Vol.8.Issue.2.2020 (April-June)

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RESEARCH ARTICLE

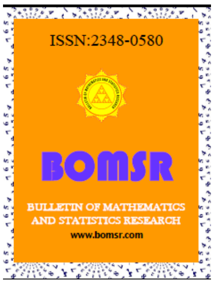
BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

FIXED POINT THEORY IN ORBITALLY COMPLETE PARTIAL METRIC SPACE

KESHAB PRASAD ADHIKARI[1], NARAYAN PRASAD PAHARI[2]
Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal
Email: keshab.kpa@gmail.com[1]; nppahari@gmail.com[2]
DOI: 10.33329/bomsr.8.2.6



ABSTRACT

So far, a bulk number of research works have been done on various types of fixed point theorems for contraction type mappings in usual metric space. The notion of partial metric space was introduced by S.G. Matthews in 1992 as a generalization of the usual metric space in which the distance of a point from itself may not be zero. The study of fixed point theory in orbitally complete partial meric space was initiated in 2011. The aim of this paper is to generalize some of the works by Karapinar studied in [1] and [2] and by Altun, et al studied in [4]. Beside these, we also show the equivalence of the balls for partial metric space defined by Matthews [7] and the other authors.

Keywords: Contraction principle, Partial metric space, Orbitally complete space, Orbitally continuous map, Fixed point.

1. Introduction and Historical Motivation

In 1922, Stefan Banach proved a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been introduced and investigated by several workers, and many more generalizations of this theorem have been established. The study of fixed point of mappings satisfying certain contractive conditions have been at the center of vigorous research activity. Many different improvements and generalizations of Banach's contraction theorem in different metric spaces are made.

The failure of a metric function in computer studies was the primary motivation behind the introductory of the partial metric. In 1994, Matthews [7] introduced the concept of partial metric space as a part of the study of denotational semantics of data flow network and proved the Banach contraction principle in such spaces. Many researchers studied and generalized the fixed point for mappings satisfying contractive conditions in complete partial metric spaces, for instances, we refer a few: Altun , et al[4], Abdeljawad & karapinar[9], Kadelburg, et al[11] and many others.

In 2011, the study of fixed points in orbitally complete partial metric spaces was initiated (see [1]). E. Karapinar[2], H.K. Nashine and E. Karapinar [3] obtained some new results on orbitally complete partial metric spaces. Recently in 2017, Popa, et al proved an unified theorem for mapping with implicit relation in orbitally complete partial metric spaces, (see [10]).

2. Preliminaries

Before proceeding with the work, we recall some of the basic notations and definitions that are used in this paper.

The concept and basic properties of partial metric (p-metric) function was introduced by Matthews [6] in 1992 as follows:

Definition 2.1. Let M be a non-empty set. A function $p: M\times M \to R_+$ is said to be a partial metric on M if for any $x, y, z \in M$, the following conditions hold:

(P_1): $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,

(P_2): $p(x,x) \le p(x,y)$,

(P_3): $p(x,y) = p(y,x)$,

(P_4): $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

The pair (M, p) is called a partial metric space. If $p(x,y) = 0$, then (P_1) and (P_2) implies $x = y$, but the converse does not always hold.

Each partial metric p on M generates a T_0 topology τ_p which has a base the family of open $p-\text{balls}$ $\{B_p(x,\varepsilon): x\in M \text{ and } \varepsilon > 0\}$ where

$$B_p(x,\varepsilon) = \{y \in M : p(x,y) < p(x,x) + \varepsilon\} \text{ for all } x \in M \text{ and } \varepsilon > 0.$$

If p is a partial metric on M , then the function

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

defines a metric on M . Further, a sequence (x_n) converges in (M, d_p) to a point $x \in M$ if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x,x).$$

Example 2.1 (see [3] , [7]). Consider $M = [0,\infty)$ with $p(x,y) = \max\{x,y\}, \forall x, y \in M$. Then (M, p) is a partial metric space. The corresponding metric is

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2\max\{x,y\} - x - y = |x-y|.$$

Example 2.2 (see [3], [7]). Let $M = \{[a,b]: a,b \in R, a \le b\}$ and define

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}.$$

Then, (M, p) is a partial metric space.

There are some generalizations of partial metrics. O'Neill [8] generalized it a bit further by admitting negative distances. The partial metric of O'Neill sense is called dualistic partial metric. Also, Heckmann [5] generalized it by omitting small self – distance axiom $p(x,x) \le p(x,y)$. The partial metric of Heckmann sense is called weak partial metric. The inequality

$$2p(x,y) \ge p(x,x) + p(y,y)$$

is satisfied for all x, y in a weak partial metric space.

Lemma 2.1 (see [7]). Let (M, p) be a partial metric space and (x_n) be a sequence in M convergent to z, where $p(z,z) = 0$. Then, $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in M$.

It is noted that the limit of a sequence in a partial metric space need not be unique.

Example 2.3. If $M = [0,\infty)$ and define $p(x,y) = \max\{x,y\}\ \forall x,y \in M$, then for $(x_n) = \{1\}$, $\lim_{n\to\infty} p(x_n, x) = x$ for $x \ge 1$. Hence the limit of $p(x_n, x)$ depends upon the value of x but not on the sequence (x_n).

Definition 2.2 (see [7]). Let (M, p) be a partial metric space.

a) A sequence (x_n) in M is a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and finite.

b) A partial metric space is said to be complete if every Cauchy sequence converges with respect to τ_p to a point $x \in M$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

c) A mapping $T: M \to M$ is said to be continuous at $x_0 \in M$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x_0,\delta)) \subset B_p(T(x_0),\varepsilon)$.

Lemma 2.2 (see [7]).

a) A sequence (x_n) is Cauchy in a partial metric space (M, p) if and only if (x_n) is Cauchy in the metric space (M, d_p).

b) A partial metric space (M, p) is complete if and only if the metric space (M, d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$,

where x is a limit of (x_n) in (M, d_p).

Definition 2.3(see [2]). Let (M, p) be a partial metric space. A mapping $T: M \to M$ is called orbitally continuous if

$$\lim_{i,j\to\infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i\to\infty} p(T^{n_i}x, z) = p(z,z)$$

implies

$$\lim_{i,j\to\infty} p(TT^{n_i}x, TT^{n_j}x) = \lim_{i\to\infty} p(TT^{n_i}x, Tz) = p(Tz,Tz) \text{ for each } x \in M.$$

Definition 2.4 (see [2]). A partial metric space is called orbitally complete if every Cauchy sequence $(T^n x)$ converges in (M, p), that is

$$\lim_{i,j\to\infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i\to\infty} p(T^{n_i}x, z) = p(z,z).$$

Lemma 2.3 (see [4]). Let (M, p) be a partial metric space, $A \subset M$ and $x_0 \in M$. Define $p(x_0, A) = \inf\{p(x_0, x) : x \in A\}$. Then $a \in \overline{A} \Leftrightarrow p(a, A) = p(a, a)$.

Theorem 2.1 (see [1]). Let T be an orbitally continuous self- map of an orbitally complete partial metric space (M, p). Suppose that T satisfies the inequality

$$\min\{p(Tx,Ty), p(x,Tx), p(y,Ty)\} \le \alpha p(x, y)$$

for some $0 \le \alpha < 1$ and all $x, y \in M$, then the sequence $(T^n x)$ converges to a fixed point of T in M.

Theorem 2.2 (see [1]). Let T be an orbitally continuous self- map of an orbitally complete partial metric space (M, p). Suppose that T satisfies the inequality

$$\frac{\min\{p(Tx,Ty).p(x, y), p(x,Tx).p(y,Ty)\}}{\min\{p(x,Tx), p(y,Ty)\}} \le \alpha p(x, y)$$

for some $0 \le \alpha < 1$ and all $x, y \in M$ such that $p(x,Tx) \ne 0$ and $p(y,Ty) \ne 0$, then the sequence $(T^n x)$ converges to a fixed point of T.

Theorem 2.3 (see [3]). Let T be an orbitally continuous self- map of an orbitally complete partial metric space (M, p). Suppose that T satisfies the inequality

$$p(Tx,Ty) \le \alpha p(x, y) + \beta \frac{p(x,Tx)p(y,Ty)}{1 + p(x, y)}$$

for all $x, y \in M, x \ne y$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$. Then T has a fixed point z in M. Moreover, $p(z,Tz) = p(Tz,Tz) = p(z, z) = 0$

Theorem 2.4 (see [2]). Let T be an orbitally continuous self- map of an orbitally complete partial metric space (M, p). Suppose that T satisfies the inequality

$$\min\{p(x,Tx), p(y,Ty), p(Tx,Ty)\} < p(x, y)$$

for all $x, y \in M, x \ne y$. If the sequence $(T^n x)$ has a cluster point $z \in M$ for some $x \in M$, then z is a fixed point of *T*.

Theorem 2.5 (see [2]). Let T be an orbitally continuous self- map of an orbitally complete partial metric space (M, p). Suppose that T satisfies the inequality

$$\min\{p^2(x,Tx), p^2(y,Ty), p(x, y).p(Tx,Ty)\} \le \alpha p(x,Tx).p(y,Ty)$$

for all $x, y \in M$ and for some $0 \le \alpha < 1$. Then for each $x \in M$, the sequence $(T^n x)$ converges to a fixed point of *T*.

Definition 2.5 (see [10]). Let $\mathcal{F}_{op}$ be the sets of all continuous functions $F(t_1,t_2,t_3,\ldots,t_5): R_+^5 \to R$ satisfying the following conditions:

(F_1): F is not increasing in variable t_5 ,

(F_2): There exists $h \in (0,1)$ such that for all $u \geq 0, v > 0,\ F(u,v,v,u,u+v) \leq 0$ implies $u \leq hv.$

Theorem 2.6 (see [10]). Let T be an orbitally continuous self- map of an orbitally complete partial metric space (M, p). Suppose that

$$F(p(Tx,Ty), p(x,y), p(x,Tx), p(y,Ty), p(x,Ty)+p(y,Tx)) \leq 0$$

for all $x, y \in M, x \neq y$ and $F \in \mathcal{F}_{op}$. Then T has a fixed point z such that

$$p(z,z) = p(Tz,Tz) = p(z,Tz) = 0.$$

Theorem 2.7 (see [4]). Let T be a self-map of a complete partial metric space (M, p). Suppose that T satisfies the inequality

$$p(Tx,Ty) \leq \phi(\max\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]\})$$

for all $x, y \in M$, where $\phi:[0,\infty) \to [0,\infty)$ is continuous, non-decreasing function such that $\phi(t) < t$ for each $t > 0$. Then T has a unique fixed point.

Corollary 2.1 (see [4]). Let T be a self-map of a complete partial metric space (M ,p). Suppose that T satisfies the inequality

$$p(Tx,Ty) \leq \lambda \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]\}$$

for all $x, y \in M$, where $0 \leq \lambda < 1.$ Then T has a unique fixed point.

Theorem 2.8 (see [4]). Let T be a self-map of a complete partial metric space (M, p). Suppose that T satisfies the inequality

$$p(Tx,Ty) \leq ap(x,y) + bp(x,Tx) + cp(y,Ty) + dp(x,Ty) + ep(y,Tx)$$

for all $x, y \in M$, where $a,b,c,d,e \geq 0$ and, if $d \geq e$, then $a+b+c+d+e<1$, if $d < e$, then $a+b+c+d+2e<1$. Then T has a unique fixed point.

Corollary 2.2 (Banach type , see [4]). Let T be a self-map of a complete partial metric space (M, p). Suppose that T satisfies the inequality

$$p(Tx,Ty) \leq ap(x,y)$$

for all $x, y \in M$, where $0 \leq a < 1$. Then T has a unique fixed point.

Corollary 2.3 (Kannan type, see [4]). Let T be a self-map of a complete partial metric space (M, p). Suppose that T satisfies the inequality

$$p(Tx,Ty) \leq bp(x,Tx) + cp(y,Ty)$$

for all $x, y \in M$, where $b, c \geq 0$ and $b+c<1$. Then T has a unique fixed point.

Corollary 2.4 (Riech type, see [4]). Let T be a self-map of a complete partial metric space (M, p). Suppose that T satisfies the inequality

$$p(Tx,Ty) \le ap(x,y)+bp(x,Tx)+cp(y,Ty)$$

for all $x, y \in M$, where $a,b,c \ge 0$ and $a+b+c<1$. Then T has a unique fixed point.

3. Main Results

In this section we shall investigate the results that generalize the Theorem 2.1 studied by Karapinar [2].

Theorem 3.9. Let T be an orbitally continuous self- map of an orbitally complete partial metric space (M, p). Suppose that T satisfies the inequality

$$\min\{ p(Tx,Ty), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]\} \le \lambda p(x,y) \qquad \dots (3.1)$$

for all $x, y \in M$ for some $\lambda \in (0,1)$, then the sequence $(T^n x)$ converges to a fixed point of T for each $x \in M$.

Proof.

Take an arbitrary point $x_0 \in M$. Let us construct a sequence

$$x_{n+1} = Tx_n, \quad n = 0,\ 1,\ 2, \dots .$$

If there exists a positive integer n such that $x_n = x_{n+1}$, then x_n is a fixed point of T. Hence we are done.

So, suppose that $x_n \ne x_{n+1}$ for each $n = 0, 1, 2, , \dots$. Substituting $x = x_n$ and $y = x_{n+1}$ in (3.1), we obtain the inequality

$$\min\{ p(Tx_n,Tx_{n+1}), p(x_n,Tx_n), p(x_{n+1},Tx_{n+1}), \frac{1}{2}[p(x_n,Tx_{n+1})+p(x_{n+1},Tx_n)]\} \le \lambda p(x_n,x_{n+1})$$

$$\Rightarrow \min\{ p(x_{n+1},x_{n+2}), p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}), \frac{1}{2}[p(x_n,x_{n+2})+p(x_{n+1},x_{n+1})]\} \le \lambda p(x_n,x_{n+1})$$

$$\Rightarrow \min\{ p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}), \frac{1}{2}[p(x_n,x_{n+2})+p(x_{n+1},x_{n+1})]\} \le \lambda p(x_n,x_{n+1}) \qquad \dots (3.2)$$

But in view of (P_4),we have

$$p(x_n,x_{n+2}) \le p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})-p(x_{n+1},x_{n+1})$$

$$\Rightarrow p(x_n,x_{n+2})+p(x_{n+1},x_{n+1}) \le p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})$$

Therefore, we obtain from (3.2) that

$$\min\{ p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}), \frac{1}{2}[p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})]\} \le \lambda p(x_n,x_{n+1})$$

$$\Rightarrow \min\{ p(x_n,x_{n+1}), p(x_{n+1},x_{n+2})\} \le \lambda p(x_n,x_{n+1}) \qquad \dots (3.3)$$

Now, if $\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_n, x_{n+1})$ for some *n*, then from (3.3) we have,

$$p(x_n, x_{n+1}) \le \lambda p(x_n, x_{n+1})$$

which is contradiction as $\lambda \in (0,1)$. Thus

$$\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_{n+1}, x_{n+2}) \ \forall\ n.$$

Then we have,

$p(x_{n+1,} x_{n+2}) \le \lambda p(x_n, x_{n+1})$ for every $n = 0, 1, 2, \ldots$. Thus, we get

$$p(x_{n+1}, x_{n+2}) \le \lambda p(x_n, x_{n+1}) \le \lambda^2 p(x_{n-1}, x_n) \le \ldots \le \lambda^{n+1} p(x_0, x_1) \quad \ldots \quad (3.4)$$

Now, we claim that (x_n) is a Cauchy sequence. Without loss of generality assume that $n > m$, then in view of (3.4) and (P_4) we have

$$p(x_n, x_m) \le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \ldots + p(x_{m+1}, x_m) - [p(x_{n-1}, x_{n-1}) + p(x_{n-2}, x_{n-2}) + \ldots + p(x_{m+1}, x_{m+1})]$$

$$\le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \ldots + p(x_{m+1}, x_m)$$

$$\le [\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m] p(x_0, x_1)$$

$$= \lambda^m \frac{1-\lambda^{n-m}}{1-\lambda} p(x_0, x_1)$$

$$\le \frac{\lambda^m}{1-\lambda} p(x_0, x_1) \ \to 0 \ \text{ as } \ m \to \infty$$

This shows that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. That is, ($x_n$) is a Cauchy sequence in (M, p). Since (M, p) is orbitally complete partial metric space, then $\left(T^n x_0\right)$ converges to a limit $z \in X$ such that

$$\lim_{n,m\to\infty} p(T^n x_0, T^m x_0) = \lim_{n\to\infty} p(T^n x_0, z) = p(z, z) = 0.$$

Now, we will show that z is a fixed point of T.

Since T is orbitally continuous, therefore

$$\lim_{n\to\infty} p(T^n x_0, z) = p(z, z) \Rightarrow \lim_{n\to\infty} p(T^{n+1} x_0, Tz) = p(Tz, Tz).$$

On the other hand, from (P_4),we have

$$p(z, Tz) \le p(z, T^{n+1} x_0) + p(T^{n+1} x_0, Tz) - p(T^{n+1} x_0, T^{n+1} x_0)$$

$$\le p(z, T^{n+1} x_0) + p(T^{n+1} x_0, Tz)$$

$$= p(z, x_{n+2}) + p(T^{n+1} x_0, Tz)$$

Using Lemma 2.1 and letting $n \to \infty$, we obtain

$$p(z, Tz) \le p(Tz, Tz).$$

But by (P_2), this is possible only if

$$p(z,Tz) = p(Tz,Tz).$$

From (3.1), we have

$$\min\{p(Tz,Tz), p(z,Tz), p(z,Tz), \frac{1}{2}[p(z,Tz)+p(z,Tz)]\} \le \lambda p(z,z)$$

$$\Rightarrow p(Tz,Tz) = p(z,Tz) \le \lambda p(z,z) = 0.$$

This implies that $p(Tz,Tz) = p(z,Tz) = p(z,z) = 0$, then from ($P_1$) we obtain $z = Tz$.

Hence, z is a fixed point of T, which completes the proof.

Now, we shall investigate the result that generalize Theorem 2 studied by Altun, et al [4].

Theorem 3.10. Let (M, p) be an orbitally complete partial metric space and $T : M \to M$ be an orbitally continuous map such that

$$p(Tx,Ty) = a[p(x,Ty) + p(y,Tx)] + bp(x,y) \qquad ...(3.5)$$

for all $x, y \in M$, where $a,b \ge 0$ and $2a+b<1$. Then T has a unique fixed point.

Proof:

Take an arbitrary point $x_0 \in M$. Let us construct a sequence (x_n) in M such that $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots$ and in general $x_n = T^n x_0$ for $n = 0,1,2,\ldots$

Now, Substituting $x = x_{n-1}$ and $y = x_n$ in (3.5), we obtain the inequality

$$\begin{aligned} p(x_n, x_{n+1}) &= p(T^n x_0, T^{n+1} x_0) \\ &= p(T(T^{n-1}x_0), T(T^n x_0)) \\ &= a[p(T^{n-1}x_0, T^{n+1}x_0) + p(T^n x_0, T^n x_0)] + bp(T^{n-1}x_0, T^n x_0) \\ &= a[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] + bp(x_{n-1}, x_n) \qquad ...(3.6) \end{aligned}$$

But from (P_4) we get

$$p(x_{n-1}, x_{n+1}) \le p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)$$

which implies

$$p(x_{n-1}, x_{n+1}) + p(x_n, x_n) \le p(x_{n-1}, x_n) + p(x_n, x_{n+1}) \qquad ...(3.7)$$

Thus, we get from (3.6) and (3.7) that

$$p(x_n, x_{n+1}) \le a[p(x_{n-1}, x_n) + p(x_n, x_{n+1})] + bp(x_{n-1}, x_n)$$

$$\therefore p(x_n, x_{n+1}) \le \lambda p(x_{n-1}, x_n), \ \forall n, \quad \text{where } \lambda = \frac{a+b}{1-a} < 1$$

Thus, we see that,

$$p(x_n, x_{n+1}) \le \lambda p(x_{n-1}, x_n) \le \lambda^2 p(x_{n-2}, x_{n-1}) \le \ldots \le \lambda^n p(x_0, x_1) \qquad ...(3.8)$$

Now, we claim that (x_n) is a Cauchy sequence. Without loss of generality assume that $n > m$. Then, using (3.8) and (P_4) we have

$$p(x_n, x_m) \le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \ldots + p(x_{m+1}, x_m) - [p(x_{n-1}, x_{n-1}) + p(x_{n-2}, x_{n-2}) + \ldots + p(x_{m+1}, x_{m+1})]$$

$$\le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \ldots + p(x_{m+1}, x_m)$$

$$\le [\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m] p(x_0, x_1)$$

$$\le \frac{\lambda^m}{1-\lambda} p(x_0, x_1) \to 0 \text{ as } m \to \infty$$

This shows that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. That is, ($x_n$) is a Cauchy sequence in (M, p). Since (M, p) is orbitally complete partial metric space, then $(T^n x_0)$ converges to a limit $z \in X$ such that

$$\lim_{n,m\to\infty} p(T^n x_0, T^m x_0) = \lim_{n\to\infty} p(T^n x_0, z) = p(z,z) = 0.$$

Now, we will show that z is a fixed point of T. For this, we prove $p(z,Tz) = 0.$ If possible, let us suppose $p(z,Tz) \ne 0.$ Then from (P_4) and (3.5) we obtain

$$p(z,Tz) \le p(z,T^{n+1}x_0) + p(T^{n+1}x_0, Tz) - p(T^{n+1}x_0, T^{n+1}x_0)$$

$$\le p(z,T^{n+1}x_0) + p(T^{n+1}x_0, Tz)$$

$$\le p(z, x_{n+1}) + a[p(T^n x_0, Tz) + p(z, T^{n+1}x_0)] + bp(T^n x_0, z)$$

$$= p(z, x_{n+1}) + a[p(x_n, Tz) + p(z, x_{n+1})] + p(x_n, z)$$

Letting $n \to \infty$, we have

$$p(z,Tz) \le ap(z,Tz)$$

$$\Rightarrow (1-a)p(z,Tz) \le 0$$

which is a contradiction. Thus, $p(z,Tz) = 0$ and so $Tz = z$. Hence z is a fixed point of T.

Uniqueness:

Let y be another fixed point of T. Then $Ty = y$ and $p(y,y) = 0.$ Then,

$$p(z,y) = p(Tz,Ty) = a[p(z,Ty) + p(y,Tz)] + bp(z,y)$$

$$= a[p(z,y) + p(y,z)] + bp(z,y)$$

$$= (2a+b)p(z,y)$$

$\therefore (1-2a-b)p(z,y) \le 0.$ This implies $p(z,y) = 0.$

Thus, we see that,

$$p(z,z) = p(y,y) = p(z,y) = 0$$

Then from (P_1), we get $z = y$ .This completes the proof.

Now, we show the equivalence of the definition of open ball for a partial metric space given by Matthews and the other authors by proving the Theorem 3.2[7].

Theorem 3.11 (see [7]). For each partial metric p, open ball $B_\varepsilon^p(a)$, and $x \in B_\varepsilon^p(a)$, there exists $\delta > 0$ such that $x \in B_\delta^p(x) \subseteq B_\varepsilon^p(a)$.

Proof:

Suppose $x \in B_\varepsilon^p(a)$, then by definition of ball, $p(x,a) < p(x,x) + \varepsilon$.

Let $\delta = \varepsilon - p(x,a) + p(x,x)$, then clearly $\delta > 0$.

Also, $p(x,x) < p(x,x) + \delta$ and so $x \in B_\delta^p(x)$.

Again, suppose that $y \in B_\delta^p(x)$, so that $p(y,x) < p(y,y) + \delta$

$\therefore p(y,x) < p(y,y) + \varepsilon - p(x,a) + p(x,x).$

or, $p(y,x) + p(x,a) - p(x,x) < p(y,y) + \varepsilon$

or, $p(y,a) < p(y,y) + \varepsilon$.

$\therefore y \in B_\varepsilon^p(a)$

Thus, $B_\delta^p(x) \subseteq B_\varepsilon^p(a)$, which completes the proof.

Acknowledgement

The first author gratefully acknowledges University Grant Commission (UGC) Nepal, for financial support during the preparation of this paper. Also the authors would like to thank for the unknown referee for his/her comments that helped us to improve this paper.

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