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LINDLEY INVERSE WEIBULL DISTRIBUTION: THEORY AND APPLICATIONS

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ABSTRACT

In this study, we have introduced a new three-parameter Lindley inverse Weibull distribution using Lindley family of distributions. The mathematical and statistical properties of the new distribution such as probability density function, cumulative distribution function, survival function, hazard rate function, quantile, the measure of skewness, and kurtosis are illustrated. The parameters of the new distribution are estimated using maximum likelihood estimation (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods. By using the maximum likelihood method, we have constructed the asymptotic confidence interval for the model parameters. All the computations are performed in R software. A real data set is analyzed for illustration and application.The potentiality of the proposed distribution is evaluated by goodness of fit in contrast with some other existing distributions using a real life data.

Keywords: Lindley-G-family, Inverse Weibull, Maximum likelihood estimation, Hazard function.

1. INTRODUCTION

In the probability distribution and applied statistics literature, we can found so many continuous univariate distributions. Most of the classical distributions have been widely used over the past decades for modeling data in several areas such as actuarial, environmental and medical sciences, life sciences, demography, economics, finance, insurance, etc. However, in many applied areas like survival analysis, finance, and insurance, there is a clear need for modified forms of more flexible distributions to model real data that can address a high degree of skewness and kurtosis.

The Weibull distribution has been extensively used in survival analysis and in applications of several different fields. For a detailed study, the learners can go through Lai et al. (2003) and Nadarajah (2009). Even though its widespread use, it has some drawbacks that is the limited shape of its hazard rate function (HRF) that can only be monotonically increasing or decreasing or constant. Usually, practical problems require a wider range of possibilities in the medium risk, for example, when the lifetime data produces a bathtub shaped hazard function such as human mortality and machine's component life cycles.

The inverse Weibull distribution has been used to model, many real-life applications for example degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as the breakdown of insulating fluid (Khan et al., 2008, Pararai et al., 2014). Akgül et al. (2016) has introduced the inverse Weibull distribution for modeling the wind speed data. Kumar & Kumar (2019) has presented the estimation of the parameters and reliability characteristics in inverse Weibull distribution based on the random censoring model.

Hence the researchers in the last few years developed various extensions and modified forms of the Weibull distribution to obtain more flexible distributions. Some generalizations of the Weibull (W) distribution are available in the statistical literature such as the exponentiated W (Mudholkar et al. 1996), additive W (Xie and Lai 1995), Marshall–Olkin extended W (Ghitany et al. 2005), modified W (Lai et al. 2003, Sarhan and Zaindin 2009), beta-W (Lee et al. 2007), beta modified W (Silva et al. 2010), transmuted W (Aryal and Tsokos 2011), Kumaraswamy inverse W (Shahbaz et al. 2012), exponentiated generalized W (Cordeiro et al. 2013), beta inverse W (Hanook et al. 2013), transmuted complementary W geometric (Afify et al. 2014), transmuted exponentiated generalized W (Yousof et al. 2015), Marshall–Olkin additive W (Afify et al. 2018), Kumaraswamy transmuted exponentiated additive W (Nofal et al. 2016), Topp-Leone generated W (Aryal et al. 2017) and Kumaraswamy complementary W geometric (Afify et al. 2017) distributions.

Recently,Okasha et al. (2017) has introduced the extended inverse Weibull distribution with reliability application Cordeiro et al. (2018) has introduced the Lindley Weibull distribution which accommodates unimodal and bathtub, and a broad variety of monotone failure rates, Basheer (2019) also introduced the alpha power inverse Weibull distribution with reliability application and Abd EL-Baset& Ghazal (2020) has presented the exponentiated additive Weibull distribution.

The one parameter Lindley distribution was developed by (Lindley, 1958) in the context of Bayesian statistics, as a counterexample to fiducial statistics. In recent years, many studies have been focused to obtain various modified forms of the baseline distribution using Lindley family presented by Zografas and Balakrishnan (2009) with more flexible density and hazard rate functions. A detailed study on the Lindley distribution was done by (Ghitany et al., 2008).

A random variable T follows Lindley distribution with parameter λ and its probability density function (PDF) is given by

$$f(t)=\frac{\lambda^2}{\lambda+1}(1+t)e^{-\lambda t};\ \ \mathrm{t}>0,\lambda>0 \tag{1.1}$$

And its cumulative density function (CDF) is

$$F(t)=1-\frac{1+\lambda+\lambda t}{1+\lambda}e^{-\lambda t};\quad t>0,\lambda>0 \tag{1.2}$$

Cakmakyapan and Ozel (2016) have introduced a new class of distributions to generate new distribution based on Lindley generator (Lindley-G) having additional shape parameter θ. The CDF and PDF of Li-G are respectively,

$$F_{L-G}\left(x;\theta,\psi\right)=1-\left[\bar{G}\left(x;\psi\right)\right]^{\theta}\left[1-\frac{\theta}{\theta+1}\ln\bar{G}\left(x;\psi\right)\right];\quad x>0,\theta>0 \tag{1.3}$$

and
$$f_{L-G}\left(x;\theta,\psi\right)=\frac{\theta^{2}}{\theta+1}g\left(x;\psi\right)\left[\bar{G}\left(x;\psi\right)\right]^{\theta-1}\left[1-\ln\bar{G}\left(x;\psi\right)\right];\quad x>0,\theta>0 \tag{1.4}$$

where $g\left(x;\psi\right)=\dfrac{dG\left(x;\psi\right)}{dx}$, $\bar{G}\left(x;\psi\right)=1-G\left(x;\psi\right)$ and ψ is the parameter space of baseline distribution.

The main objective of this work is to develop a more flexible model by adding just one extra parameter to the inverse Weibull distribution to achieve a better fit to real data. We explore the properties of the LIW distribution and its applicability. The rest of the article is organized as follows. In Section 2, the proposed Lindley inverse Weibull (LIW) distribution is derived and we explore some mathematical and statistical properties of the LIW distribution such as a reliability function, hazard rate function, quantile function, and skewness and kurtosis. In Section3, we have presented some well-known estimation methods namely maximum likelihood estimation (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods. Further we also construct the asymptotic confidence interval for MLEs. In section 4 the application of the proposed model is explored by taking a real data set used by earlier researchers. Concluding remarks are presented in Section 5.

2. The Lindley inverse Weibull (LIW) distribution

In the literature of probability models and applied statistics, the inverse Weibull distribution has established some attention. Keller et al. (1982) study the shapes of the density and failure rate functions for the basic inverse model The inverse Weibull distribution with parameters α (scale parameter) and β (shape parameter) with cumulative distribution function (CDF) and the probability density function (PDF) of a random variable X are respectively given by

$$G\left(x;\alpha,\beta\right)=\exp\left(-\alpha x^{-\beta}\right);\ \ x\geq 0,\ \ \alpha>0,\ \ \beta>0 \tag{2.1}$$

and

$$g\left(x\right)=\alpha\beta x^{-(\beta+1)}\exp\left(-\alpha x^{-\beta}\right);\ \ x\geq 0,\ \ \alpha>0,\ \ \beta>0 \tag{2.2}$$

Using (2.1) and (2.2) in (1.3) and (1.4) we get the CDF and PDF of Lindley inverse Weibull (LIW) distribution as,

$$F(x;\alpha,\beta,\theta)=1-\left(1-e^{-\alpha x^{-\beta}}\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\};\ \ x>0 \tag{2.3}$$

and

$$f(x;\alpha,\beta,\theta)=\alpha\beta\left(\frac{\theta^{2}}{\theta+1}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left(1-e^{-\alpha x^{-\beta}}\right)^{\theta-1}\left\{1-\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\};\ \ x>0 \tag{2.4}$$

respectively. Where α = scale parameter, β and θ are shape parameters of the LIW distribution.

The reliability function of LIW (α, β, θ) is

$$R(x;\alpha,\beta,\theta)=1-F\left(x\right)=\left(1-e^{-\alpha x^{-\beta}}\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\};\quad x>0 \qquad (2.5)$$

1.1 Hazard rate function

The hazard rate function is

$$h(x;\alpha,\beta,\theta)=\frac{f(x)}{1-F\left(x\right)}=\frac{\alpha\beta\left(\frac{\theta^{2}}{\theta+1}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left\{1-\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\}}{\left(1-e^{-\alpha x^{-\beta}}\right)\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\}};\quad x>0 \qquad (2.6)$$

1.2 Quantile function

The quantile function of LIW (α, β, θ) can be expressed as,

$$\theta\ln\left(1-e^{-\alpha x^{-\beta}}\right)+\ln\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\}-\ln\left(1-p\right)=0;\quad 0<p<1 \qquad (2.7)$$

1.3 Random Deviate Generation

The random deviate can be generated from LIW (α, β, θ) by

$$\theta\ln\left(1-e^{-\alpha x^{-\beta}}\right)+\ln\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\}-\ln\left(1-v\right)=0\ ;\quad 0<v<1 \qquad (2.8)$$

Solving (2.8) for x we get the expression for the random deviate generation, where *v* has the *U* (0, 1) distribution. For model choice based on information criterion, the values of AIC, BIC, CAIC and HQIC can be used.

1.4 Skewness and Kurtosis of LIW distribution

In descriptive statistics, the measures of skewness and kurtosis play a significant role in data analysis. The coefficient of Bowley'sskewness measure based on quartiles is given by

$$S_{BL}=\frac{Q\left(3/4\right)+Q\left(1/4\right)-2Q\left(1/2\right)}{Q\left(3/4\right)-Q\left(1/4\right)}, \qquad (2.9)$$

and the coefficient of Moor's kurtosis measures based on octiles Moors (1988) is given by

$$K_{MOORS}=\frac{Q\left(0.875\right)-Q\left(0.625\right)+Q\left(0.375\right)-Q\left(0.125\right)}{Q\left(0.75\right)-Q\left(0.25\right)}, \qquad (2.10)$$

Plots of probability density function and hazard rate function of LIW(α, β, θ) with different values of parameters are shown in Figure 1.

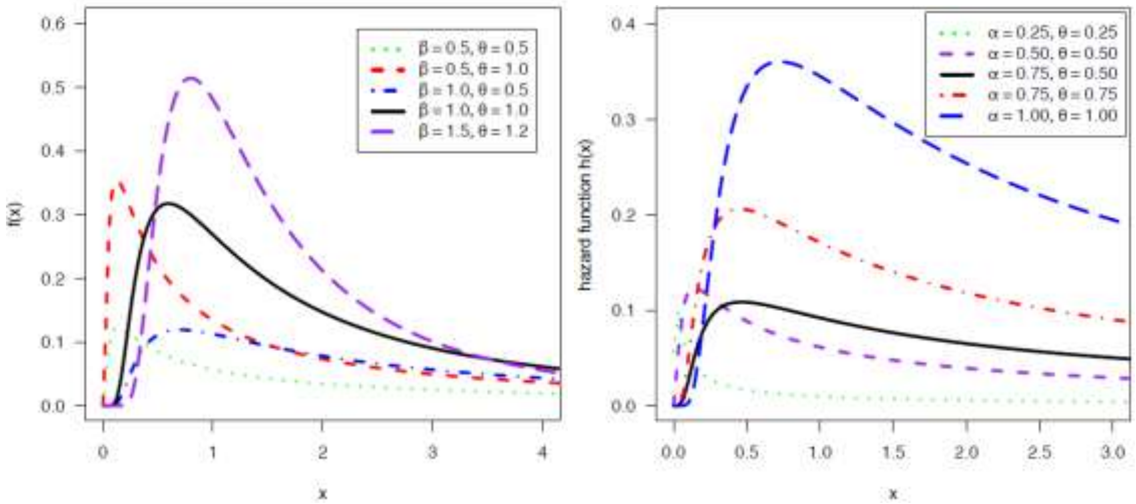

Figure 1. Plots of PDF (left panel) and hazard function (right panel) for different values of α, β and θ.

3. METHODS OF ESTIMATION

The objective of estimation is to approximate the value of a model parameter based on sample information. The estimation theory deals with the basic problem of inferring some relevant features of a random experiment based on the observation of the experiment outcomes. There are so many methods for estimating unknown parameters of the model. We have considered four types of estimation methods such as the maximum likelihood (MLE), ordinary least squares (LSE), and the Cramer-von Mises (CVM) method.

3.1. Maximum Likelihood Estimation (MLE)

In this section, we have illustrated the maximum likelihood estimators (MLE's) of the LIW (α, β, θ) distribution. Let $\underset{\sim}{x} = (x_1,\ldots..,x_n)$ be the observed values of size 'n' from LIW(α, λ, θ), then the likelihood function for the parameter vector $\Theta=\left(\alpha,\beta,\theta\right)^T$ can be written as,

$$L\left(\Theta\right)=\alpha\beta\left(\frac{\theta^2}{\theta+1}\right)\prod_{i=1}^{n}x_i^{-(\beta+1)}e^{-\alpha x_i^{-\beta}}\left(1-e^{-\alpha x_i^{-\beta}}\right)^{\theta-1}\left\{1-\ln\left(1-e^{-\alpha x_i^{-\beta}}\right)\right\}$$

It is easy to deals with log-likelihood function as,

$$\ln L\left(\Theta\right)=2n\ln\theta-n\ln(1+\theta)+n\ln\alpha+n\ln\beta-(\beta+1)\sum_{i=1}^{n}\ln x_i-\alpha\sum_{i=1}^{n}x_i^{-\beta}+(\theta-1)\sum_{i=1}^{n}\ln(1-e^{-\alpha x_i^{-\beta}}) \quad (3.1)$$

The elements of the score function $Z\left(\Theta\right)=\left(Z_\alpha,Z_\beta,Z_\theta\right)$ are obtained as

$$Z_\alpha=\frac{n}{\alpha}-\sum_{i=1}^{n}x_i^{-\beta}+(\theta-1)\sum_{i=1}^{n}e^{-\alpha x_i^{-\beta}}x_i^{-\beta}+\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}}{A(x_i)(1-\ln A(x_i))}$$

$$\begin{aligned}Z_\beta=\frac{n}{\beta}-\sum_{i=1}^{n}\ln x_i+\alpha\sum_{i=1}^{n}x_i^{-\beta}\ln x_i-\alpha(\theta-1)\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}\ln x_i}{A(x_i)}\\-\alpha\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{-\beta}}x_i^{-\beta}\ln x_i}{A(x_i)(1-\ln A(x_i))}\end{aligned} \quad (3.2)$$

$$Z_\theta = \frac{2n}{\theta} - \frac{n}{(1+\theta)} + \sum_{i=1}^{n} \ln A(x_i)$$

where $A(x_i) = 1 - e^{-\alpha x_i^{-\beta}}$

Equating Z_α, Z_β *and* Z_θ to zero and solving these non-linear equations simultaneously gives the MLE $\hat{\Theta} = \left(\hat{\alpha}, \hat{\beta}, \hat{\theta}\right)$ of $\Theta = \left(\alpha, \beta, \theta\right)^T$. These equations cannot be solved analytically and by using the computer software R, Mathematica, Matlab, or any other programs and Newton-Raphson's iteration method, one can solve these equations.

Let us denote the parameter vector by $\Theta = \left(\alpha, \beta, \theta\right)^T$ and the corresponding MLE of Θ as $\hat{\Theta} = \left(\hat{\alpha}, \hat{\beta}, \hat{\theta}\right)$, then the asymptotic normality results in, $\left(\hat{\Theta} - \Theta\right) \rightarrow N_3\left[0, \left(I\left(\Theta\right)\right)^{-1}\right]$ where $I\left(\Theta\right)$ is the Fisher's information matrix given by,

$$I\left(\Theta\right) = -\begin{pmatrix} E\left(Z_{\alpha\alpha}\right) & E\left(Z_{\alpha\beta}\right) & E\left(Z_{\alpha\theta}\right) \\ E\left(Z_{\alpha\beta}\right) & E\left(Z_{\beta\beta}\right) & E\left(Z_{\beta\theta}\right) \\ E\left(Z_{\alpha\theta}\right) & E\left(Z_{\beta\theta}\right) & E\left(Z_{\theta\theta}\right) \end{pmatrix}$$

Further differentiating (3.2) we get,

$$Z_{\alpha\alpha} = -\frac{n}{\alpha^2} + (\theta-1)\sum_{i=1}^{n} x_i^{-2\beta} e^{-\alpha x_i^{-\beta}} + \sum_{i=1}^{n} \frac{e^{-\alpha x_i^{-\beta}} x_i^{-2\beta}}{A(x_i)(1-\ln A(x_i))}$$
$$-\sum_{i=1}^{n} \frac{e^{-2\alpha x_i^{-\beta}} x_i^{-2\beta}}{\left[A(x_i)\right]^2 (1-\ln A(x_i))} - \sum_{i=1}^{n} \frac{e^{-2\alpha x_i^{-\beta}} x_i^{-2\beta}}{\left[A(x_i)\right]^2 (1-\ln A(x_i))^3}$$

$$Z_{\beta\beta} = -\frac{n}{\beta^2} - \alpha\sum_{i=1}^{n} (\ln x_i)^2 x_i^{-\beta} - \alpha(\theta-1)\sum_{i=1}^{n} \frac{(\ln x_i)^2 x_i^{-\beta} e^{-\alpha x_i^{-\beta}} (\alpha x_i^{-\beta} + e^{-\alpha x_i^{-\beta}} - 1)}{\left[A(x_i)\right]^2}$$
$$-\alpha\left(\alpha+1\right)\sum_{i=1}^{n} \frac{(\ln x_i)^2 x_i^{-2\beta} e^{-2\alpha x_i^{-\beta}} (\alpha x_i^{-\beta} + e^{-\alpha x_i^{-\beta}} - 1)}{\left[A(x_i)\right]^2 (1-\ln A(x_i))^2}$$

$$Z_{\theta\theta} = -\frac{2n}{\theta^2} + \frac{n}{(1+\theta)^2}$$

$$Z_{\alpha\beta} = x_i^{-\beta}\sum_{i=1}^{n}\left(\ln x_i\right) + \sum_{i=1}^{n}\left\{1 + \left(1-\ln A(x_i)\right)^{-1} (A(x_i))^{-1}\right\}\left\{x_i^{-\beta} e^{-\lambda x_i^{-\beta}} (\lambda x_i^{-\beta} - 1)\left(\ln x_i\right)\right\}$$
$$+\lambda\sum_{i=1}^{n} e^{-\lambda x_i^{-\beta}} x_i^{-\beta}\left(1-\ln A(x_i)\right)^{-1} (A(x_i))^{-2}\left\{x_i^{-\beta}\left(\ln x_i\right) - \frac{e^{-\lambda x_i^{-\beta}}}{\left(1-\ln(A(x_i))\right)}\right\}$$

$$Z_{\alpha\theta} = -\sum_{i=1}^{n} \frac{x_i^{-\beta+1} e^{-\alpha x_i^{-\beta}}}{(1-e^{-\alpha x_i^{-\beta}})}$$

$$Z_{\beta\theta}=\lambda\sum_{i=1}^{n}\frac{x_i^{-\beta}\,e^{-\alpha x_i^{-\beta}}\left(\ln x_i\right)}{A(x_i)}$$

The observed fisher information matrix $O\left(\hat{\Theta}\right)$ as an estimate of the information matrix $I\left(\Theta\right)$ given by

$$O\left(\hat{\Theta}\right)=-\begin{pmatrix} Z_{\alpha\alpha} & Z_{\alpha\beta} & Z_{\alpha\theta} \\ Z_{\alpha\beta} & Z_{\beta\beta} & Z_{\beta\theta} \\ Z_{\alpha\theta} & Z_{\beta\theta} & Z_{\theta\theta} \end{pmatrix}_{\left|\left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)\right.} = -H\left(\Theta\right)_{\left|\left(\Theta=\hat{\Theta}\right)\right.}$$

where H is the Hessian matrix.

The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H\left(\Theta\right)_{\left|\left(\Theta=\hat{\Theta}\right)\right.}\right]^{-1}=\begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{cov}(\hat{\alpha},\hat{\theta}) \\ \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\theta},\hat{\beta}) \\ \operatorname{cov}(\hat{\alpha},\hat{\theta}) & \operatorname{cov}(\hat{\theta},\hat{\beta}) & \operatorname{var}(\hat{\theta}) \end{pmatrix}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β, and θ can be constructed as,

$\hat{\alpha}\pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\alpha})}$, $\hat{\beta}\pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\beta})}$ and $\hat{\theta}\pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\theta})}$, where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

3.2. Method of Least-Square Estimation (LSE)

The ordinary least square estimators and weighted least square estimators are introduced by (Swain et al., 1988) to estimate the parameters of Beta distributions. The least-square estimators of the unknown parameters for α, β, and θ of LIW distribution can be obtained by minimizing

$$L\left(X;\alpha,\beta,\theta\right)=\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]^2 \tag{3.3.1}$$

with respect to unknown parameters α, β, and θ.

Let $F(X_{(i)})$ denotes the CDF of the ordered random variables $X_{(1)} < X_{(2)} < \ldots\ldots < X_{(n)}$, where $\left\{X_1, X_2, \ldots\ldots, X_n\right\}$ is a random sample of size n from a CDF (2.3). Therefore, the least square estimators of α, β, and θ say $\hat{\alpha},\hat{\beta}$ *and* $\hat{\theta}$ respectively, can be obtained by minimizing

$$L\left(X;\alpha,\beta,\theta\right)=\sum_{i=1}^{n}\left[1-\left(1-e^{-\alpha x^{-\beta}}\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\}-\frac{i}{n+1}\right]^2 \tag{3.3.2}$$

with respect to α, β, and θ.

To obtain the least square estimates of α, β, and θ, we have to solve the following two nonlinear equations simultaneously by equating to zero,

$$\frac{\partial L}{\partial \alpha}=-2\sum_{i=1}^{n}\left[1-\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(V(x_i)\right)\right\}-\frac{i}{n+1}\right]\left(1-e^{\alpha x_i^{-\beta}}\right)^{-1}$$

$$\left[\left\{\theta x_i^{-\beta}\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln V(x_i)\right\}\right\}+\left(V(x_i)\right)^{\theta}\left(\frac{\theta}{\theta+1}\right)x_i^{-\beta}\right]$$

$$\frac{\partial L}{\partial \beta}=-2\sum_{i=1}^{n}\left[1-\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(V(x_i)\right)\right\}-\frac{i}{n+1}\right]$$

$$\left[\left\{\alpha\theta x_i^{-\beta}e^{-\alpha x_i^{-\beta}}\left(V(x_i)\right)^{\theta-1}\ln(x)\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln V(x_i)\right\}\right\}+\left(V(x_i)\right)^{\theta}\alpha x_i^{-\beta}\left(\frac{\theta}{\theta+1}\right)x_i^{-\beta}\ln(x)\right]$$

$$\frac{\partial L}{\partial \theta}=-2\sum_{i=1}^{n}\left[1-\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(V(x_i)\right)\right\}-\frac{i}{n+1}\right]$$

$$\left[\left\{\left(V(x_i)\right)^{\theta}\ln(V(x_i))\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln V(x_i)\right\}\right\}-\frac{1}{(\theta+1)^2}\left(V(x_i)\right)^{\theta}\ln(V(x_i))\right]$$

where $V(x_i)=1-e^{-\alpha x_i^{-\beta}}$

3.4. Method of Cramer-Von-Mises (CVM)

We interested in Cramér-von-Mises type minimum distance estimators, (Macdonald 1971) because it provides empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The CVM estimators of α, β, and θ are obtained by minimizing the function

$$H_{CVM}\left(\alpha,\beta,\lambda\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\mid\alpha,\beta,\lambda\right)-\frac{2i-1}{2n}\right]^2 \quad (3.4.1)$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\left(1-e^{-\alpha x^{-\beta}}\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\alpha x^{-\beta}}\right)\right\}-\frac{2i-1}{2n}\right]^2;x>0$$

To obtain the CVM estimators of α, β, and θ, we have to solve the following two nonlinear equations simultaneously by equating to zero,

$$\frac{\partial H_{CVM}\left(\alpha,\beta,\lambda\right)}{\partial \alpha}=-2\sum_{i=1}^{n}\left[1-\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(V(x_i)\right)\right\}-\frac{2i-1}{2n}\right]\left(1-e^{\alpha x_i^{-\beta}}\right)^{-1}$$

$$\left[\left\{\theta x_i^{-\beta}\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln V(x_i)\right\}\right\}+\left(V(x_i)\right)^{\theta}\left(\frac{\theta}{\theta+1}\right)x_i^{-\beta}\right]$$

$$\frac{\partial H_{CVM}\left(\alpha,\beta,\lambda\right)}{\partial\beta}=-2\sum_{i=1}^{n}\left[1-\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(V(x_i)\right)\right\}-\frac{2i-1}{2n}\right]$$
$$\left[\left\{\alpha\theta x_i^{-\beta}e^{-\alpha x_i^{-\beta}}\left(V(x_i)\right)^{\theta-1}\ln(x)\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln V(x_i)\right\}\right\}+\left(V(x_i)\right)^{\theta}\alpha x_i^{-\beta}\left(\frac{\theta}{\theta+1}\right)x_i^{-\beta}\ln(x)\right]$$

$$\frac{\partial H_{CVM}\left(\alpha,\beta,\lambda\right)}{\partial\theta}=-2\sum_{i=1}^{n}\left[1-\left(V(x_i)\right)^{\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(V(x_i)\right)\right\}-\frac{2i-1}{2n}\right]$$
$$\left[\left\{\left(V(x_i)\right)^{\theta}\ln(V(x_i))\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln V(x_i)\right\}\right\}-\frac{1}{(\theta+1)^2}\left(V(x_i)\right)^{\theta}\ln(V(x_i))\right]$$

where $V(x_i)=1-e^{-\alpha x_i^{-\beta}}$

4. APPLICATION WITH REAL DATASET AND RESULT

In this section, we illustrate the applicability of LIW distribution using a real dataset used by earlier researchers .we have taken 100 observations on waiting times (in minutes) before the customer received service in a bank (Ghitany et al. 2008).

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5

The plots of profile log-likelihood function for the parameters α, β and θ have been displayed in Figure 2, and noticed that the ML estimates can be uniquely determined.

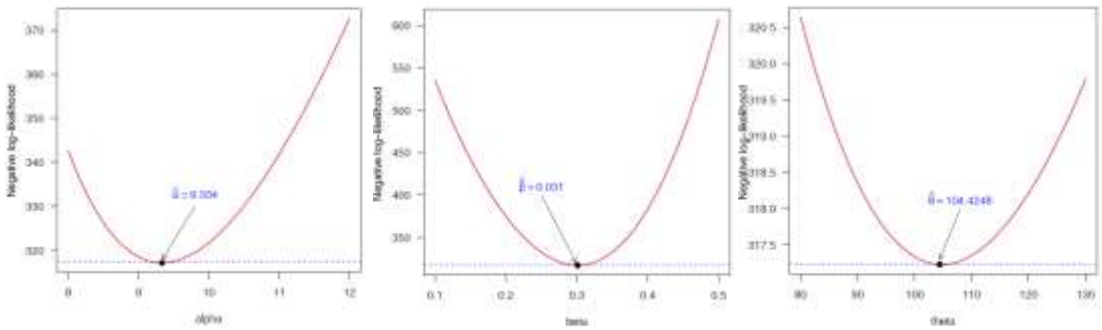

Figure 2. Graph of Profile log-likelihood function for the parameters α, β and θ.

The maximum likelihood estimates are calculated directly by using optim() function in R software (R Core Team, 2020) and (Rizzo, 2008) by maximizing the likelihood function (3.1). We have obtained $\hat{\alpha}$ = 9.3340, $\hat{\beta}$ = 0.3010, $\hat{\theta}$ = 104.4248 and corresponding Log-Likelihood value is -317.2356. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α, β, and θ.

Table 1: MLE, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	9.3340	0.5078	(8.3387, 10.3293)
beta	0.3010	0.0217	(0.2584, 0.3436)
theta	104.4248	2.4923	(99.5398, 109.3098)

In Table 2 we have displayed the estimated value of the parameters of Lindley inverse Weibull distribution using MLE, LSE and CVE method and their corresponding negative log-likelihood, AIC, BIC and AICC criterion.

Table 2: Estimated parameters, log-likelihood, AIC, BIC, AICC and HQIC

Method of Estimation	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	-LL	AIC	BIC	AICC	HQIC
MLE	9.3340	0.3010	104.4248	317.2356	640.4711	648.2866	640.716	643.6342
LSE	8.9289	0.3291	63.4914	317.499	640.9981	648.8136	641.243	644.1611
CVE	9.0867	0.3262	70.4662	317.4243	640.8486	648.6642	641.0935	644.0117

Table 3: The KS, AD and CVM statistics with p-value

Method of Estimation	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
MLE	0.0382(0.9986)	0.0189(0.9980)	0.0189(0.9980)
LSE	0.0363(0.9994)	0.0177(0.9987)	0.0177(0.9987)
CVE	0.0345(0.9998)	0.0171(0.9990)	0.0171(0.9990)

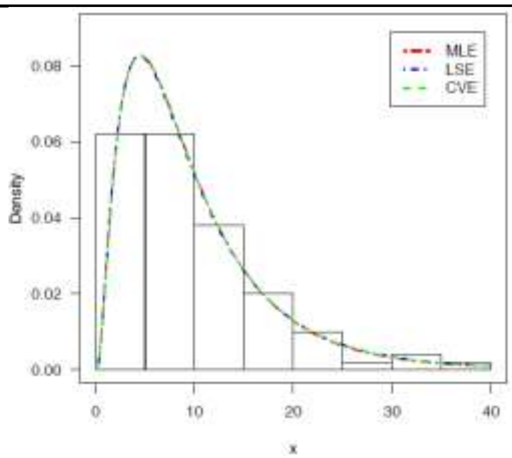

Figure 3. The Histogram and the density function of fitted distributions of estimation methods MLE, LSE and CVM.

To illustrate the goodness of fit of the Lindley inverse Weibull distribution, we have taken some well known distribution for comparison purpose which are listed blew,

I. Power Lindley distribution:

The probability density function of power Lindley distribution (Ghitany et al., 2013) with parameters αand β is

$$f_{PL}(x) = \frac{\alpha\beta^2}{\beta+1}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\beta x^{\alpha}} \qquad ; x \ge 0, \alpha > 0, \beta > 0.$$

II. Weibull distribution:

The probability density function of Weibull (W) distribution is

$$f_W\left(x\right) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(x/\lambda)^{\theta}};\ \lambda\theta > 0, x \ge 0$$

III. Flexible Weibull Extension (FWE) distribution:

The density of Flexible Weibull (FW) distribution (Bebbington, 2007) with parameters αand βis

$$f_{FW}(x) = \left(\alpha + \frac{\beta}{x^2}\right) \exp\left(\alpha x - \frac{\beta}{x}\right) \exp\left\{-\exp\left(\alpha x - \frac{\beta}{x}\right)\right\} \quad ; x \ge 0, \alpha > 0, \beta > 0.$$

IV. The inverse Weibull (IW) distribution

The probability density function (PDF) of a random variable X of IW (Keller et al., 1982)is given by

$$g\left(x\right) = \alpha\beta x^{-(\beta+1)} \exp\left(-\alpha x^{-\beta}\right);\ \ x \ge 0,\ \alpha > 0,\ \beta > 0$$

To assess the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot Q-Q and P-P plots. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the P-P plot emphasizes the lack-of-fit. From Figure 4 it is proven that the LIW model fits the data very well.

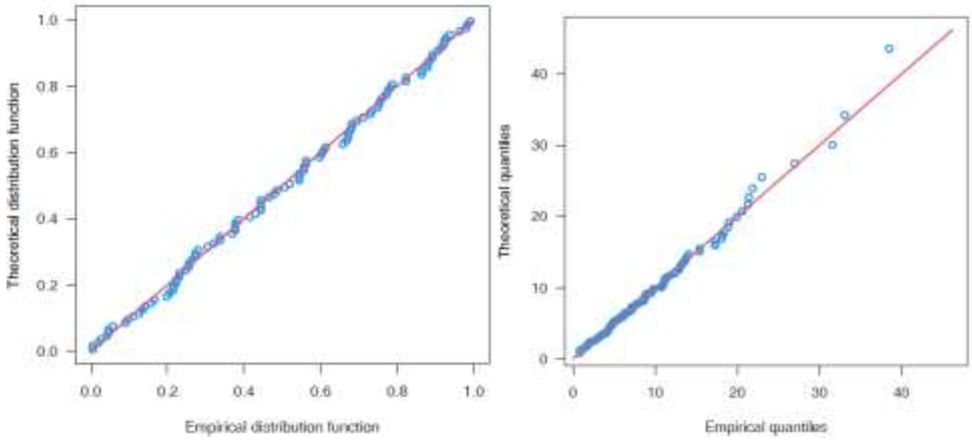

Figure 4. The P-P plot (left panel) and Q-Q plot (right panel) of LIW distribution

For the assessment of potentiality of the proposed model we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4: Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
LIW	317.2356	640.4711	648.2866	640.7160	643.6342
Power Lindley	318.3186	640.6372	645.8475	640.7609	642.7459
Weibull	318.7307	641.4614	646.6717	641.5851	643.5701
Flexible Weibull	321.2682	646.5363	651.7467	646.6600	648.6450
Inverse Weibull	334.3810	672.7620	677.9723	672.8857	674.8707

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of LIW and some selected distributions are presented in Figure 5.

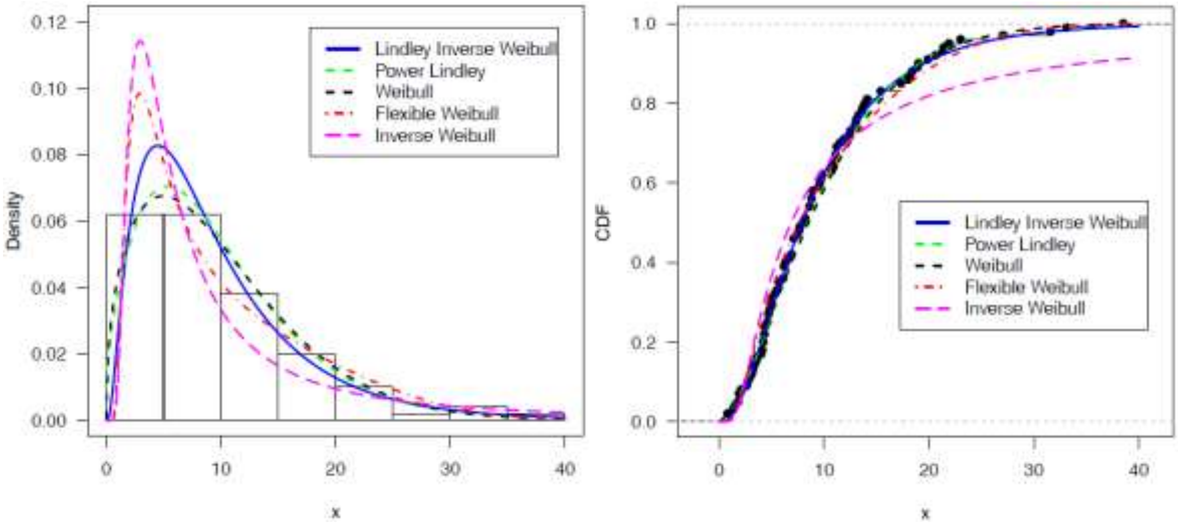

Figure 5. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the LIW distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 5. It is observed that the LIW distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the LIW distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5: The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
LIW	0.0382(0.9986)	0.0189(0.9980)	0.1527(0.9984)
Power Lindley	0.0520(0.9498)	0.0458(0.9025)	0.3028(0.9359)
Weibull	0.0578(0.8920)	0.0611(0.8084)	0.4058(0.8426)
Flexible Weibull	0.0849(0.4717)	0.1116(0.5316)	0.7710(0.5021)
Inverse Weibull	0.1167(0.1313)	0.4272(0.0611)	2.8925(0.0311)

5. CONCLUSION

In this study, we have studied the three-parameter Lindley inverse Weibull(LIW) distribution. For our study, we provided the PDF, the CDF, and the shapes of the hazard function. The shape of the PDF of the LIW model is unimodal and positively skewed, while the hazard function of the LIW model is increasing. The P-P and Q-Q plots showed that the purposed distribution is quite better for fitting the real dataset. Finally, using a real data set we have explored some well-known estimation methods namely maximum likelihood estimation (MLE), least-square (LSE), and Cramer-Von-Mises (CVM) methods. Further we also construct the asymptotic confidence interval for MLEs. We conclude that MLE is the best estimation method as compared to the LSE and CVM methods. The application illustrate that the proposed model provides consistently better fit then other underling models. We expect that this model will contribute in the field of survival analysis.

REFERENCES

[1]. Abd EL-Baset, A. A., & Ghazal, M. G. M. (2020). Exponentiated additive Weibull distribution. *Reliability Engineering & System Safety*, *193*, 106663.

[2]. Afify, A. Z., Cordeiro, G. M., Butt, N. S., Ortega, E. M., & Suzuki, A. K. (2017). A new lifetime model with variable shapes for the hazard rate. *Brazilian Journal of Probability and Statistics*, *31*(3), 516-541.

[3]. Afify, A. Z., Nofal, Z. M., & Butt, N. S. (2014). Transmuted complementary Weibull geometric distribution. *Pakistan Journal of Statistics and Operation Research*, 435-454.

[4]. Akgül, F. G., Şenoğlu, B., &Arslan, T. (2016). An alternative distribution to Weibull for modeling the wind speed data: Inverse Weibull distribution. *Energy Conversion and Management*, *114*, 234-240.

[5]. Aryal, G. R., Ortega, E. M., Hamedani, G. G., &Yousof, H. M. (2017). The Topp-Leone generated Weibull distribution: regression model, characterizations and applications. *International Journal of Statistics and Probability*, *6*(1), 126-141.

[6]. Aryal, G. R., &Tsokos, C. P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics*, *4*(2), 89-102.

[7]. Basheer, A. M. (2019). Alpha power inverse Weibull distribution with reliability application. *Journal of Taibah University for Science*, *13*(1), 423-432.

[8]. Bebbington, M., Lai, C. D., &Zitikis, R. (2007). A flexible Weibull extension. *Reliability Engineering & System Safety*, *92*(6), 719-726.

[9]. Cakmakyapan, S., &Gamze, O. Z. E. L. (2016). The Lindley family of distributions: Properties and applications. *Hacettepe Journal of Mathematics and Statistics*, *46*(6), 1113-1137.

[10]. Cordeiro, G. M., Ortega, E. M., & da Cunha, D. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, *11*(1), 1-27.

[11]. Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Cakmakyapan, S., &Ozel, G. (2018). The Lindley Weibull distribution: properties and applications. *Anais da Academia Brasileira de Ciências*, *90*(3), 2579-2598.

[12]. Ghitany, M. E., Al-Hussaini, E. K., & Al-Jarallah, R. A. (2005). Marshall–Olkin extended Weibull distribution and its application to censored data. *Journal of Applied Statistics*, *32*(10), 1025-1034.

[13]. Ghitany, M.E., Atieh, B., Nadarajah, S. (2008). Lindley distribution and its application. Math. Comput. Simul. **78**, 493–506.

[14]. Ghitany, M. E., Al-Mutairi, D. K., and Aboukhamseen, S. M., (2013). Estimation of the reliability of a stress-strength system from power Lindley distributions, *Communications in Statistics - Simulation and Computation*, **78**, 493-506.

[15]. Hanook, S., Shahbaz, M. Q., Mohsin, M., &GolamKibria, B. M. (2013). A note on beta inverse-Weibull distribution. *Communications in Statistics-Theory and Methods*, *42*(2), 320-335.

[16]. Keller, A. Z., Kamath, A. R. R., &Perera, U. D. (1982). Reliability analysis of CNC machine tools. *Reliability engineering*, *3*(6), 449-473.

[17]. Khan, M. S., Pasha, G. R., & Pasha, A. H. (2008). Theoretical analysis of inverse Weibull distribution. *WSEAS Transactions on Mathematics*, *7*(2), 30-38.

[18]. Kumar, K., & Kumar, I. (2019). Estimation in inverse Weibull distribution based on randomly censored data. *Statistica*, *79*(1), 47-74.

[19]. Lai, C. D., Xie, M., & Murthy, D. N. P. (2003). A modified Weibull distribution. *IEEE Transactions on reliability*, *52*(1), 33-37.

[20]. Lindley, D.V. (1958). Fiducial distributions and Bayes' theorem, Journal of the Royal Statistical Society Series B, 20, p. 102-107.

[21]. Mudholkar, G. S., Srivastava, D. K., &Kollia, G. D. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *Journal of the American Statistical Association*, *91*(436), 1575-1583.

[22]. Nadarajah, S. (2009). Bathtub-shaped failure rate functions. *Qual Quant* 43: 855-863.

[23]. Nofal, Z. M., Afify, A. Z., Yousof, H. M., Granzotto, D. C., &Louzada, F. (2016). Kumaraswamy transmuted exponentiated additive Weibull distribution. *International Journal of Statistics and Probability*, *5*(2), 78-99.

[24]. Okasha, H. M., El-Baz, A. H., Tarabia, A. M. K., &Basheer, A. M. (2017). Extended inverse Weibull distribution with reliability application. *Journal of the Egyptian Mathematical Society*, *25*(3), 343-349.

[25]. Pararai, M., Warahena-Liyanage, G., &Oluyede, B. O. (2014). A new class of generalized inverse Weibull distribution with applications. *Journal of Applied Mathematics and Bioinformatics*, *4*(2), 17.

[26]. R Core Team (2020). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

[27]. Sarhan, A. M., &Zaindin, M. (2009). Modified Weibull distribution. *APPS. Applied Sciences*, *11*, 123-136.

[28]. Shahbaz, M. Q., Shahbaz, S., & Butt, N. S. (2012). The Kumaraswamy-Inverse Weibull Distribution. *Shahbaz, MQ, Shahbaz, S., & Butt, NS (2012). The Kumaraswamy–Inverse Weibull Distribution. Pakistan journal of statistics and operation research*, *8*(3), 479-489.

[29]. Silva, G. O., Ortega, E. M., &Cordeiro, G. M. (2010). The beta modified Weibull distribution. *Lifetime data analysis*, *16*(3), 409-430.

[30]. Xie, M., & Lai, C. D. (1996). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. *Reliability Engineering & System Safety*, *52*(1), 87-93.

[31]. Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., &Hamedani, G. (2015). The transmuted exponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operation Research*.

[32]. Lee, C., Famoye, F., &Olumolade, O. (2007). Beta-Weibull distribution: some properties and applications to censored data. *Journal of modern applied statistical methods*, *6*(1), 17.

[33]. Zografos, K., &Balakrishnan, N. (2009). On families of beta-and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, *6*(4), 344-362.