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A Simple Analysis of Customers Behaviour on M/M/1/k Queues

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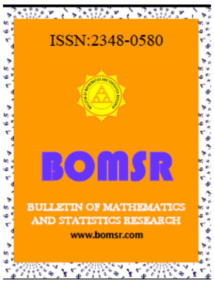

ABSTRACT

In this paper, the finite buffer single server Markovian queuing system is analyzed with the additional restriction that customers can balk as well as renege. State dependent balking and Markovian reneging is considered. A few redesigned performance measures along with sensitivity analysis on server utilization have been presented. An example with design connotation rounds up the paper.

Keywords: Balking, Customer impatience, Finite buffer queue, Reneging.
2010 Mathematics Subject Classification: 60K25, 68M20, 90B22.

1. Introduction

The discipline of queuing theory had its origin in 1908 when Erlang (1909) published his fundamental paper analyzing telephone traffic. Since then models of various types with different characteristics have been analyzed in literature. Of the different characteristics of a queuing system, a particular one relates to customer behavior. Specifically, a customer on its arrival to a queuing system has to arrive at a decision to join the queue or not and second, if it joins, it has to decide if it is willing to wait as long as it is necessary to obtain service. In our modern fast-paced life where people place a premium on time, it is conceivable that an arriving customer may have negative thoughts on joining the system if it is required to wait in the queue. Often this would be a function of length of the queue. Additionally, even if one were to decide to join the system, it is but likely, that one would not be prepared to wait as long as it is necessary to obtain service.

The phenomenon of customers arriving at a queuing system and refusing to join is known as balking. Haight (1957) has provided a rationale, which might influence a person to balk. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency so that a queue of certain length will not be joined to indifference where a non-zero queue is also joined. Once a customer joins, the phenomenon of customers leaving the system before completing the act of receiving service is known as reneging.

In this paper, we shall analyze these two aspects of customer behavior in relation to the Markovian single server finite buffer queue. From the managerial point of view, a few redesigned performance measures will also be presented. Implications of change in system parameters vis-à-vis server utilization will also be discussed.

Mainly, reneging can be of two types- reneging till beginning of service (henceforth referred to as R_BOS) and reneging till end of service (henceforth referred to as R_EOS). R_BOS can be observed in queuing systems where a customer can renege only as long as in queue. Once it begins receiving service, it cannot renege. A common example is the barbershop. A customer can renege while he is waiting in queue. However once service commences i.e. hair cut begins, the customer cannot leave till hair cutting is over. On the other hand, R_EOS can be observed in queuing systems where a customer can renege not only while waiting in queue but also while receiving service. A common example is the hospital emergency room/O.T. handling critical patients. Such patients may expire (i.e. renege) while waiting in queue for doctors to attend to them. They may also expire while receiving service i.e. while being attended to by doctors.

Reneging considered in literature is mostly of deterministic type. In deterministic reneging, each customer is assumed to have a fixed patience time after which he becomes a 'lost' customer. An early work on reneging was by Barrer (1957) where he considered deterministic reneging with single server Markovian arrival and service rates. Customers were selected for service on random basis. In his subsequent work, Barrer (1958) also considered deterministic reneging in a multi-server scenario with FCFS discipline. Reneging considered was of both R_BOS and R_EOS type. Another early work was by Haight (1959). Ghosal (1963) considered a D/G/1 model with deterministic reneging. Gavish and Schweitzer (1977) also considered a deterministic reneging model with the additional assumption that arrivals can be labeled by their service requirement before joining the queue and arriving customers are admitted only if their waiting plus service time do not exceed some fixed amount. This assumption is met in communication systems. Choudhury (2008) analyzed a single server Markovian queuing system with the added complexity of customers who are prone to giving up whenever its waiting time is larger than a random threshold-his patience time. He assumed that these individual patience times were independent and identically distributed exponential random variables. Reneging till beginning of service was considered. A detailed and lucid derivation of the distribution of virtual waiting time in the system was presented. Some performance measures were also presented.

Few other attempts at modeling reneging phenomenon include those by Baccelli et al. (1984), Kok and Tijms (1985), Martin and Artalejo (1995), Boots and Tijms(1999), Bae et al. (2001), Choi et al. (2001), Choi Kim and Zhu (2004), Zhang et al. (2005), Singh et al. (2007), Altman and Yechiali (2008), Kim et al. (2008) and Liu and Kulkarni (2008).

An early work on balking was by Haight (1957). Haghighi et al. (1986) considered a multiserver queuing model with balking as well as reneging. Each customer had a balking probability which was independent of the state of the system. The arrival, service and reneging distribution were Markovian. Reneging discipline considered was R_BOS. Liu et al (1987) considered an infinite server Markovian queuing system with reneging of type R_BOS. Customers had a choice of individual service or batch service: batch service being preferred by the customer. Brandt et al (1998) considered a S-server system with two FCFS queues, where the arrival rates at the queues and the service may depend on number of customers 'n' being in service or in the first queue, but the service rate was assumed to be constant for n>s. The customers in the first queue were assumed to be impatient customers with deterministic reneging. Wang et al (1999) considered the machine repair problem in which failed machines balk with probability (1-b) and renege according to a negative exponential distribution. Another work using the concepts of balking and reneging in machine interference queue has been carried out by Al-Seedy and Al-Ibraheem (2001). There have been a few papers, which considered both balking as well as reneging. Here mention may be made of the papers by Haghighi et al. (1986), Zhang et al. (2005), El- Paoumy (2008), El- Sherbiny (2008), Shawky and El-Paoumy (2008), El-Paoumy (2009).

In this paper we discuss the analysis of M/M/1/k model with the additional restriction that customer may balk as well as renege. Both types of reneging R_BOS and R_EOS are discussed separately. To the best of our knowledge, an analysis with both types of reneging in addition to state dependent balking in this model has not been carried out. Importance of the queuing model stems from the fact that in the classical M/M/1 model," it is assumed that the system can accommodate any number of units. In practice, this may seldom be the case. We have thus to consider the situation such that the system has limited waiting space and can hold a maximum number of k units (including the one being served)" {Medhi (1994)}. As for balking, we assume that each customer has a state dependent balking probability. It will be assumed that if the customer on arrival observes the system to be in state 'i', the probability that he will balk is 'i/k', i=1,2,…,k. With this set up, the finite buffer restriction can also be seen as the state from which customer balks with probability 1(=k/k). There is no balking from an empty system. Each customer has a random patience time following exp (ν) distribution. This patience time commences from the time it joins the systems. In case the reneging distribution is R_BOS, the customer will renege i.e. leave the system in case service does not begin before expiry of this patience time. In case of R_EOS, the customer would renege in case service is not over before the expiry of the patience time. Thus in case of R_EOS, the customer may depart either from the queue or from the service with partial and incomplete service whereas in case of R_BOS, the customer can renege only from the queue.

The arrival and service pattern are assumed to be Markovian with rates λ and μ respectively.

The rest of the paper is structured as follows. In section 2, we obtain the system state probabilities. In section 3, performance measures are presented. In section 4, we carried out sensitivity analysis. A numerical example is given in section 5. Section 6 concludes the paper. Some derivations are given in the appendix placed in section 7.

2. The System State Probabilities:

In this section, we derive the steady state probabilities are derived. Under R_BOS, let p_n denotes the probability that there are 'n' customers in the system. Applying the Markov process theory, we obtain the following set of steady- state equations:

$$\lambda\, p_0 = \mu\, p_1 , \tag{2.1}$$

$$\lambda\left(1-\frac{n-1}{k}\right)p_{n-1} + (\mu + n\nu)p_{n+1} = \lambda\left(1-\frac{n}{k}\right)p_n + \{\mu + (n-1)\nu\}p_n \quad ;1 \le n \le k-1, \tag{2.2}$$

$$\lambda\left(1-\frac{k-1}{k}\right)p_{k-1} = \{\mu + (k-1)\nu\}\, p_k ,$$

Solving recursively, we get (under R_BOS)

$$p_n = \frac{\lambda^n \prod_{r=1}^{n-1}\left(1-\frac{r-1}{k}\right)}{\prod_{r=1}^{n}\{\mu+(n-1)\nu\}} p_0 \quad ; \text{n=1, 2,..., k.} \tag{2.3}$$

where p_0 is obtained from the normalizing condition $\sum_{n=0}^{k} p_n = 1$ and is given as

$$p_0 = \left[1+\sum_{n=1}^{k}\frac{\lambda^n \prod_{r=1}^{n-1}\left(1-\frac{r-1}{k}\right)}{\prod_{r=1}^{n}\{\mu+(r-1)\nu\}}\right]^{-1} . \tag{2.4}$$

Under R_EOS, let q_n denote the probability that there are n customers in the system.

Proceeding similarly, we obtain the following set of steady state equations.

$$\lambda q_0 = (\mu+\nu)q_1 \quad , \tag{2.5}$$

$$\lambda\left(1-\frac{n-1}{k}\right)q_{n-1} + \{\mu+(n+1)\nu\}q_{n+1} = \lambda\left(1-\frac{n}{k}\right)q_n + (\mu+n\nu)q_n \quad : \quad 1 \le n \le k-1, \tag{2.6}$$

$$\lambda\left(1-\frac{k-1}{k}\right)q_{k-1} = (\mu+k\nu)q_k$$

solving, we get (under R_EOS)

$$q_n = \frac{\lambda^n \prod_{r=1}^{n-1}\left(1-\frac{r-1}{k}\right)}{\prod_{r=1}^{n}(\mu+r\nu)} q_0 \qquad ;\text{n} = 1,2,\ldots,\text{k} \tag{2.7}$$

where

$$q_0 = \left[1+\sum_{n=1}^{k}\frac{\lambda^n \prod_{r=1}^{n-1}\left(1-\frac{r-1}{k}\right)}{\prod_{r=1}^{n}(\mu+r\nu)}\right]^{-1} \tag{2.8}$$

3. Performance Measures:

An important measure is 'L', which denotes the mean number of customers in the system. To obtain an expression for the same, we note that L=P′(1) where

$$P'(1) = \frac{d}{ds}P(s)\,|_{s=1}$$

Here P(S) is the p.g.f. of the steady state probabilities. The derivation of P′(1) is given in the appendix. From (7.1.1) and (7.2.1), the mean system size under two reneging rules are

$$L_{(R\_BOS)} = \frac{\mathrm{k}}{(\lambda + \mathrm{k}\nu)}\{\lambda - (\mu - \nu)(1 - p_0)\},$$

$$L_{(R\_EOS)} = \frac{\mathrm{k}}{(\lambda + \mathrm{k}\nu)}\{\lambda - (1 - q_0)\mu\}.$$

Mean queue size can now be obtained and are given by

$$L_{q(R\_BOS)} = P'(1) - (1 - p_0)$$
$$= \frac{\mathrm{k}}{(\lambda + \mathrm{k}\nu)}\{\lambda - (\mu - \nu)(1 - p_0)\} - (1 - p_0),$$

Similarly,

$$L_{q(R\_EOS)} = P'(1) - (1 - q_0)$$
$$= \frac{\mathrm{k}}{(\lambda + \mathrm{k}\nu)}\{\lambda - (1 - q_0)\mu\} - (1 - q_0)$$

Customers arrive at the system at the rate of λ. However all the customers who arrive do not join the system either because of balking or because of finite buffer restriction. As noted earlier, the restriction due to finite buffer can also be seen as balking with probability 1. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\lambda^{e}{}_{(R\_BOS)} = \lambda \sum_{n=0}^{k}\left(1 - \frac{n}{k}\right)p_n$$
$$= \lambda\left\{1 - \frac{1}{k}p'(1)\right\}$$
$$= \frac{\lambda\{k\nu + (1 - p_0)(\mu - \nu)\}}{(\lambda + k\nu)}, \quad (3.1)$$

$$\lambda^{e}{}_{(R\_EOS)} = \lambda \sum_{n=0}^{k}\left(1 - \frac{n}{k}\right)q_n$$
$$= \lambda\left\{1 - \frac{1}{k}p'(1)\right\}$$
$$= \frac{\lambda\{k\nu + (1 - q_0)\mu\}}{(\lambda + k\nu)}. \quad (3.2)$$

It is relevant to note here that unlike in the traditional M/M/1 model where it is important that λ<μ, the same need not hold in M/M/1/k model as customers arriving after maximum buffer size has been reached are turned back.

We have assumed that each of the customers who join the system has a reneging distribution (of type R_BOS or R_EOS) following exp(ν). Clearly then, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rate (avg rr) is given by

$$\begin{aligned}\text{Avgrr}_{(\text{R\_BOS})} &= \sum_{n=2}^{k}(n-1)\nu p_n \\ &= \nu\{\text{p}'(1)-(1-p_0)\} \\ &= \nu\left[\frac{k\{\lambda-(1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)}-(1-p_0)\right]\end{aligned} \tag{3.3}$$

$$\begin{aligned}\text{Avgrr}_{(\text{R\_EOS})} &= \sum_{n=1}^{k} n\,\nu q_n \\ &= \nu P'(1) \\ &= \frac{\nu k}{(\lambda+k\nu)}\{\lambda - (1 - \text{q}_0)\mu\}\ .\end{aligned} \tag{3.4}$$

From the point of view of system management, customers who balk or renege represent business lost. In totality, customers are lost to the system in three ways, due to finite buffer, due to reneging and due to balking. The management would like to know the proportion of total customers lost in order to have an idea of total business lost. Customers are lost due to reneging and due to balking (restriction due to finite buffer being seen as balking). The rate of loss due to balking is $\lambda - \lambda^{e}$ (R_BOS and R_EOS according to the reneging rule). λ^{e} is given in (3.1) and (3.2).

Hence the mean rate at which customers are lost (under R_BOS) is

$$\begin{aligned}&\lambda-\lambda^{e}{}_{(R\_BOS)}+avg\ rr_{(R\_BOS)} \\ &= \frac{\lambda\{\lambda+(1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)}+\nu\{p'(1)-(1-p_0)\} \\ &= \frac{\lambda\{\lambda+(1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)}+\nu\left[\frac{k\{\lambda-(1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)}-(1-p_0)\right]\end{aligned} \tag{3.5}$$

and mean rate at which customers are lost (Under R_EOS):

$$\begin{aligned}&\lambda-\lambda^{e}{}_{(R\_EOS)}+avg\,rr_{(R\_EOS)} \\ &= \frac{\lambda\{\lambda-(1-q_0)\mu\}}{(\lambda+k\nu)}+\frac{\nu k}{(\lambda+k\nu)}\{\lambda-(1-q_0)\mu\} \\ &= \lambda-(1-q_0)\mu\end{aligned} \tag{3.6}$$

To the system manger, this rate is of interest as it helps in the determination of proportion of customers lost which is an important measure of business lost. This proportion (Under R_BOS) is given by

$$\frac{\lambda - \lambda^{e}{}_{(R\_BOS)} + avg\ rr_{(R\_BOS)}}{\lambda}$$

$$= \frac{\lambda + (1-p_0)(\mu-\nu)}{(\lambda+k\nu)} + \frac{\nu}{\lambda}\{p'(1) - (1-p_0)\}$$

$$= \frac{\lambda + (1-p_0)(\mu-\nu)}{(\lambda+k\nu)} + \frac{\nu}{\lambda}\left[\frac{k\{\lambda-(1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)} - (1-p_0)\right]$$

and the proportion (under R_EOS) is given by

$$\frac{\lambda - \lambda^{e}{}_{(R\_EOS)} + avg\ rr_{(R\_EOS)}}{\lambda}$$

$$= 1 - \frac{\mu}{\lambda}(1-q_0)$$

The proportion of customer completing receipt of service can now be easily determined from the above proportion.

All the customers who join the system do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server's point of view, this provides a measure of the amount of work he has to do. Let us call the rate at which customers reach the service station as λ^s .Then under R_BOS

$\lambda^s{}_{(R\_BOS)} = \lambda^e{}_{(R\_BOS)}$(1-proportion of customers lost due to reneging out of those joining the system)

$$= \lambda^{e}{}_{(R\_BOS)}\left\{1 - \sum_{n=2}^{k}(n-1)\nu p_n / \lambda^{e}{}_{(R\_BOS)}\right\}$$

$$= \frac{\lambda\{k\nu + (1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)} - \nu\{P'(1) - (1-p_0)\}$$

$$= \frac{\lambda\{k\nu + (1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)} - \nu\left[\frac{k\{\lambda-(1-p_0)(\mu-\nu)\}}{(\lambda+k\nu)} - (1-p_0)\right]$$

In case of R_EOS, one needs to recall that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Then

$\lambda^s{}_{(R\_EOS)} = \lambda^e{}_{(R\_EOS)}$(1-proportion of customers lost due to reneging from the queue out of those joining the system)

$$= \lambda^{e}{}_{(R\_EOS)}\left\{1 - \sum_{n=2}^{k}(n-1)\nu q_n / \lambda^{e}{}_{(R\_EOS)}\right\}$$

$$= \lambda^{e}{}_{(R\_EOS)} - \nu\{L_{R\_EOS} - (1-q_0)\}$$

$$= \frac{\lambda\{k\nu + (1-q_0)\mu\}}{(\lambda+k\nu)} - \nu\left[\frac{k\{\lambda-(1-q_0)\mu\}}{(\lambda+k\nu)} - (1-q_0)\right]$$

4. Sensitivity Analysis

It is interesting to examine and understand how server utilization varies in response to change in system parameters. The four system parameters of interest are $\lambda, \mu, \nu, k.$ We place below the effect of change in these system parameters on server utilization. For this purpose, we shall follow the following notational convention in the rest of this section.

$p_n(\lambda,\mu,\nu,k)$ and $q_n(\lambda,\mu,\nu,k)$ will denote the probability that there are 'n' customers in a system with parameters λ,μ,ν,k in steady state under R_BOS and R_EOS respectively.

It can be shown that

i) If $\lambda_1 > \lambda_0$ then

$$\frac{p_0(\lambda_1,\mu,\nu,k)}{p_0(\lambda_0,\mu,\nu,k)} < 1$$

$$\Rightarrow \frac{(\lambda_0-\lambda_1)}{\mu} + \frac{\left(1-\frac{1}{k}\right)}{\mu(\mu+\nu)}\left(\lambda_0^{\,2}-\lambda_1^{\,2}\right) + \ldots + \frac{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\ldots\left(1-\frac{k-1}{k}\right)}{\mu(\mu+\nu)\ldots\{\mu+(k-1)\nu\}}\left(\lambda_0^{\,k}-\lambda_1^{\,k}\right) < 0$$

which is true and hence $p_0 \downarrow as\ \lambda \uparrow$.

ii) If $\mu_1 > \mu_0$ then

$$\frac{p_0(\lambda,\mu_1,\nu,k)}{\mathrm{p}_0(\lambda,\mu_0,\nu,k)} > 1$$

$$\Rightarrow \lambda\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right) + \lambda^2\left(1-\frac{1}{k}\right)\left\{\frac{1}{\mu_0(\mu_0+\nu)}-\frac{1}{\mu_1(\mu_1+\nu)}\right\} + \ldots. +$$

$$\lambda^k\left(1-\frac{1}{k}\right)\ldots\left(1-\frac{k-1}{k}\right)\left\{\frac{1}{\mu_0(\mu_0+\nu)\ldots\{\mu_0+(k-1)\nu\}}-\frac{1}{\mu_1(\mu_1+\nu)\ldots\{\mu_1+(k-1)\nu\}}\right\} > 0$$

which is true and hence $p_0 \uparrow \ as\ \mu \uparrow$.

iii) If $\nu_1 > \nu_0$ then

$$\frac{p_0(\lambda,\mu\ ,\nu_1,k)}{\mathrm{p}_0(\lambda,\mu,\nu_0,k)} > 1$$

$$= \lambda\left(\frac{1}{\mu}-\frac{1}{\mu}\right) + \lambda^2\left(1-\frac{1}{k}\right)\left\{\frac{1}{\mu(\mu+\nu_0)}-\frac{1}{\mu(\mu+\nu_1)}\right\} + \ldots. +$$

$$\lambda^k\left(1-\frac{1}{k}\right)\ldots\left(1-\frac{k-1}{k}\right)\left\{\frac{1}{\mu(\mu+\nu_0)\ldots\{\mu+(k-1)\nu_0\}}-\frac{1}{\mu(\mu+\nu_1)\ldots\{\mu+(k-1)\nu_1\}}\right\} > 0$$

which is true and hence $p_0 \uparrow as\ \nu \uparrow$.

iv) If $k_1 > k_0$ then

$$\frac{p_0(\lambda,\mu\ ,\nu,k_1)}{\mathrm{p}_0(\lambda,\mu,\nu,k_0)} < 1$$

$$\Rightarrow \sum_{n=1}^{k_0}\frac{\lambda^n\prod_{r=1}^{n}\left(1-\frac{r-1}{k_0}\right)}{\prod_{r=1}^{n}\{\mu+(r-1)\nu\}} - \sum_{n=1}^{k_1}\frac{\lambda^n\prod_{r=1}^{n}\left(1-\frac{r-1}{k_1}\right)}{\prod_{r=1}^{n}\{\mu+(r-1)\nu\}} < 0$$

which is true and hence $p_0 \downarrow as\ k \uparrow$.

The following can similarly be shown.

v) $q_0 \downarrow as\ \lambda \uparrow$

vi) $q_0 \uparrow as\ \mu \uparrow$

vii) $q_0 \uparrow as\ \nu \uparrow$

viii) $q_0 \downarrow as\ k \uparrow$

Managerial implications of the above results are obvious.

6. Numerical Example:

To illustrate the use of our results, we apply them to a queuing scenario. We quote below an example from Taha (2003, page 610).

'The time for barber Joe to give a haircut is exponential with mean of 12 minutes. Because of his popularity, customers usually arrive (according to a Poisson distribution) at a rate much higher than Joe can handle 6 customers per hour. Joe really will feel comfortable if the arrival rate is effectively reduced to about 4 customers per hour. To accomplish this goal, he came up with the idea of providing limited seating in the waiting area so that newly arriving customers would go elsewhere when they discover that all the seats are taken. How many seats should Joe provide to accomplish his goal?'

This is a design problem where the system manager (Joe, the barber) desires a system design in respect of size of the waiting area (number of chairs for waiting customers).

Here λ=6/hr and μ=5/hr. As required by Joe, we examine the effect of limited seating arrangement in the waiting area with different choices of k. Though not explicitly stated, it is necessary to assume reneging and balking. Customers these days are very hard pressed for time. Prompt customer service being the expectation, it is all the more reasonable to assume that customers are all of reneging type. Since Joe has not collected data on customer reneging rate in his shop, let us therefore consider alternative possible Markovian reneging rates of 120 min (ν=0.5), 100 min (ν=0.6) and 80 min (ν=0.75). Given the fact that service in a barber shop is being analyzed, clearly the reneging rule would be R_BOS. We further assume that the probability of balking by an arriving customer is 'i/k', i= 1,2,…,k where i is the state of the customer observes the system to be in on its arrival.

Various performance measures of interest computed under different scenarios are given in Table 1, 2 and 3. These measures were arrived at using a FORTRAN 77 program coded by the authors. Different choices of k were considered. Results relevant with regard to Joe's desire to limit arrival rate of customers into his service station to something around 4/hr are presented in the tables

Table1: Performance Measures assuming λ=6/hr, μ=5/hr and ν=0.5/hr.

Performance Measure	Size of Waiting Area		
	6 (k=7)	7 (k=8)	8 (k=9)
λ^s (i.e. arrival rate of customers reaching service station)	3.94535	3.99785	4.0408
Effective arrival rate(λ^e)	4.45315	4.55884	4.64956
Fraction of time server is idle	0.21093	0.20043	0.19184
Average length of queue	1.01559	1.12197	1.21750
Average length of system	1.80466	1.92154	2.02566
Mean reneging rate	0.50779	0.56099	0.60875
Rate of loss due to balking and finite buffer	1.54685	1.44116	1.35044
Mean rate of customers lost	2.05464	2.00214	1.95919
Proportion of customers lost due to reneging, balking and finite buffer	0.34244	0.33369	0.32653

Table2: Performance Measures assuming λ=6/hr, μ=5/hr and ν=0.6/hr.

Performance Measure	Size of Waiting Area		
	7 (k=8)	8 (k=9)	9 (k=10)
λ^s (i.e. arrival rate of customers reaching service station)	3.96596	4.00663	4.0404
Effective arrival rate(λ^e)	4.60558	4.69781	4.77780
Fraction of time server is idle	0.20681	0.19867	0.19191
Average length of queue	1.06603	1.15196	1.22890
Average length of system	1.85923	1.95329	2.03699
Mean reneging rate	0.63962	0.69118	0.73734
Rate of loss due to balking and finite buffer	1.39442	1.30219	1.22220
Mean rate of customers lost	2.03404	1.99337	1.95954
Proportion of customers lost due to reneging, balking and finite buffer	0.33901	0.33223	0.32659

Table3: Performance Measures assuming λ=6/hr, μ=5/hr and ν=0.75/hr.

Performance Measure	Size of Waiting Area		
	8 (k=9)	9 (k=10)	10 (k=11)
λ^s (i.e. arrival rate of customers reaching service station)	3.95966	3.99093	4.01728
Effective arrival rate(λ^e)	4.76034	4.84102	4.91145
Fraction of time server is idle	0.20807	0.20181	0.19654
Average length of queue	1.06756	1.13345	1.19222
Average length of system	1.85949	1.93163	1.99567
Mean reneging rate	0.80067	0.85008	0.89417
Rate of loss due to balking and finite buffer	1.23966	1.15898	1.08855
Mean rate of customers lost	2.04034	2.00906	1.98272
Proportion of customers lost due to reneging, balking and finite buffer	0.34005	0.33484	0.33045

Since a larger waiting area would also entail additional expenditure/ investment, Joe needs to examine how the performance measures differ across different choices of k. In case the reneging behavior of customer follows exp (0.5) distribution, it appears from the tables 1 that an ideal choice of k could be 8 (seating space in waiting area =7) with λ^s=3.99786. In case the reneging distribution is exp (0.6), then k=9 appears to be close to Joe's target with λ^s=4.00663 (table 2). Table 3 states that an ideal choice of k could be k=10 with λ^s=3.99094 if the reneging distribution is exp (0.75).

Two interesting observations can be made from the tables. To a layman, Joe's aim of reducing λ from 6 to 4 effectively boils down to turning away one third of his customers. Our analysis confirms the same. In each of the three scenarios examined (in three tables), the percentage of customers lost due to reneging together with finite buffer at the level of ideal choice of k hovers very close to one third at 33.36%, 33.22% and 33.48%. Second, at the level of λ^s nearest to 4, the fraction of time Joe would be idle (p_0) is almost constant at 20% in the three scenarios. These results stand to reason.

6. Conclusion

In this paper, we have analyzed a M/M/1/k queuing system with state dependent balking and markovian reneging. A number of performance measures have been presented. Quite often, system parameter undergoes change. This could be due to changes in the environment in which the system is operating or it could be deliberate on the part of management. The effect of such changes on server utilization has been analyzed. A numerical example with design connotations has been presented to demonstrate results derived. It is our belief that the results presented will be of use to practitioners of queuing theory.

7. Appendix

7.1. Derivation of P′(1) under R_BOS:

From equation (2.2) we have

$$\lambda\left(1-\frac{n-1}{k}\right)p_{n-1}+(\mu+n\nu)p_{n+1}=\lambda\left(1-\frac{n}{k}\right)p_n+\{\mu+(n-1)\nu\}p_n \quad .$$

Multiplying both sides of the equation by s^n and summing over n

$$\lambda s\sum_{n=1}^{k-1}\left(1-\frac{n-1}{k}\right)p_{n-1}s^{n-1}+\frac{1}{s}\sum_{n=1}^{k-1}(\mu+n\nu)p_{n+1}s^{n+1}=\lambda\sum_{n=1}^{k-1}\left(1-\frac{n}{k}\right)p_n s^n+\sum_{n=1}^{k-1}\{\mu+(n-1)\nu\}p_n s_n$$

$$\Rightarrow \lambda s\sum_{n=1}^{k-1}\left(1-\frac{n-1}{k}\right)p_{n-1}s^{n-1}-\lambda\sum_{n=1}^{k-1}\left(1-\frac{n}{k}\right)p_n s^n=\sum_{n=1}^{k-1}\{\mu+(n-1)\nu\}p_n s_n-\frac{1}{s}\sum_{n=1}^{k-1}(\mu+n\nu)p_{n+1}s_{n+1}$$

$$\Rightarrow \lambda s\left\{p_0 s^0+(1-\frac{1}{k})p_1 s^1+\left(1-\frac{2}{k}\right)p_2 s^2+\left(1-\frac{3}{k}\right)p_3 s^3+..\left(1-\frac{k-2}{k}\right)p_{k-2}s^{k-2}\right\}-$$

$$\lambda\left\{\left(1-\frac{1}{k}\right)p_1 s^1+\left(1-\frac{2}{k}\right)p_2 s^2+..\left(1-\frac{k-1}{k}\right)p_{k-1}s^{k-1}\right\}$$

$$=\left[(\mu+0\nu)p_1 s^1+(\mu+\nu)p_2 s^2+...\{\mu+(k-2)\nu\}p_{k-1}s^{k-1}\right]-\frac{1}{s}\left[(\mu+\nu)p_2 s^2+(\mu+2\nu)p_3 s^3+...+\{\mu+(k-1)\nu\}p_k s^k\right]$$

$$\Rightarrow \lambda s\left\{\left(p_0 s^0+p_1 s^1+....+p_{k-2}s^{k-2}\right)-\frac{s}{k}\left(p_1 s^0+2p_2 s+3p_3 s^2+...+(k-2)p_{k-2}s^{k-3}\right)\right\}$$

$$-\lambda\left\{\left(p_1 s^1+p_2 s^2+...+p_{k-1}s^{k-1}\right)-\frac{s}{k}\left(p_1 s^0+2p_2 s^1+...+(k-1)p_{k-1}s^{k-2}\right)\right\}$$

$$=\left\{\mu\left(p_1 s^1+p_2 s^2+...+p_{k-1}s^{k-1}\right)+\nu s\left(p_2 s+2p_3 s^2+...+(k-2)p_{k-1}s^{k-2}\right)\right\}$$

$$-\frac{1}{s}\left\{\mu\left(p_2 s^2+...+p_{k-1}s^{k-1}+p_k s^k\right)+\nu s\left(p_2 s+2p_3 s^2+...+(k-1)p_k s^{k-1}\right)\right\}$$

$$\Rightarrow \lambda s\left[\left\{p(s)-p_{k-1}s^{k-1}-p_k s^k\right\}-\frac{s}{k}\left\{p'(s)-(k-1)p_{k-1}s^{k-2}-kp_k s^{k-1}\right\}\right]-\lambda\left[\left\{p(s)-p_0-p_k s^k\right\}-\frac{s}{k}\left\{p'(s)-kp_k s^{k-1}\right\}\right]$$

$$=\left[\mu\left\{p(s)-p_0-p_k s^k\right\}+\nu s\left\{p'(s)-p_1-kp_k s^{k-1}\right\}-\nu\left\{p(s)-p_0-p_1 s-p_k s^k\right\}\right]$$

$$-\frac{1}{s}\left[\mu\{p(s)-p_0-p_1 s\}+\nu s\{p'(s)-p_1\}-\nu\{p(s)-p_0-p_1 s\}\right]$$

$$\Rightarrow \lambda s\left\{p(s)-p_{k-1}s^{k-1}-p_k s^k\right\}-\frac{\lambda s^2}{k}\left\{p'(s)-(k-1)p_{k-1}s^{k-2}-kp_k s^{k-1}\right\}-\lambda\left\{p(s)-p_0-p_k s^k\right\}+\frac{\lambda s}{k}\left\{p'(s)-kp_k s^{k-1}\right\}$$

$$=\mu\left\{p(s)-p_0-p_k s^k\right\}+\nu s\left\{p'(s)-p_1-kp_k s^{k-1}\right\}-\nu\left\{p(s)-p_0-p_1 s-p_k s^k\right\}-\frac{\mu}{s}\{p(s)-p_0-p_1 s\}$$

$$-\nu\{p'(s)-p_1\}+\frac{\nu}{s}\{p(s)-p_0-p_1 s\}$$

$$\Rightarrow \lambda sp(s)-\lambda sp_{k-1}s^{k-1}-\lambda sp_k s^k-\frac{\lambda s^2}{k}p'(s)+\frac{\lambda s^2(k-1)}{k}p_{k-1}s^{k-2}+\lambda s^2 p_k s^{k-1}-\lambda p(s)+\lambda p_0+\lambda p_k s^k+\frac{\lambda s}{k}p'(s)$$
$$-\lambda sp_k s^{k-1}$$
$$=\mu p(s)-\mu p_0-\mu p_k s^k+\nu sp'(s)-\nu skp_k s^{k-1}-\nu p(s)+\nu p_0+\nu p_1 s+\nu p_k s^k-\frac{\mu}{s}p(s)+\frac{\mu}{s}p_0+\mu p_1-\nu p'(1)+\nu p_1$$
$$+\frac{\nu}{s}p(s)-\frac{\nu}{s}p_0-\nu p_1$$
$$\Rightarrow \frac{\lambda s}{k}(1-s)p'(s)+\nu(1-s)p'(s)=p(s)\left(\lambda-\lambda s+\mu-\frac{\mu}{s}+\frac{\nu}{s}-\nu\right)+\lambda s\left(1-\frac{k-1}{k}\right)\frac{k\{\mu+(k-1)\nu\}}{\lambda}p_k s^{k-1}-\lambda p_0$$
$$-\mu p_0-\mu p_k s^k-\nu kp_k s^k+\nu p_0+\nu p_k s^k+\frac{\mu}{s}p_0+\mu p_1-\frac{\nu}{s}p_0$$
$$\Rightarrow (1-s)p'(s)\left(\frac{\lambda s}{k}+\nu\right)=p(s)\left(\lambda-\lambda s+\mu-\frac{\mu}{s}+\frac{\nu}{s}-\nu\right)+\{\mu+(k-1)\nu\}p_k s^k-\lambda p_0-\mu p_0-\mu p_k s^k-\nu kp_k s^k$$
$$+\nu p_0+\nu p_k s^k+\frac{\mu}{s}p_0+\mu p_1-\frac{\nu}{s}p_0$$
$$\Rightarrow \frac{(1-s)(\lambda s+k\nu)}{k}p'(s)=\frac{(1-s)(\lambda s-\mu+\nu)}{s}p(s)+\frac{(1-s)(\mu-\nu)}{s}p_0$$
$$\Rightarrow p'(s)=\frac{(1-s)(\lambda s-\mu+\nu)k}{s(1-s)(\lambda s+k\nu)}p(s)+\frac{(1-s)(\mu-\nu)k}{s(1-s)(\lambda s+k\nu)}p_0$$
$$=\frac{k}{s(\lambda s+k\nu)}\{(\lambda s-\mu+\nu)p(s)+(\mu-\nu)p_0\}$$

Now

$$\lim_{s\to 1-} p'(s)=\lim_{s\to 1-}\frac{k}{s(\lambda s+k\nu)}\{(\lambda s-\mu+\nu)p(s)+(\mu-\nu)p_0\}$$
$$\Rightarrow p'(1)=\frac{k}{(\lambda+k\nu)}\{(\lambda-\mu+\nu)p(1)+(\mu-\nu)p_0\} \qquad (7.1.1)$$
$$=\frac{k}{(\lambda+k\nu)}\{\lambda-(\mu-\nu)(1-p_0)\}$$

7.2. Derivation Of Q′(1) under R_EOS:

From equation (2.6) we have

$$\lambda\left(1-\frac{n-1}{k}\right)q_{n-1}+\{\mu+(n+1)\nu\}q_{n+1}=\lambda\left(1-\frac{n}{k}\right)q_n+(\mu+n\nu)q_n \qquad :1\le n\le k-1$$

Multiplying both sides of this equation by s^n and summing over n

$$\lambda s\sum_{n=1}^{k-1}\left(1-\frac{n-1}{k}\right)q_{n-1}s^{n-1}-\lambda\sum_{n=1}^{k-1}\left(1-\frac{n}{k}\right)q_n s^n=\sum_{n=1}^{k-1}(\mu+n\nu)q_n s^n-\frac{1}{s}\sum_{n=1}^{k-1}\{\mu+(n+1)\nu\}q_{n+1}s^{n+1}$$

Proceeding similar to the previous subsection 7.1, we obtain

$$Q'(1) = \frac{k}{(\lambda + k\nu)}\{\lambda - (1 - q_0)\mu\} \tag{7.2.1}$$

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