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ON A CONJECTURE OF ERDOS-STRAUS

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ABSTRACT

We discuss about a famous conjecture of Erdos-Strass, for which there are some always incomplete demonstrations or with mistakes. We think that it remains an open problem.

1. Introduction

In *Number Theory* there is a famous conjecture of Erdos-Straus which states that, for all positive integers *n≥2*, the rational number *4/n* can be expressed as the sum of three positive *unit fractions*. Paul Erdos and E.G.Straus formulated the following conjecture in 1948:

For every positive integer n≥2 there exist positive integers x,y,z such that:

$$\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

This problem attracted the attention of many mathematicians, as L.Bernstein [1], L.A.Rosati [4], S.E. Salez [5], K.Yamamoto [6], and many others.

It is not clear if there exists a demonstration of the truth or the falsity of this conjecture. There are many cases in which the authors adfirm to have proved that the conjecture is true and others in which the authors prove that it is false. We will see that this conjecture is always open.

In what follows, if for a positive interger *n'*, there exist positive integers *x',y',z'* such that: *4/n' = 1/x' + 1/y' +1/z'*, then we will say that (*x',y',z'*) *is a solution of the conjecture, associated with n'*.

2. Remark

A possible remark is can be why it is requested that *x,y,z* must be *positive.*

The answer is that, without this condition, the problem can be solved easily.

Indeed, we can see that, for *n≡0,1,2,3 mod 4*:

$$\frac{4}{4k}=\frac{1}{k}+\frac{1}{2k}-\frac{1}{2k}\ ,$$

$$\frac{4}{4k+1}=\frac{1}{2k}+\frac{1}{2k}-\frac{1}{k(4k+1)},$$

$$\frac{4}{4k+2}=\frac{1}{2k}+\frac{1}{2(k+1)}-\frac{1}{2k(k+1)(2k+1)},$$

$$\frac{4}{4k+3}=\frac{1}{2(k+1)}+\frac{1}{2k+1}-\frac{1}{2(k+1)(2k+1)(4k+3)}.$$

3. Some results

In this section we see some cases in which it is proved that the conjecture is true.

First of all, observe that in [5] S.E.Salez proved by computer searches that the truth of the conjecture up to $n \leq 10^{17}$.

Theorem 3.1: *If (x',y',z') is a solution of the conjecture associated with n', then, for every positive integer h, also (hx',hy',hz') is a solution of the conjecture and it is associated with hn'.*

Proof: Easily, if:

$$\frac{4}{n'}=\frac{1}{x'}+\frac{1}{y'}+\frac{1}{z'}$$

then:

$$\frac{4}{hn'}=\frac{1}{hx'}+\frac{1}{hy'}+\frac{1}{zh'}$$

for every positive integer *h.* Hence, the statement follows.

Theorem 3.2: *For n≡0,2,3 mod 4, the conjecture is true.*

Proof: Let *n≡0,2,3 mod 4.*

If *n=4k,* then for: *x=2k, y=4k, z=4k,* it is possible to verity that the conjecture is true.

If *n=4k+2*, then for: *x=2k+1, y=4k+2, z=4k+2,* it is possible to verity that the conjecture is true.

If *n=4k+3*, then for: *x=2k+2, y=2k+2, z=(k+1)(4k+3),* it is possible to verity that the conjecture is true. The case *n=4k+1* is not solved.

Theorem 3.3 [3]: *For n≡0,2,3,4,5 mod 6, the conjecture is true.*

Proof: Let *n≡0,2,3,4,5 mod 6.*

If *n=6k,* then for: *x=3k, y=6k, z=6k,* it is possible to verify that the conjecture is true.

If *n=6k+2*, then for: *x=3k+1, y=6k+2, z=6k+2,* it is possible to verify that the conjecture is true.

If *n=6k+3*, then for: *x=6k+3, y=2k+2, z=(2k+1)(2k+2),* it is possible to verify that the conjecture is true.

If *n=6k+4*, then for: *x=3k+2, y=6k+4, z=6k+4,* it is possible to verify that the conjecture is true.

If *n=6k+5*, then for: *x=6k+5, y=2k+2, z=(6k+5)(2k+2),* it is possible to verify that the conjecture is true. The case *n=6k+1* is not solved.

Theorem 3.4: *For every n≥2, n even or n≡3,5,7,9,11 mod 12, the conjecture is true.*

Proof: Observe that:

- if *n≡5 mod* 12, it is *n≡1 mod 4*, but also *n≡5 mod 6*;
- if *n≡7 mod* 12, it is *n≡1 mod 6*, but also *n≡3 mod 4*;
- if *n≡9 mod* 12, it is *n≡1 mod 4*, but also *n≡3 mod 6*;
- if *n≡3 mod* 12, it is *n≡3 mod 4* and also *n≡3 mod 6*;
- if *n≡11 mod* 12, it is *n≡3 mod 4* and also *n≡5 mod 6*.

For *n even, n≥2*, see Theorems 3.2, 3.3.

4. Some apparent demonstrations

In [7] it is proved that *the conjecture of Erdos-Straus is false*.

Following the demonstration of the author, he proves by induction that for *n=2* the equation has solution, but in all the other cases the equation does not admit any solution. It is clear, reading the proof, that it is not valid and that, therefore, the statement cannot be accepted.

In [3] it is proved that *the conjecture of Erdos-Straus is true*.

The author divides the proof into the cases: *n≡0,1,2,3,4,5 mod 6*. For *n≡0,2,3,4,5 mod 6*, he gives the demonstrations of Theorem 3.3.

For *n≡1 mod 6*, if *n=6k+1*, the validity of the proof depends on the fact that it is always possible to choose positive integers *b,c* such that

$$\frac{b(6k+1)(k+c)}{(k+c)(4b-1)-b(6k+1)}$$

is a positive integer.

But it seems that the proof of this last sentence is not immediate, supposed that it is true. Certainly, if:

For every positive integer k, there exist positive integers b,c such that the function

$$F(b,c,k)=\frac{b(6k+1)(k+c)}{(k+c)(4b-1)-b(6k+1)}$$

is a positive integer, then the conjecture of Erdos-Straus is true.

But this is not proved.

5. The function *F(b,c,k)*

In this section we examine some properties of the function *F*(*b,c,k*), which can be useful to prove the conjecture of Erdos-Straus.

First of all, as we have seen in the previous section, it is possible to adfirm that:

Theorem 5.1: *If for every positive integer k there exist positive integers b,c, such that F(b,c,k) is a positive integer, then the conjecture of Erdos-Straus is true.*

It follows that:

Theorem 5.2: *If k+c and b are both odd, then F(b,c,k) is not an integer.*

Proof: Immediately, if the positive integers *k+c* and *b* are both odd, then *numerator* of *F* is odd and *denominator* of *F* is even. Hence *F* cannot be an integer.

To study what Theorem 5.1 adfirms, a possible way to follow is to consider the case for which the denominator of *F* can be equal to one. For this, it can be useful the following result.

Theorem 5.3: *If k is even and k ≤ 12, then denominator of F cannot be equal to 1*.

Proof: Let *k=2h,* for any *h=1,2,3,4,5,6.* Suppose that the *denominator* of *F* is 1:

$$(k+c)(4b-1)-b(6k+1)=1.$$

It follows that:

$$c(4b-1)=b(2k+1)+k+1,$$

hence:

$$c=h+\frac{b+3h+1}{4b-1},$$

where:

$$\frac{b+3h+1}{4b-1}\geq 1, \qquad for\ b\leq h.$$

Since:

$h=1 \quad \Rightarrow \quad c=1+\frac{b+4}{4b-1}$,

and $b=1 \Rightarrow c=1+\frac{5}{3}$;

$h=2 \quad \Rightarrow \quad c=2+\frac{b+7}{4b-1}$,

and $b=1 \Rightarrow c=2+\frac{8}{3}$, $b=2 \Rightarrow c=2+\frac{9}{7}$;

$h=3 \quad \Rightarrow \quad c=3+\frac{b+10}{4b-1}$,

and $b=1 \Rightarrow c=3+\frac{11}{3}$, $b=2 \Rightarrow c=3+\frac{12}{7}$,

$b=3 \Rightarrow c=3+\frac{13}{11}$;

$h=4 \quad \Rightarrow \quad c=4+\frac{b+13}{4b-1}$,

and $b=1 \Rightarrow c=4+\frac{14}{3}$, $b=2 \Rightarrow c=4+\frac{15}{7}$,

$b=3 \Rightarrow c=4+\frac{16}{11}$, $b=4 \Rightarrow c=4+\frac{17}{15}$;

$h=5 \quad \Rightarrow \quad c=5+\frac{b+16}{4b-1}$,

and $b=1 \Rightarrow c=5+\frac{17}{3}$, $b=2 \Rightarrow c=5+\frac{18}{7}$,

$b=3 \Rightarrow c=5+\frac{19}{11}$, $b=4 \Rightarrow c=5+\frac{20}{15}$,

$b=5 \Rightarrow c=5+\frac{21}{19}$;

$h=6 \quad \Rightarrow \quad c=6+\frac{b+19}{4b-1}$,

and $b=1 \Rightarrow c=6+\frac{20}{3}$, $b=2 \Rightarrow c=6+\frac{21}{7}$,

$b=3 \Rightarrow c=6+\frac{22}{11}$, $b=4 \Rightarrow c=6+\frac{23}{15}$,

$b=5 \Rightarrow c=6+\frac{24}{19}$, $b=6 \Rightarrow c=6+\frac{25}{23}$;

we can verify that, in every case, for $b \le h \le 6$, the parameter *c* is not a positive integer and, from this, the statement follows.

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