Vol.8.Issue.4.2020 (Oct-Dec)

http://www.bomsr.com
Email:editorbomsr@gmail.com

RESEARCH ARTICLE

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

LINDLEY INVERSE EXPONENTIAL DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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DOI: 10.33329/bomsr.8.4.1

ABSTRACT

In this study, a two-parameter Lindley inverse exponential distribution is presented. Some mathematical and statistical properties of the distribution namely the shapes of the probability density, cumulative density and hazard rate functions, survival function, hazard function, quantile function, the skewness, and kurtosis measures are derived and established. To estimate the model parameters, we have employed three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises (CVM) methods. A real data set is considered to explore the applicability and suitability of the proposed distribution. Also, AIC, BIC, AICC and HQIC are calculated to assess the validity of the Lindley inverse exponential model.
Keywords: Cramer-Von-Mises, Generalized Exponential (GE) distribution, Inverse exponential distribution, Least-square estimation, Lindley distribution

1. Introduction

Researchers in the last few years has developed various extensions and modified form of the Lindley distribution which was developed by (Lindley, 1958) in the context of Bayesian statistics, as a counterexample to fiducial statistics. A detailed study on the Lindley distribution was done by (Ghitany et al., 2008), Mazucheli & Achcar (2011) have been used the Lindley distribution to competing risks lifetime data.

Let R denote a random variable that follows Lindley distribution with parameter μ and its probability density function (PDF) is given by

$$f(r)=\frac{\beta}{\beta+1}(1+r)e^{-\beta r};\ r>0,\beta>0 \tag{1.1}$$

And its cumulative density function (CDF) is

$$F(r)=1-\frac{1+\beta+\beta r}{1+\beta}e^{-\beta r};\ r>0,\beta>0 \tag{1.2}$$

Some of the modifications in the literature of Lindley distribution are given by (Ghitany et al., 2008a) showed that the Lindley distribution is fairly similar to the exponential distribution. Gupta and Singh (2013) investigated the estimation of the parameters using hybrid censored data. The estimation of the model parameters for censored samples by (Krishna and Kumar, 2011).

Towards the modified theoretical distribution, Ghitany et al. (2011) introduced weighted Lindley distribution having two parameters and has shown that it is appropriate in modeling survival data for a mortality study. Nadarajah et al. (2011) have introduced generalized Lindley, extended Lindley by (Bakouch et al., 2012), Ashour & Eltehiwy (2015) for exponentiated power Lindley, Bhati et al. (2015) Lindley–Exponential distribution. Ieren et al. (2018) have introduced modeling lifetime data with Weibull-Lindley distribution. Ibrahim et al. (2019) has introduced a new extension of Lindley distribution.

Also, we have found some continuous-discrete mixed approaches, the discrete Poisson-Lindley have introduced by (Sankaran, 1970). Zamani and Ismail (2010) introduced negative binomial Lindley distribution. A new weighted Lindley distribution is introduced by (Asgharzadeh et al., 2016).

A new class of distributions to generate new distribution based on Lindley distribution having additional shape parameter θ has introduced by (Cakmakyapan and Ozel, 2016). The CDF of Lindley generator can be expressed as,

$$F_{L-G}(y;\theta,\lambda)=1-\left[\bar{G}(y;\lambda)\right]^{\theta}\left[1-\frac{\theta}{\theta+1}\ln\bar{G}(y;\lambda)\right];\ y>0,\theta>0 \tag{1.3}$$

and

$$f_{L-G}(y;\theta,\lambda)=\frac{\theta^2}{\theta+1}g(y;\lambda)\left[\bar{G}(y;\lambda)\right]^{\theta-1}\left[1-\ln\bar{G}(y;\lambda)\right];\ y>0,\theta>0 \tag{1.4}$$

where $g(y;\lambda)=\frac{dG(y;\lambda)}{dy}, \bar{G}(y;\lambda)=1-G(y;\lambda)$

The primary purpose of this paper is to achieve a more flexible distribution by adding just one extra parameter to the inverse exponential distribution using (1.3) and (1.4) to achieve a better fit to real data. We investigate the properties of the L-IE distribution and elucidate its applicability. The contents of the proposed study are organized as follows. The new Lindley inverse exponential distribution is introduced and various distributional properties are discussed in Section 2. To estimate the model parameters, we have employed four well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises (CVM) methods in Section 3. In Section 4 a real data set has been analyzed to explore the applications and suitability of the proposed distribution. In this section, we present the ML estimators of the parameters and approximate confidence intervals also for the above-mentioned method of estimation, AIC, BIC, AICC and HQIC are calculated to assess the validity of the L-IE model. Lastly, Section 5 ends up with some general concluding remarks.

2. The Lindley Inverse exponential (L-IE) distribution:

Using (1.3) and (1.4) we are introducing a new distribution where the baseline distribution is the inverse exponential distribution. The Inverse Exponential (IE) distribution has been introduced by (Keller & Kamath, 1982) and it has been studied and discussed as a lifetime model. If a random variable $Y \sim IE(\lambda)$ then the variable $U = \frac{1}{Y}$ will have an inverse exponential distribution and its CDF and PDF can be written as,

$$H(y) = e^{-\lambda/y}; \;\; \lambda > 0, y > 0 \tag{2.1}$$

and $$h(y) = \frac{\lambda}{y^2} e^{-\lambda/y}; \;\; \lambda > 0, y > 0 \tag{2.2}$$

Utilizing (2.1) and (2.2) in (1.3) and (1.4) we get the CDF and PDF of Lindley inverse exponential distribution as follows

$$F(x) = 1-(1-e^{-\lambda/x})^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln(1-e^{-\lambda/x})\right\}; \;\; \theta > 0, \lambda > 0, x > 0 \tag{2.3}$$

and $$f(x) = \left(\frac{\theta^2}{1+\theta}\right)\frac{\lambda}{x^2} e^{-\lambda/x}(1-e^{-\lambda/x})^{\theta-1}\left\{1-\ln(1-e^{-\lambda/x})\right\}; \;\; \theta > 0, \lambda > 0, x > 0 \tag{2.4}$$

where λ and θ are scale and shape parameters of the L-IE distribution.

Reliability function

The reliability function of Lindley inverse exponential (L-IE) distribution is

$$R(x) = 1 - F(x)$$

$$= (1-e^{-\lambda/x})^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln(1-e^{-\lambda/x})\right\}; \theta > 0, \lambda > 0, x > 0 \tag{2.5}$$

Hazard function

The failure rate function of L-IE distribution can be defined as,

$$h(x) = \frac{f(x)}{R(x)} = \frac{\lambda\theta^2\left\{1-\ln(1-e^{-\lambda/x})\right\}}{x^2 e^{\lambda/x}(1+\theta)(1-e^{-\lambda/x})\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln(1-e^{-\lambda/x})\right\}}; \;\; \theta,\lambda > 0, \; x > 0 \tag{2.6}$$

In Figure 1, we have displayed the plots of the PDF and hazard rate function of L-IE distribution for different values of λ and θ.

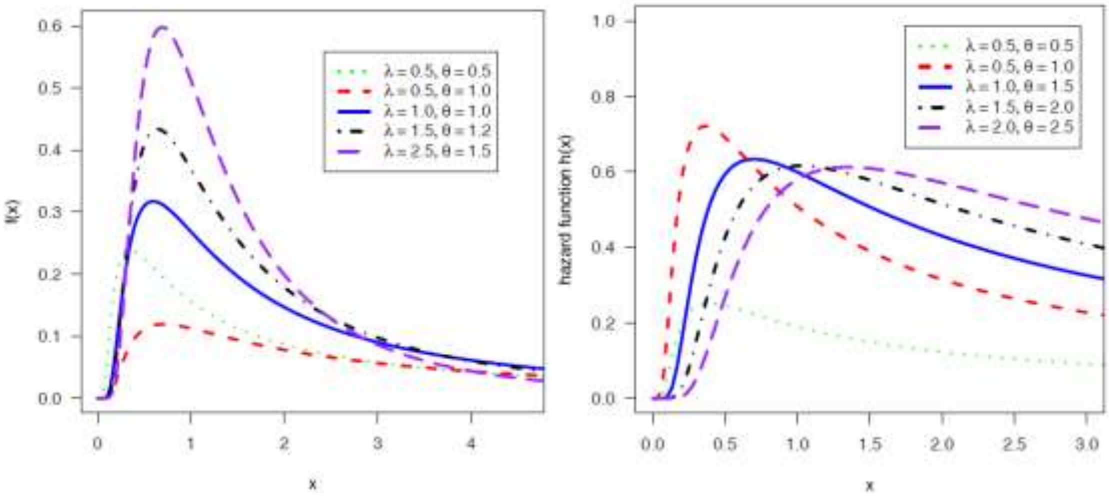

Figure 1. Plots of PDF (left panel) and hazard function (right panel) for different values of λ and θ.

Quantile function:

The quantile function of L-IE distribution can be given by

$$p-1+(1-e^{-\lambda/x})^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln(1-e^{-\lambda/x})\right\}=0\ ;\ 0<p<1 \quad (2.7)$$

Skewness and Kurtosis:

The Skewness and Kurtosis based on quantile function are,

Bowley's coefficient of skewness is

$$\Upsilon_{Sk}=\frac{Q(3/4)+Q(1/4)-2Q(2/4)}{Q(3/4)-Q(1/4)}, \text{ and} \quad (2.8)$$

Coefficient of kurtosis based on octiles given by (Moors, 1988) is

$$M_{Ku}=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(3/4)-Q(1/4)} \quad (2.9)$$

3. Methods of estimation

In this segment, we have presented some well-known estimation methods for estimating parameters of the proposed model, which are as follows

3.1. Maximum Likelihood Estimates

For the estimation of the parameter, the maximum likelihood method is the most commonly used method (Casella & Berger, 1990). Let, $x_1, x_2, ...x_n$ be a random sample from $L-IE(\lambda,\theta)$ and the likelihood function, $L(\lambda,\theta)$ is given by,

$$L(\varphi; x_1, x_2......x_n)=f(x_1, x_2,......x_n/\varphi)=\prod_{i=1}^{n} f(x_i/\varphi)$$

$$L(\lambda,\theta)=\left(\lambda\frac{\theta^2}{1+\theta}\right)\prod_{i=1}^{n}x_i^{-2}e^{-\lambda/x_i}(1-e^{-\lambda/x_i})^{\theta-1}\left\{1-\ln(1-e^{-\lambda/x_i})\right\};\quad \theta,\lambda>0,x>0$$

Now log-likelihood density is

$$l=2n\ln\theta-n\ln(1+\theta)-n\ln\lambda-2\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\frac{\lambda}{x_i}+(\theta-1)\sum_{i=1}^{n}\ln(1-e^{-\lambda/x_i})+\sum_{i=1}^{n}\ln\left\{1-\ln(1-e^{-\lambda/x_i})\right\} \quad (3.1.1)$$

Differentiating (3.1) with respect to λ and θ we get,

$$\frac{\partial l}{\partial\lambda}=-\frac{n}{\lambda}-\sum_{i=1}^{n}\frac{1}{x_i}+(\theta-1)\sum_{i=1}^{n}\frac{e^{-\lambda/x_i}}{x(1-e^{-\lambda/x_i})}-\sum_{i=1}^{n}\frac{e^{-\lambda/x_i}}{x_i\left\{1-\ln(1-e^{-\lambda/x_i})\right\}(1-e^{-\lambda/x_i})} \quad (3.1.2)$$

$$\frac{\partial l}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{1+\theta}+\sum_{i=1}^{n}\ln(1-e^{-\lambda/x}) \quad (3.1.3)$$

Equating (3.1.2) and (3.1.3) to zero and solving simultaneously for λ and θ, we get the maximum likelihood estimate $\hat{\lambda}\ and\ \hat{\theta}$ of the parameters λ and θ. By using computer software like R, Matlab, etc for maximization of (3.1.1) we can obtain the estimated value of λ and θ. For the interval estimation of λ and θ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for λ and θ can be obtained as,

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$$

Where

$$A_{11}=\frac{\partial^2 l}{\partial\lambda^2}=-\frac{n}{\lambda^2}+\sum_{i=1}^{n}\frac{e^{-\lambda/x_i}}{x_i^2(1-e^{-\lambda/x_i})^2}\left[\frac{\theta-1}{x_i}-\frac{1-e^{-\lambda/x_i}-\ln(1-e^{-\lambda/x_i})}{\left\{1-\ln(1-e^{-\lambda/x_i}\right\}^2}\right]$$

$$A_{22}=\frac{\partial^2 l}{\partial\theta^2}=-\frac{2n}{\theta^2}+\frac{n}{(1+\theta)^2}$$

$$A_{12}=\frac{\partial^2 l}{\partial\theta\partial\lambda}=\sum_{i=1}^{n}\frac{e^{-\lambda/x_i}}{x_i(1-e^{-\lambda/x_i})}$$

$$A_{21}=\frac{\partial^2 l}{\partial\lambda\partial\theta}=\sum_{i=1}^{n}\frac{e^{-\lambda/x_i}}{x_i(1-e^{-\lambda/x_i})}$$

Let $\kappa=(\lambda,\theta)$ denote the parameter space and the corresponding MLE of κ $\hat{\kappa}=(\hat{\lambda},\hat{\theta})$ as, then $(\hat{\kappa}-\kappa)\rightarrow N_2\left[0,(A(\kappa))^{-1}\right]$ where $A(\kappa)$ denotes the Fisher's information matrix. The Newton-Raphson algorithm is used in order to maximize the likelihood and create the observed information matrix and hence the variance-covariance matrix is obtained as,

$$\left[A(\kappa)\right]^{-1}=\begin{pmatrix}\text{var}(\hat{\lambda}) & \text{cov}(\hat{\lambda},\hat{\theta})\\ \text{cov}(\hat{\theta},\hat{\lambda}) & \text{var}(\hat{\theta})\end{pmatrix} \quad (3.1.4)$$

Therefore, approximate 100(1-α) % confidence intervals for λ and θ can be constructed using the asymptotic normality of MLEs as,

$$\hat{\lambda} \pm z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\lambda})} \text{ and } \hat{\theta} \pm z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\theta})} \qquad (3.1.5)$$

where $z_{\alpha/2}$ denotes the upper percentile of standard normal variate.

3.2. Method of Least-Square Estimation (LSE)

Swain et al. (1988) proposed the ordinary least square estimators and weighted least square estimators to estimate the parameters of Beta distributions. Here we have applied the same technique for the L-IE distribution. The least-square estimators of the unknown parameters λ and θ of L-IE distribution can be attained by minimizing

$$A(X;\lambda,\theta)=\sum_{k=1}^{n}\left[G(X_k)-\frac{k}{n+1}\right]^2 \qquad (3.2.1)$$

with respect to unknown parameters λ and θ.

Consider $G(X_k)$ denotes the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \ldots\ldots < X_{(n)}$, where $\{X_1, X_2, \ldots\ldots, X_n\}$ represents a random sample of size n from a distribution function G(.). The least-square estimators of λ and θ say $\hat{\lambda}$ *and* $\hat{\theta}$ respectively, can be obtained by minimizing

$$A(X;\lambda,\theta)=\sum_{k=1}^{n}\left[1-(1-e^{-\lambda/x_k})^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln(1-e^{-\lambda/x_k})\right\}-\frac{k}{n+1}\right]^2 \qquad (3.2.2)$$

with respect to λ and θ.

Differentiating (3.2.2) with respect to λ and θ we get,

$$\frac{\partial A}{\partial \lambda}=2\frac{\theta}{1+\theta}\sum_{k=1}^{n}\left[1-(C(x_k))^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln C(x_k)\right\}-\frac{k}{n+1}\right]\frac{\{\theta\ln C(x_k)+2\}(C(x_k))^{\theta}}{x_k\left(1-e^{\lambda/x_k}\right)}$$

$$\frac{\partial A}{\partial \theta}=2\sum_{k=1}^{n}\left[1-(C(x_k))^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln C(x_k)\right\}-\frac{k}{n+1}\right]$$
$$(C(x_k))^{\theta}\ln C(x_k)\left[-1-\frac{1}{(1+\theta)^2}\left\{\left[\theta^2\ln C(x_k)+\theta\ln C(x_k)+1\right]\right\}\right]$$

where $C(x_k)=1-e^{-\lambda/x_k}$

The weighted least square estimators can be attained by minimizing

$$A(X;\lambda,\theta)=\sum_{k=1}^{n}w_k\left[G(X_{(k)})-\frac{k}{n+1}\right]^2$$

with respect to λ and θ. The weights w_k are $w_k=\frac{1}{Var(X_{(k)})}=\frac{(n+1)^2(n+2)}{k(n-k+1)}$

Hence, the weighted least square estimators of λ and θ respectively can be obtained by minimizing,

$$A(X;\lambda,\theta)=\sum_{k=1}^{n}\frac{(n+1)^2(n+2)}{k(n-k+1)}\left[1-(1-e^{-\lambda/x_k})^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln(1-e^{-\lambda/x_k})\right\}-\frac{k}{n+1}\right]^2 \quad (3.2.3)$$

with respect to λ and θ.

3.3. Method of Cramer-Von-Mises (CVM)

The CVM estimators of λ and θ are obtained by minimizing the function

$$M(X;\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[G(x_{i:n}\mid\lambda,\theta)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-(1-e^{-\lambda/x_k})^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln(1-e^{-\lambda/x_k})\right\}-\frac{2i-1}{2n}\right]^2 \quad (3.3.1)$$

Differentiating (3.3.1) with respect to λ and θ we get,

$$\frac{\partial M}{\partial\lambda}=2\frac{\theta}{1+\theta}\sum_{k=1}^{n}\left[1-(C(x_k))^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln C(x_k)\right\}-\frac{2i-1}{2n}\right]\frac{\{\theta\ln C(x_k)+2\}(C(x_k))^{\theta}}{x_k\left(1-e^{\lambda/x_k}\right)}$$

$$\frac{\partial M}{\partial\theta}=2\sum_{k=1}^{n}\left[1-(C(x_k))^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln C(x_k)\right\}-\frac{2i-1}{2n}\right]$$

$$(C(x_k))^{\theta}\ln C(x_k)\left[-1-\frac{1}{(1+\theta)^2}\left\{\left[\theta^2\ln C(x_k)+\theta\ln C(x_k)+1\right]\right\}\right]$$

Solving $\frac{\partial M}{\partial\lambda}=0$ *and* $\frac{\partial M}{\partial\theta}=0$ simultaneously we get the CVM estimators.

4. Application with a real dataset

The data given below are obtained from an accelerated life test comprising of 59 conductors, (Nelson & Doganaksoy, 1995). The failures can befall in microcircuits due to the movement of atoms in the conductors in the circuit; this is connoted to as electro-migration. The failure times are in hours and with no censored observations.

6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

The contour plot and fitted CDF with empirical distribution function (EDF) are presented in Figure 2, Kumar & Ligges (2011).

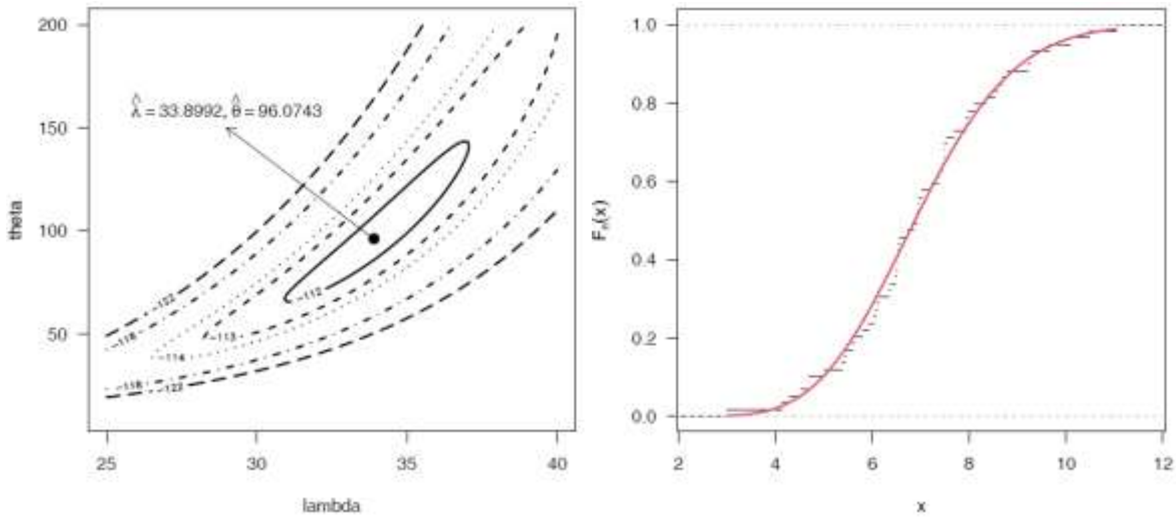

Figure 2. Contour plot (left panel) and the fitted CDF with empirical distribution function (right panel).

The MLEs are calculated directly by using optim() function (Ming, 2019) in R software (R Core Team, 2020) and (Rizzo, 2008) by maximizing the likelihood function (3.1.1). We have obtained $\hat{\lambda}$ = 33.8992 and $\hat{\theta}$ = 96.0743 and the corresponding Log-Likelihood value is -111.6267. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for λ and θ.

Table 1: MLEs, SE and 95% confidence interval

Parameter	**MLE**	**SE**	**95% ACI**
lambda	33.8992	0.9991	(31.9410, 35.8574)
theta	96.0743	2.9763	(90.2406, 101.908)

We have displayed the graph of the profile log-likelihood function of λ and θ in Figure 3 and observed that the MLEs are unique.

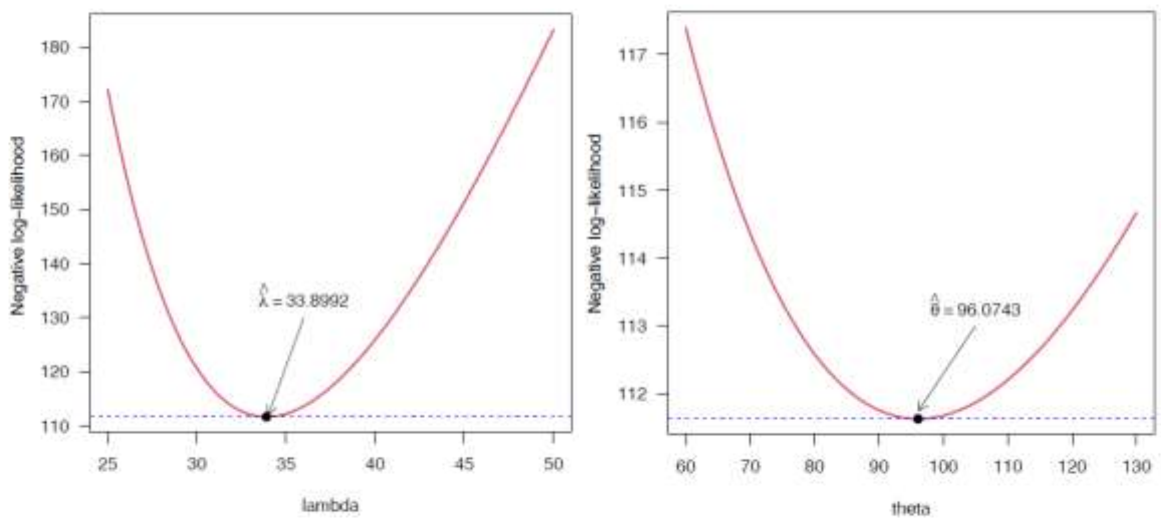

Figure 3. Graph of profile log-likelihood function of λ and θ.

In Table 2 we have displayed the estimated value of the parameters of Lindley inverse exponential distribution using MLE, LSE and CVE method and their corresponding negative log-likelihood, AIC, BIC AICC and HQIC information criterion.

Table 2: Estimated parameters, negative log-likelihood, AIC, BIC, AICC and HQIC

Method of Estimation	$\hat{\lambda}$	$\hat{\theta}$	**-LL**	**AIC**	**BIC**	**AICC**	**HQIC**
MLE	33.8992	96.0743	-111.6267	227.2534	231.4085	227.4603	228.8754
LSE	35.1689	113.5768	-111.6908	227.3816	231.5367	227.5885	229.0036
CVE	36.1518	130.7198	-111.8489	227.6978	231.8529	227.9047	229.3198

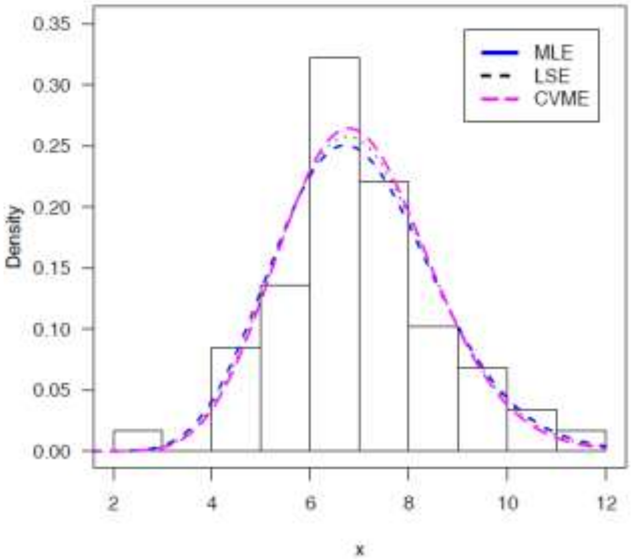

Figure 4. The Histogram and the density function of fitted distributions of estimation methods MLE, LSE and CVM.

Table 3: The KS, AD and CVM statistics with p-value

Method of Estimation	***KS(p-value)***	***AD(p-value)***	***CVM(p-value)***
MLE	0.0627(0.9630)	0.0331(0.9662)	0.2056(0.9888)
LSE	0.0555(0.9888)	0.0279(0.9833)	0.2014(0.9901)
CVE	0.0515(0.9954)	0.0267(0.9863)	0.2278(0.9808)

To illustrate the goodness of fit of the Lindley inverse exponential distribution, we have taken some well known distribution for comparison purpose which are listed blew,

I. Weighted Lindley distribution (W-Lindley):

II. The weighted Lindley distribution has presented by (Ghitany et al., 2011) whose PDF is

$$f(t)=\frac{\theta^{\alpha+1}}{(\alpha+\theta)\Gamma(\alpha)}t^{\alpha-1}(1+t)e^{-\theta t}\quad ;t\ge 0,\,\alpha>0,\theta>0.$$

III. Chen distribution:

Chen (2000) has introduced Chain distribution having probability density function (PDF) as

$$f(x;\lambda,\theta)=\lambda\beta x^{\theta-1}e^{x\theta}\,exp\left\{\lambda\left(1-e^{x^{\theta}}\right)\right\}\quad ;(\lambda,\theta)>0,x>0.$$

IV. Weibull distribution:

The probability density function of Weibull (W) distribution is

$$f_W(x)=\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-(x/\lambda)^{\theta}};\ \lambda\theta>0,x\ge 0$$

V. Generalized Exponential (GE) distribution:

The probability density function of generalized exponential distribution (Gupta & Kundu, 1999) is.

$$f_{GE}(x;\alpha,\lambda)=\alpha\,\lambda\,e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1};(\alpha,\lambda)>0,x>0$$

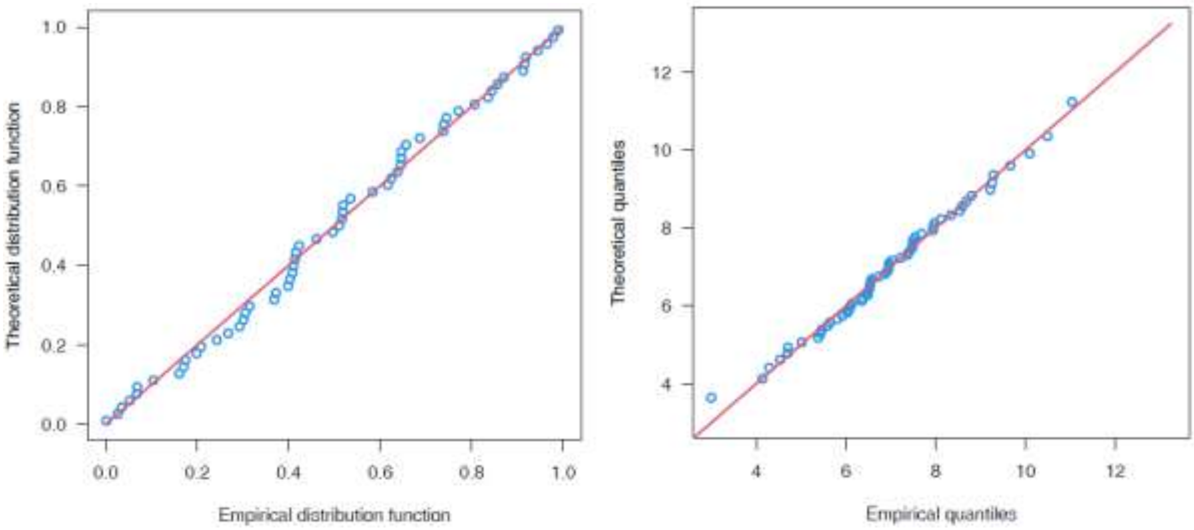

Figure 4. The P-P plot (left panel) and Q-Q plot (right panel) of L-IE distribution

For the judgment of potentiality of the proposed model we have presented the value of Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4: Negative Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
LIE	111.6267	227.2534	231.4085	227.4603	228.8754
WL	111.8202	227.6403	231.7954	227.8546	229.2623
Weibull	112.4973	228.9946	233.1496	229.2088	230.6165
GE	114.9473	233.8946	238.0497	234.1089	235.5166
Chen	116.3874	236.7748	240.9299	236.9891	238.3968

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of L-IE and some selected distributions are presented in Figure 5.

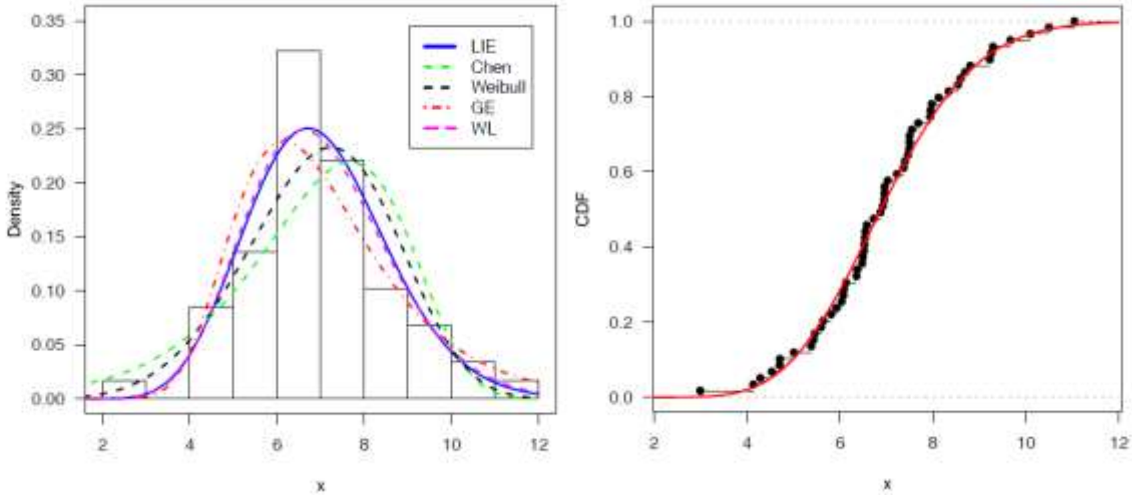

Figure 5. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the L-IE distribution with other competing distributions, we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 5. It is observed that the L-IE distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the L-IE distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5: The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
LIE	0.0627(0.9630)	0.0331(0.9662)	0.2056(0.9888)
WL	0.0708(0.9085)	0.0390(0.9398)	0.2352(0.9776)
Weibull	0.0956(0.6194)	0.0840(0.6707)	0.4773(0.7693)
GE	0.1042(0.5103)	0.1173(0.5079)	0.7368(0.5282)
Chen	0.1238(0.3006)	0.1913(0.2855)	1.1741(0.2774)

5. Conclusions

In this work, a two-parameter Lindley inverse exponential (L-IE) distribution is introduced. Some mathematical and statistical properties of the L-IE distribution are presented such as the shapes of the probability density, cumulative density and hazard rate functions, survival function, hazard function quantile function, the skewness, and kurtosis measures are derived and established and found that the proposed model is flexible and inverted bathtub shaped hazard function. To estimate the model parameters, we have employed four well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises (CVM) methods and we concluded that the MLEs are quite better than LSE, and CVM. A real data set is considered to explore the applicability and suitability of the proposed distribution and found that the proposed model is quite better than other lifetime model taken into consideration.

References

[1]. Asgharzadeh, A., Bakouch, H. S., Nadarajah, S., & Sharafi, F. (2016). A new weighted Lindley distribution with the application. *Brazilian Journal of Probability and Statistics*, *30*(1), 1-27.

[2]. Ashour, S. K., & Eltehiwy, M. A. (2015). Exponentiated power Lindley distribution. *Journal of advanced research*, *6*(6), 895-905.

[3]. Bakouch, H.S., Al-Zahrani, B.M., Al-Shomrani, A.A., Marchi, V.A. & Louzada, F. (2012). An extended Lindley distribution. J. Korean Stat. Soc. **41**, 75–85.

[4]. Bhati, D., Malik, M. A., & Vaman, H. J. (2015). Lindley–Exponential distribution: properties and applications. *Metron*, *73*(3), 335-357.

[5]. Cakmakyapan, S., & Gamze, O. Z. E. L. (2016). The Lindley family of distributions: Properties and applications. *Hacettepe Journal of Mathematics and Statistics*, *46*(6), 1113-1137.

[6]. Casella, G., & Berger, R. L. (1990). Statistical Inference. Brooks/ *Cole Publishing Company,* California.

[7]. Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statistics & Probability Letters,* 49, 155-161.

[8]. Ghitany, M., Al-Mutairi, D. And Nadarajah, S. (2008a). Zero-truncated Poisson–Lindley distribution and its application. Math Comput Simulat 79: 279-287.

[9]. Ghitany, M.E., Atieh, B., Nadarajah, S. (2008). Lindley distribution and its application. Math. Comput. Simul. **78**, 493–506.

[10]. Ghitany, M. E., Alqallaf, F., Al-Mutairi, D. K., & Husain, H. A. (2011). A two-parameter weighted Lindley distribution and its applications to survival data. *Mathematics and Computers in simulation*, *81*(6), 1190-1201.

[11]. Gupta, R. D. & Kundu, D. (1999). Generalized exponential distributions, *Australian and New Zealand Journal of Statistics,* 41(2), 173 - 188.

[12]. Gupta, P.K. And Singh, B. (2013). Parameter estimation of Lindley distribution with hybrid censored data. Int J Syst Assur Eng Manag 4: 378-385.

[13]. Ibrahim, M., Yadav, A. S., Yousof, H. M., Goual, H., & Hamedani, G. G. (2019). A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. *Communications for Statistical Applications and Methods*, *26*(5), 473-495.

[14]. Ieren, T. G., Oyamakin, S. O., & Chukwu, A. U. (2018). Modeling lifetime data with Weibull-Lindley distribution. *Biometrics and Biostatistics International Journal*, *7*(6), 532-544.

[15]. Killer, A.Z. & Kamath, A.R. (1982). Reliability analysis of CNC Machine Tools, Reliability Engineering, 3, 449-473.

[16]. Krishna, H. And Kumar, K. (2011). Reliability estimation in Lindley distribution with progressively type II right censored sample. Math Comput Simulat 82: 281-294.

[17]. Kumar, V. and Ligges, U. (2011). reliaR: A package for some probability distributions, http://cran.r-project.org/web/packages/reliaR/index.html.

[18]. Lindley, D.V. (1958). Fiducial distributions and Bayes' theorem, Journal of the Royal Statistical Society Series B, 20, p. 102-107.

[19]. Mazucheli, J. & Achcar, J. (2011). The Lindley distribution applied to competing risks lifetime data. Comput Meth Prog Bio 104: 188-192.

[20]. Ming Hui, E. G. (2019). *Learn R for applied statistics*. Springer, New York.

[21]. Moors, J. J. A. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, *37*(1), 25-32.

[22]. Nadarajah, S., Bakouch, H.S. And Tahmasbi, R. (2011). A generalized Lindley distribution. Sankhya B 73: 331-359.

[23]. Nelson, W., & Doganaksoy, N. (1995). Statistical analysis of life or strength data from specimens of various sizes using the power-(log) normal model. *Recent Advances in Life-Testing and Reliability*, 377-408.

[24]. Sankaran, M. (1970). The discrete Poisson–Lindley distribution. Biometrics 26: 145-149.

[25]. Srivastava, A. K., & Kumar, V. (2011). Analysis of software reliability data using exponential power model. *IJACSA Editorial*.

[26]. Swain, J. J., Venkatraman, S. & Wilson, J. R. (1988), 'Least-squares estimation of distribution functions in johnson's translation system', Journal of Statistical Computation and Simulation 29(4), 271–297.

[27]. R Core Team (2020). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

[28]. Rizzo, M. L. (2008). Statistical computing with R. Chapman & Hall/CRC.

[29]. Zamani, H. & Ismail, N. (2010). Negative binomial-Lindley distribution and its application. J Math Stat 6: 4-9.