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ON THE DIOPHNATINE EQUATION $2^x + p^y = z^2$

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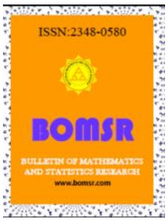

ABSTRACT

In 2013, Somchit Chotchaisht [4], studied the Diophantine equation $2^x + 11^y = z^2$ and posed an open problem to find all the solutions of the equation $2^x + p^y = z^2$ for any odd prime p. In this paper, we give the solutions of titled equation except the case when both x and y are positive odd integers together.

Keywords: Exponential equation, Diophantine Equation, Catalan's conjecture

1. INTRODUCTION

In recent time many papers are devoted to the study of the Diophantine equations. In 2011, A. Suvarnamani [1], published a paper on finding the solutions of the titled equation. But in 2013, Somchit Chotchaisthit [4], pointed some misleading arguments in [1], and gave all solutions of the equation $\mathbf{2^x + 11^y = z^2}$. In the same paper, he stated that finding solutions of titled equation is still an open problem.

In this paper, we give solutions of the titled equation for any odd prime $\boldsymbol{p}$ greater than 3 and $\boldsymbol{x}$ and $\boldsymbol{y}$ are not both positive odd integers together. If $\boldsymbol{p = 3}$, then the solutions are given in [2]. We use Catalan's conjecture which is proved by P. Mihailescu [3] in 2004.

2. PRELIMINARIES

Proposition 2.1 : *(The Catalan's conjecture)* $\mathbf{(3,2,2,3)}$ *is a unique solution* $(\boldsymbol{a,b,x,y})$ *of the Diophantine equation* $\boldsymbol{a^x - b^y = 1}$ *where* $\boldsymbol{a, b, x}$ *and* $\boldsymbol{y}$ *are integers with* $\boldsymbol{\min\{a,b,x,y\} > 1}$.

Proof : See in [3]

Lemma 2.2 *: The equation* $1 + p^y = z^2$*, where* p *is odd prime greater than 3 and* y *and* z *are non negative integers has no solution*

Proof : Let p be odd prime and y and z be non negative integers such that $1 + p^y = z^2$. Thus z must be greater than 1. We proceed in the following cases:

Case 1: $y = 0$*,* then we get $z^2 = 2$ which has no solution.

Case 2 : $y = 1$, then we get $1 + p = z^2$, which is possible only if $z = 2$ and $p = 3$, but p is greater than 3, a contradiction.

Case 3 : $y > 1$, then we have $z^2 - p^y = 1$, by Catalan's conjecture, we get $p = 2$, but this is a contradiction.

3. RESULTS

Theorem 3.1: *The Diophantine equation*

$$2^x + p^y = z^2 \tag{1}$$

where p *is odd prime greater than 3 and* x, y *and* z *are non negative integers, both* x *and* y *are not positive odd integers together , then*

1. *for every prime of the form* $p = 2^{m+1} + 1$*, Equation (1) has the solutions of the form* $(x, y, z) = (2m, 1, 2^m + 1)$*.*
2. *for every odd prime* p*, Equation (1) have the solutions of the form* $(x, y, z) = (3, 0, 3)$
3. *for every Mersenne prime* p*,* $p = 2^q - 1$*, has a solution of the form* $(x, y, z) = (q + 2, 2, p + 2)$

Proof : Let p be odd prime greater than 3 and x, y and z are non negative integers, both x and y are not odd integers together, such that $2^x + p^y = z^2$. Thus z must be greater than 1.

If $x = 0$ then by Lemma 2.2, Equation (1) has no solution. So, let $x \geq 1$. We proceed in two cases:

Case 1 : x is even, let $x = 2m$ where m is positive integer. Equation becomes

$$2^{2m} + p^y = z^2$$

$\boldsymbol{Subcase\ 1.1}$: $y = 0$, then we get $z^2 - 2^{2m} = 1$. By Catalan's conjecture, $2m = 3$ which is not possible and thus no solution in this case.

$\boldsymbol{Subcase\ 1.2}$: $y = 1$, then $p = (z + 2^m)(z - 2^m)$, since p is prime, we get $z - 2^m = 1$, then

$z = 2^m + 1$, and $p = 2^{m+1} + 1$. Thus, for primes of the form $p = 2^{m+1} + 1$, the solutions are given by $(x, y, z) = (2m, 1, 2^m + 1)$.

$\boldsymbol{Subcase\ 1.3}$: $y > 1$, then $p^y = (z + 2^m)(z - 2^m)$. Thus, there are non negative integers α and β such that $p^\alpha = z + 2^m$ and $p^\beta = z - 2^m$, where $\alpha + \beta = y$ and $\alpha > \beta$.

Then

$$p^\beta(p^{\alpha - \beta} - 1) = 2^{m+1}$$

If $\beta \neq 0$, then taking above equation modulo p, we get $0 \equiv 2^{m+1} (mod\ p)$. This is possible only if $p = 2$ but this is contradiction.

Let $\boldsymbol{\beta = 0}$, then we get $\boldsymbol{p^{\alpha} - 2^{m+1} = 1}$ by Catalan's conjecture $\boldsymbol{p = 3}$, but this is contradiction.

Case 2 : $\boldsymbol{x}$ is odd, then $\boldsymbol{y}$ must be either $\boldsymbol{0}$ or even.

Subcase 2.1 : $\boldsymbol{y = 0}$. Then $\boldsymbol{2^x = (z+1)(z-1)}$. Thus there are non negative integers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $\boldsymbol{2^{\alpha} = z+1}$ and $\boldsymbol{2^{\beta} = z-1}$, where $\boldsymbol{\alpha + \beta = x}$ and $\boldsymbol{\alpha > \beta}$. Then $\boldsymbol{2^{\beta}(2^{\alpha-\beta} - 1) = 2}$, we get $\boldsymbol{\beta = 1}$ and $\boldsymbol{2^{\alpha-\beta} = 2}$ and thus $\boldsymbol{\alpha = 2}$. Hence for every prime $\boldsymbol{p}$, we get the solutions of the form $\boldsymbol{(x, y, z) = (3, 0, 3)}$.

Subcase 2.2 : $\boldsymbol{y > 1}$, $\boldsymbol{y}$ is even say $\boldsymbol{y = 2m}$.

Then we get $\boldsymbol{2^x = (z + p^m)(z - p^m)}$

Thus there are non-negative integers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $\boldsymbol{2^{\alpha} = z + p^m}$ and $\boldsymbol{2^{\beta} = z - p^m}$, where $\boldsymbol{\alpha + \beta = x}$ and $\boldsymbol{\alpha > \beta}$. Then $\boldsymbol{2^{\beta}(2^{\alpha-\beta} - 1) = 2p^m}$, we get $\boldsymbol{\beta = 1}$ and $\boldsymbol{2^{\alpha-1} - p^m = 1}$. Now if $\boldsymbol{\min\{\alpha - 1, m\} > 1}$, then by Catalan's conjecture, there is no solution. It is easy to check that if $\boldsymbol{\alpha - 1 = 0}$ or $\boldsymbol{\alpha - 1 = 1}$, there is no solution.

Let $\boldsymbol{m = 1}$, then $\boldsymbol{p = 2^{\alpha-1} - 1}$ which is Mersenne prime. Thus if $\boldsymbol{p}$ is Mersenne prime, $\boldsymbol{p = 2^q - 1}$, then the solutions are given by $\boldsymbol{(x, y, z) = (q + 2, 2, 2 + p)}$

This completes the proof.

4. CONCLUSION

In this paper, we gave all the solutions of titled equation except the case when both $\boldsymbol{x}$ and $\boldsymbol{y}$ are odd positive integers together. Thus finding all the solutions of the titled equation in non-negative integers is still an open problem.

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