, the range and the adjoint operator of A are denoted by $N(A)$,$R(A)$ and A^*. Let $T\in B(H,K)$, if there exists an operator $T^+\in B(H,K)$ satisfying the following four operator equations

$$TT^+T=T,T^+TT^+=T^+,TT^+=(TT^+)^*,T^+T=(T^+T)^*,$$

then T^+ is called the Moore-Penrose inverse of T. It is well known that T has the Moore-Penrose inverse if and only if $R(T)$ is closed and the Moore-Penrose inverse of T is unique. Moreover, TT^+ is the orthogonal projection from K onto $R(T)$ and T^+T is the orthogonal projection from H onto $R(T^*)$ (see[1,2]).

For given operators $A,B\in B(H,K)$ and $C\in B(H)$, we are interested in finding the solution $X\in B(H)$ of the equation

$$A^{*}XB+B^{*}X^{*}A=C.(1)$$

This kind of equations has been studied by several authors because of its multiple applications in different areas, for example, control theory and sampling. The equation $A^{*}X+X^{*}A=C$ was studied for matrices by Braden [3], and for the Hilbert space operators by D S. Djordjević[4]. More general equations $AX-XB^{T}=C$ and $AX-XF=BY$ are considered in [5] and [6]. Yuan [7] studied the solvability of the operator equation $A^{*}XB+B^{*}X^{*}A=C$ for finite matrices under the condition that $R(B^{*})$ is contained in $R(A^{*})$. In this article, Using the block operator matrix technique and the generalized inverse of operators, we study the solvability of Eq.(1) in infinite Hilbert spaces. When $R(B^{*})\subseteq R(A^{*})$, $R(A^{*})\subseteq R(B^{*})$, $R(B^{*})\subseteq N(A)$ or $R(A^{*})\subseteq N(B)$, we give the sufficient and necessary conditions for the existences of solutions to Eq.(1) and the general forms of these solutions.

2 Main results and proofs

To prove the main results of this paper, we begin with the following lemma.

Lemma 2.1. Let $A,B\in B(H,K)$ be invertible and $C\in B(H)$. Then the following statements are equivalent:

(a) There exists a solution $X\in B(K)$ of Eq.(1).

(b) $C=C^{*}$

If (a)or(b)is satisfied , then any solution of Eq.(1) has the form

$$X=\frac{1}{2}(A^{*})^{-1}CB^{-1}+(A^{*})^{-1}ZB^{-1},(2)$$

where$Z\in B(H)$ satisfies $Z^{*}=-Z$.

Proof.$(a)\rightarrow(b)$:Clearly.

$(b)\rightarrow(a)$: It is easy to see that any operator X of the form (2) is a solution of Eq.(1). On the other hand, let X be any solution of Eq.(1). Then

$$\begin{aligned}A^{*}XB&=\frac{1}{2}(A^{*}XB+B^{*}X^{*}A)+\frac{1}{2}(A^{*}XB-B^{*}X^{*}A)\\&=\frac{1}{2}C+\frac{1}{2}(A^{*}XB-B^{*}X^{*}A)\end{aligned}.$$

Since A^{*},B are invertible, so $X=\frac{1}{2}(A^{*})^{-1}CB^{-1}+(A^{*})^{-1}[\frac{1}{2}(A^{*}XB-B^{*}X^{*}A)]B^{-1}$.

Taking $Z=\frac{1}{2}(A^{*}XB-B^{*}X^{*}A)$, we get $Z^{*}=-Z$.

Now, we solve Eq.(1) in the case when A and B have closed ranges.

Theorem 2.1. Let $A,B\in B(H,K)$ have closed ranges and $C\in B(H)$. If $R(B^*)\subseteq R(A^*)$, then the following statements are equivalent:

(a) There exists a solution $X\in B(K)$ of Eq.(1).

(b) $C=C^*$, $R(C)\subseteq R(A^*)$, $A^+AC(I-A^+A)=0$, and $(A^+A-B^+B)C(A^+A-B^+B)=0$.

If (a)or(b)is satisfied , then any solution of Eq.(1) has the form

$$X=\frac{1}{2}(A^*)^+B^+BCB^++(A^*)^+(A^+A-B^+B)CB^+ -(A^*)^+B^+BZB^++AA^+Y(I-BB^+)+(I-AA^+)Y,\quad(3)$$

where $Z\in B(H)$ satisfies $B(Z^*+Z)B^*=0$, $Y\in B(K)$ is arbitrary.

Proof.(a)$\to$(b): Obviously, $C=C^*$. Since $R(B^*)\subseteq R(A^*)$, so $R(C)=R(A^*XB)\subseteq R(A^*)$. Also,

$$\begin{aligned}A^+AC(I-A^+A)&=A^+A(A^*XB+B^*X^*A)(I-A^+A)\\&=A^*XB(I-A^+A)+B^*X^*A(I-A^+A)=0'\end{aligned}$$

$$\begin{aligned}(A^+A-B^+B)C(A^+A-B^+B)&=(A^+A-B^+B)(A^*XB+B^*X^*A)(A^+A-B^+B)\\&=(A^+A-B^+B)A^*XB(A^+A-B^+B)\\&+(A^+A-B^+B)B^*X^*A(A^+A-B^+B)\\&=0\end{aligned}.$$

(b)$\to$(a): Let X be any operator of the form (3), then

$$\begin{aligned}A^*XB+B^*X^*A&=[\tfrac{1}{2}B^+BCB^+B+(A^+A-B^+B)CB^+B+B^+BZB^+B]\\&+[\tfrac{1}{2}B^+BCB^+B+B^+BC(A^+A-B^+B)+B^+BZ^*B^+B].\\&=A^+ACB^+B+B^+BCA^+A-B^+BCB^+B\\&=A^+ACA^+A+A^+AC(I-A^+A)=A^+AC=C\end{aligned}$$

On the other hand, suppose that X is a solution of Eq.(1). Since $R(A)$,$R(B)$ are closed, we have the matrix forms of A,X and B with respect to the proper space decompositions,

$$A=\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix}R(A^*)\\ N(A)\end{pmatrix}\to\begin{pmatrix}R(A)\\ N(A^*)\end{pmatrix};$$

$$X=\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}:\begin{pmatrix}R(B)\\ N(B^*)\end{pmatrix}\to\begin{pmatrix}R(A)\\ N(A^*)\end{pmatrix};$$

$$B=\begin{pmatrix}B_1 & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix}R(A^*)\\ N(A)\end{pmatrix}\to\begin{pmatrix}R(B)\\ N(B^*)\end{pmatrix};$$

where $A_1\in B(R(A^*),R(A))$ is invertible, $R(B_1)$ is closed. Since

$$C=\begin{pmatrix} C_{11}C_{12} \\ C_{21}C_{22} \end{pmatrix}:\begin{pmatrix} R(A^{*}) \\ N(A) \end{pmatrix}\to\begin{pmatrix} R(A^{*}) \\ N(A) \end{pmatrix},$$

then$A^{*}XB+B^{*}X^{*}A=C$ implies

$$A_1^{*}X_{11}B_1+B_1^{*}X_{11}^{*}A_1=C_{11},$$

Where $X_{11}\in B(R(B),R(A))$ is unknown and $C_{11}^{*}=C_{11}$. Let $Y_1=X_{11}^{*}A_1$, then we solve the equation

$$B_1^{*}Y_1+Y_1^{*}B_1=C_{11}.$$

Since $R(B^{*})\subseteq R(A^{*})$, we obtain $R(A^{*})=R(B^{*})\oplus(R(A^{*})\ominus R(B^{*}))$. Now, B_1,Y_1,C_{11} have the following operator matrix forms

$$B_1=\begin{pmatrix} B_{11}0 \end{pmatrix}:\begin{pmatrix} R(B^{*}) \\ R(A^{*})\ominus R(B^{*}) \end{pmatrix}\to R(B);$$

$$Y_1=\begin{pmatrix} Y_{11}Y_{12} \end{pmatrix}:\begin{pmatrix} R(B^{*}) \\ R(A^{*})\ominus R(B^{*}) \end{pmatrix}\to R(B);$$

$$C_{11}=\begin{pmatrix} C_{11}^{0}C_{12}^{0} \\ C_{21}^{0}C_{22}^{0} \end{pmatrix}:\begin{pmatrix} R(B^{*}) \\ R(A^{*})\ominus R(B^{*}) \end{pmatrix}\to\begin{pmatrix} R(B^{*}) \\ R(A^{*})\ominus R(B^{*}) \end{pmatrix};$$

whereB_{11} is invertible. Hence

$$B_1^{*}Y_1+Y_1^{*}B_1=\begin{pmatrix} B_{11}^{*}Y_{11}+Y_{11}^{*}B_{11} & B_{11}^{*}Y_{12} \\ Y_{12}^{*}B_{11} & 0 \end{pmatrix}=\begin{pmatrix} C_{11}^{0}C_{12}^{0} \\ C_{21}^{0}C_{22}^{0} \end{pmatrix}.$$

Because B_{11} is invertible and $(C_{11}^{0})^{*}=C_{11}^{0}$, so by Lemma 2.1 we have

$$Y_{11}=\frac{1}{2}(B_{11}^{*})^{-1}C_{11}^{0}+(B_{11}^{*})^{-1}Z_{11},Y_{12}=(B_{11}^{*})^{-1}C_{12}^{0},$$

where$Z_{11}\in B(R(B^{*}))$ and $Z_{11}^{*}=-Z_{11}$. Noting $X_{11}^{*}A_1=Y_1=Y_{11}+Y_{12}$, then

$$X_{11}=(A_1^{*})^{-1}[\frac{1}{2}C_{11}^{0}+(C_{12}^{0})^{*}]B_{11}^{-1}-(A_1^{*})^{-1}Z_{11}B_{11}^{-1}.$$

Hence the solution X has the following form

$$X=\begin{pmatrix} (A_1^{*})^{-1}[\frac{1}{2}C_{11}^{0}+(C_{12}^{0})^{*}]B_{11}^{-1}-(A_1^{*})^{-1}Z_{11}B_{11}^{-1} & X_{12} \\ X_{21} & X_{22}; \end{pmatrix}$$

whereX_{12},X_{21}, and X_{22} can be taken arbitrary. Now, we express the solution X in terms of the Moore-Penrose inverses of A and B. Let

$$Y=\begin{pmatrix} Y_{11}X_{12} \\ X_{21}X_{22} \end{pmatrix}:\begin{pmatrix} R(B) \\ N(B^{*}) \end{pmatrix}\to\begin{pmatrix} R(A) \\ N(A^{*}) \end{pmatrix}$$

and

$$Z=\begin{pmatrix} Z_{11} Z_{12} \\ Z_{21} Z_{22}\end{pmatrix}:\begin{pmatrix} R(B^*) \\ N(B)\end{pmatrix}\to\begin{pmatrix} R(B^*) \\ N(B)\end{pmatrix}$$

whereY_{11} is arbitrary and $B(Z+Z^*)B^*=0$. Then

$$\frac{1}{2}(A^*)^+B^+BCB^++(A^*)^+(A^+A-B^+B)CB^+$$

$$=\begin{pmatrix} (A_1^*)^{-1}[\frac{1}{2}C_{11}^0+(C_{12}^0)^*]B_{11}^{-1} & 0 \\ 0 & 0\end{pmatrix},$$

$$(A^*)^+B^+BZB^+=\begin{pmatrix} (A_1^*)^{-1}Z_{11}B_{11}^{-1} & 0 \\ 0 & 0\end{pmatrix},$$

and

$$AA^+Y(I-BB^+)+(I-AA^+)Y=\begin{pmatrix} 0 & X_{12} \\ X_{21} & X_{22}\end{pmatrix}.$$

Consequently, X has the form (3).

Theorem 2.2. Let $A,B\in B(H,K)$ have closed ranges and $C\in B(H)$. If $R(A^*)\subseteq R(B^*)$, then the following statements are equivalent:

(a) There exists a solution $X\in B(K)$ of Eq.(1).

(b) $C=C^*$, $R(C)\subseteq R(B^*)$, $B^+BC(I-B^+B)=0$, and $(B^+B-A^+A)C(B^+B-A^+A)=0$.

If (a)or(b)is satisfied , then any solution of Eq.(1) has the form

$$\begin{aligned} X= & \frac{1}{2}(A^*)^+CA^+AB^++(A^*)^+C(B^+B-A^+A)B^+ \\ & +(A^*)^+ZA^+AB^++(I-AA^+)YBB^++Y(I-BB^+)\end{aligned} .(4)$$

where$Z\in B(H)$ satisfies $A(Z^*+Z)A^*=0$, $Y\in B(K)$ is arbitrary.

Proof. Obviously, Eq.(1) is equivalent to

$$B^*X^*A+A^*(X^*)^*B=C,$$

where$X^*\in B(K)$ is the unknown operator. Since $R(A^*)\subseteq R(B^*)$, then by Theorem 2.1 we have

$$\begin{aligned} X^*= & \frac{1}{2}(B^*)^+A^+ACA^++(B^*)^+(B^+B-A^+A)CA^+ \\ & -(B^*)^+A^+AZA^++BB^+Y(I-AA^+)+(I-BB^+)Y\end{aligned},$$

where$Z\in B(H)$ satisfies $B(Z^*+Z)B^*=0$, $Y\in B(K)$ is arbitrary.

Consequently, X has the form (4).

Remark 2.1 As a special case of Eq.(1), we consider the solvability of the operator equation

$$A^*X+X^*A=C.(5)$$

In Theorem 2.2, let $B=I$, clearly $R(A^*)\subseteq R(B^*)$, then we obtain the following Corollary, which is the result of D S. Djordjević in [4].

Corollary 2.1. Let $A\in B(H,K)$ have closed range and $C\in B(H)$. Then the following statements are equivalent:

(a) There exists a solution $X\in B(H,K)$ of Eq.(5).

(b) $C=C^*$ and $(I-A^+A)C(I-A^+A)=0$.

If (a)or(b)is satisfied , then any solution of Eq.(5) has the form

$$X=\frac{1}{2}(A^*)^+CA^+A+(A^*)^+C(I-A^+A)+(A^*)^+ZA^+A+(I-AA^+)Y,$$

where $Z\in B(H)$ satisfies $A(Z^*+Z)A^*=0$, $Y\in B(H,K)$ is arbitrary.

Theorem 2.3. Let $A,B\in B(H,K)$ have closed ranges and $C\in B(H)$. If $R(B^*)\subseteq N(A)$, then the following statements are equivalent:

(a) There exists a solution $X\in B(K)$ of Eq.(1).

(b) $C=C^*$, $R(C)\subseteq H_0$, $B^+BCB^+B=0$, and $A^+ACA^+A=0$, where $H_0=R(A^*)\oplus R(B^*)$.

If (a)or(b)is satisfied , then any solution of Eq.(1). has the form

$$X=(A^*)^+CB^++AA^+Y(I-BB^+)+(I-AA^+)Y,(6)$$

where $Y\in B(K)$ is arbitrary.

Proof. If $R(B^*)\subseteq N(A)$, then we have the space decomposition $H=R(A^*)\oplus R(B^*)\oplus(N(A)\ominus R(B^*))$. Let A,B have the following operator matrix forms

$$A=\begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}:\begin{pmatrix} R(A^*) \\ R(B^*) \\ N(A)\ominus R(B^*) \end{pmatrix}\to\begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix};$$

$$B=\begin{pmatrix} 0 & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}:\begin{pmatrix} R(A^*) \\ R(B^*) \\ N(A)\ominus R(B^*) \end{pmatrix}\to\begin{pmatrix} R(B) \\ N(B^*) \end{pmatrix};$$

where A_1,B_1 are invertible operators. Also, X has the similar operator matrix form as in Theorem 2.1. Consequently, Eq.(1) is equivalent to the following equation

$$\begin{pmatrix} 0 & {A_1}^* X_{11} B_1 & 0 \\ {B_1}^* {X_{11}}^* A_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}, \quad (7)$$

with respect to the space decomposition $H=R(A^*)\oplus R(B^*)\oplus(N(A)\ominus R(B^*))$. Noting that $R(C)\subseteq H_0$ is equivalent to $C_{31}=0,C_{32}=0,C_{33}=0$ and $C_{11}\oplus 0\oplus 0=A^+ACA^+A, 0\oplus C_{22}\oplus 0=B^+BCB^+B$ it is easy to prove that condition (a) is equivalent to (b) in Theorem 2.3.

From equation (7), we can obtain any solution of Eq.(1) has the form

$$X=\begin{pmatrix} ({A_1}^*)^{-1} C_{12} {B_1}^{-1} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

X_{12},X_{21},and X_{22} can be taken arbitrary.

Using the Moore-Penrose inverses of A and B to express X, we obtain X has the form (6).

Theorem 2.4. Let $A,B\in B(H,K)$ have closed ranges and $C\in B(H)$. If $R(A^*)\subseteq N(B)$, then the following statements are equivalent:

(a) There exists a solution $X\in B(K)$ of Eq.(1).

(b) $C=C^*$, $R(C)\subseteq H_0$, $B^+BCB^+B=0$, and $A^+ACA^+A=0$, where $H_0=R(A^*)\oplus R(B^*)$.

If (a)or(b)is satisfied , then any solution of Eq.(1) has the form

$X=(B^*)^+CA^++BB^+Y(I-AA^+)+(I-BB^+)Y$,

where $Y\in B(K)$ is arbitrary.

Proof. By exactly similar arguments, we obtain the analogue of Theorem 2.3, in which B is replaced by A.

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