Vol.8.Issue.3.2020 (July -Sept)
©KY PUBLICATIONS

http://www.bomsr.com
Email:editorbomsr@gmail.com

RESEARCH ARTICLE

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

GENERALIZED DERIVATIONS IN RITT ALGEBRA

Dr. K.L. Kaushik
Associate Professor in Mathematics, Aggarwal College, Ballabgarh
Faridabad, India
DOI: 10.33329/bomsr.8.3.27

ABSTRACT

We prove that $f(x^n) = f(e)x^n + D(x^n)$ $\forall f \in GD(U)$, the set of all generalized derivations on an arbitrary non-associative algebra U over a fixed field *K.* Finally we prove a theorem in Ritt algebra from which Kaplansky's [3] two lemmas page 12 can be derived immediately.

Keywords Non-associative algebra, Generalized derivations, Radical ideal, Generalized Ritt algebra,

Introduction

We have used Havala [1] definition

"Let R be a ring. The additive map $f : R \to R$ will be called a generalized derivation if $\exists$ a derivation d of R s.t. $f(xy) = f(x)y + xd(y)$ $\forall x, y \in R$"

Let U be an arbitrary non-associative algebra over a field K and GD(U) be the set of all generalized derivations on U then in section 1.

We proved that if $ab \in I$ then $af(b) \in I$ and $f(a)b \in I$, where I be any radical generalized differential ideal.

Finally we prove in Theorem 2.2 "If I is a generalized differential Ideal in a generalized Ritt algebra and a be any element with $a^n \in I$. Then $(f(a))^{2n-1} \in I$. From this theorem we immediately derive Kaplansky's [3] two lemmas page 12 as Corollaries".

1 Generalized Differential Algbera

Let U be an arbitrary non-associative algebra over a field K. A generalized derivation f in U is a linear mapping of U onto U satisfying

$f(xy) = f(x)y + xD(y)\ \forall x,y \in U$

where D is the derivation in U.

Let GD(U) be the set of all generalized derivation of U.

1. In this section we prove a Lemma 1.4 which we shall use in next section 2.

Definitions

1.1 (Generalized Differential Ring) A commutative ring with unit element together with a generalized derivation is called Generalized differential ring.

1.2 (Generalized differential Ideal) Let I be an Ideal in a generalized differential ring A. Then I is said to be generalized differential Ideal if for $a \in I \Rightarrow f(a) \in I$ or $f(I) \subset I$, where f is the generalized derivation on A.

1.3 Radical Ideal An Ideal I in A is said to be Radical Ideal if $a^n \in I \Rightarrow a \in I$.

Now we prove

1.4 Lemma Let I be any radical generalized differential Ideal and if $ab \in I$ then $af(b) \in I$ and $f(a)b \in I$.

Proof We have $f(ab) = f(a)b + aD(b) \in I$

Also $ab \in I \Rightarrow (aD(b))^2 \in I$ and hence $aD(b) \in I$ Finally we get $f(a)b \in I$. Similarly $af(b) \in I$.

Corollary 1.4.1. Now we immediately get $aD(b) \in I$ and $D(a)b \in I$.

Lemma 1.5. Let R be a non commutative ring and $f : R \to C$ be a generalized derivation then $f = 0$.

Proof Since $f : R \to C$ is a generalized derivation, Then

$$f(xy) = f(x)y + xD(y)$$

$$\Rightarrow f(xy)y = f(x)yy + xD(y)y$$

$$\Rightarrow yf(xy) = yf(x)y + yxD(y)$$

Now $f(xy)$ commutes with y

$$\Rightarrow f(xy)y = yf(xy)$$

$$\Rightarrow f(x)yy + xD(y)y = yf(x)y + yxD(y)$$

Also $f(x)y$ commutes with y

$$\Rightarrow f(x)y + xD(y)y = f(x)yy + yxD(y)$$

$$\Rightarrow xD(y)y + (-yxD(y)) = 0$$

$$\Rightarrow D(y) = 0\ (\because R \text{ is non commutative})$$

$$\Rightarrow D = 0$$

$$\Rightarrow f(xy) = f(x)y\ \forall x,y \in R$$

$$\Rightarrow f = 0\ (\because f : R \to C \text{ and} R \text{ is non commutative})$$

2. Generalized RITT algebra Definition

A generalized differential ring containing the field of rationals is called generalized Ritt algebra.

Now we prove a proposition 2.1, though simple but in our opinion, would be very useful in getting deep results in generalized derivation.

We know that for every integer $n \geq 1$ and for every x in a ring R

$$D(x^n) = nx^{n-1}D(x)$$

Now we generalizes it in the following proposition.

Proposition 2.1 If GD(U) has unit element e then $f(x^n) = f(e)x^n + D(x^n)\ \forall f \in$ GD(U).

Proof Note that $D(e) = 0$ as $D(e) = D(e^2) = 2eD(e) = 2D(e)$. Also $(f-D)(x) = f(e)x$

$\forall x \in$ GD(U). Now

$$\begin{aligned} (f-D)(x) &= (f-D)(ex) \\ &= f(ex) - D(ex) \\ &= f(e)x + eD(x) - D(x) \\ (f-D)(x) &= f(e)x \end{aligned}$$

Now

$$\begin{aligned} f(x^2) &= f(xx) \\ &= f(x)x + xD(x) \\ &= f(x)x - D(x)x + D(x)x + xD(x) \\ &= ((f-D)(x))x + D(x^2) \\ &= (f(e)(x))x + D(x^2) \\ f(x^2) &= f(e)x^2 + D(x^2) \end{aligned}$$

Again

$$\begin{aligned} f(x^3) &= f(x^2x) \\ &= f(x^2)x + x^2D(x) \\ &= (f(e)x^2 + D(x^2))x + x^2D(x) \\ &= f(e)x^3 + D(x^2)x + x^2D(x) \\ \Rightarrow f(x^3) &= f(e)x^3 + D(x^3) \end{aligned}$$

Then by induction on n, we get

$$f(x^n) = f(e)x^n + D(x^n)$$

Corollary 2.1.1. If $f \in$ GD(U) Then f(1) $\neq 0$, A = any ring.

Proof Now $1 \in A$

$$\Rightarrow aa^{-1} = 1$$

$$\Rightarrow f(aa^{-1}) = f(1)$$

$$\Rightarrow f(a)a^{-1} + aD(a^{-1}) = f(1)$$

(1) $\Rightarrow f(a)a^{-1} - a^{-1}D(a) = f(1)$

Also

$D(aa^{-1}) = D(1)$

$\Rightarrow D(a)a^{-1} - a^{-1}D(a) = 0$

$\Rightarrow D(a)a^{-1} = a^{-1}D(a)$

Putting in (1) we get

$(f - D)aa^{-1} = f(1)$

We know that $(f - D)x = f(e)x$

$\Rightarrow (f - D)a = f(1)a$

$f(1)aa^{-1} = f(1)$

$\Rightarrow f(1)(aa^{-1} - 1) = 0$

$\Rightarrow f(1) \neq 0 \qquad (\because aa^{-1} \neq 0)$

Hence proved.

We use it to prove the nice Theorem 2.2 which generalizes Kapalansky's [3] results page 12 (let I be a differential Ideal in a Ritt algebra and let a be an element with $a^n \in I$ then $(a')^{2n-1} \in I$)

Theorem 2.2 Let I be the generalized differential Ideal in a Generalized Ritt algebra if a be any element with $a^n \in I$ then $(f(a))^{2n-1} \in I$.

Proof

$$f(a^n) = f(e)a^n + D(a^n)$$

$$= f(e)a^n + na^{n-1}D(a)$$

since $a^n \in I$

$$\Rightarrow f(a^n) = f(e)a^n + na^{n-1}D(a) \in I$$

$$\Rightarrow f(e)a^n \in I$$

$$\Rightarrow f(e)aa^{n-1} \in I$$

$$\Rightarrow (f - D)(a)a^{n-1} \in I$$

$$\Rightarrow f(a)a^{n-1} \in I$$

By induction on k, we get

$$\left(f(a)\right)^{2k-1} a^{n-k} \in I$$

$$\Rightarrow f\left((f(a))^{2k-1}a^{n-k}\right) \in I$$

$$\Rightarrow f\left(f(a))^{2k-1}a^{n-k} + (f(a))^{2k-1}D(a^{n-k})\right) \in I$$

Multiplying by *a,* we get

$$\begin{aligned}
&\Rightarrow\; f\,(f(a))^{2k-1}\,a^{n-k+1} \in I\\
&\Rightarrow\; \left[f(e)(f(a))^{2k-1} + D((f(a)^{2k-1})\right]a^{n-k+1} \in I\\
&\Rightarrow\; f(e)(f(a))^{2k-1}a^{n-k+1} \in I\\
&\Rightarrow\; f(e)a(f(a))^{2k-1}a^{n-k} \in I\\
&\Rightarrow\; (f-D)(a)(f(a))^{2k-1}a^{n-k} \in I\\
&\Rightarrow\; (f(a))^{2k}a^{n-k} \in I
\end{aligned}$$

Multiplying by $f(e)$

$\Rightarrow f(e)a(f(a))^{2k}a^{n-k+1} \in I$

$\Rightarrow (f-D)(a)(f(a))^{2k}a^{n-k-1} \in I$

$\Rightarrow (f(a))^{2k+1}\,a^{n-k+1} - D(a)(f(a))^{2k}\,a^{n-k+1} \in I$ (֍)

Now

$$\begin{aligned}
f(a^n) &= f(e)a^n + D(a^n)\\
\Rightarrow f(a) &= f(e)a + D(a)\\
\Rightarrow f(a) - f(e)a &= D(a)
\end{aligned}$$

Putting this value of $D(a)$ in 2nd term of (֍), we get

$f(a)^{2k+1}a^{n-k-1} \in I$

$\Rightarrow f(a)^{2k-1}\,a^{n-k} \in I$

Putting $k = n$

$\Rightarrow \boxed{(f(a))}^{2n-1} \boxed{\in I}$ Hence proved.

Corollary 2.3 In a generalized Ritt algebra, radical Ideal of a generalized differential Ideal is a generalized differential Ideal.

Corollary 2.4 Kaplansky's [3] results page 12 is immediately follows by replacing *f* by *D.*

Conclusion

In this paper, we proved the most important result "If I is a generalized differential Ideal in a generalized Ritt algebra and a be any element with $a^n \in I$. Then $(f(a))^{2n-1} \in I$." from which Kaplansky's [3] two lemmas page 12 comes out as Corollaries".

References

[1] Havala, B. *generalized Derivations in rings,* Communication in Algbera 26 (4), 11471166 (1998).

[2] Jacobson, N. *Lie Algebras,* Interscience Publishers, New York (1992).

[3] [3] Kaplansky, I. *An introduction to differential algebra,* Hermann, Paris (1957)

[4] [4] Jacobson Nathan, Lie algebras. Interscience Publishers,New York (1979).