Vol.8.Issue.4.2020 (Oct-Dec)
©KY PUBLICATIONS

http://www.bomsr.com
Email:editorbomsr@gmail.com

RESEARCH ARTICLE

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

GENERALIZED HYERS-ULAM TYPE STABILITY OF THE TYPE FUNCTIONAL EQUATION WITH $2k$-VARIABLE IN NON-ARCHIMEDEAN SPACE

LY VAN AN
Faculty of Mathematics Teacher Education; Tay Ninh University; Ninh Trung;
Ninh Son; Tay Ninh Province; V ietnam. Email address : lyvanan145@gmail:com
DOI: 10.33329/bomsr.8.4.27

ABSTRACT

In this paper, we prove the generalized Hyers–Ulam stability of the following Cauchy type additive functional equation and the quadratic type functional equation in non–Archimedean space: We will show that the solution of the first and second equation are the additive and quadratic mappings

Keywords: Generalized Hyers Ulam stability, Cauchy functional equation; quadratic equation non-Archimedean space:
Mathematics Subject Classifcation: 39B22 39B82 46S10:

1. Introduction

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be a normed spaces on the same field $\mathbb{K}$; and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a mapping. We use the notation $\|.\|$ for the norms on both $\boldsymbol{X}$ and $\boldsymbol{Y}$: In this paper, we investigate some functional equation when $\boldsymbol{X}$ is a additive semigroup and Y is a non-Archimedean Banach space or when $\boldsymbol{X}$ is a is a additive group and $\boldsymbol{Y}$ is a non-Archimedean Banach space. In fact, when $\boldsymbol{X}$ is a is a additive semigroup and $\boldsymbol{Y}$ is a non-Archimedean Banach space. we solve and prove the Hyers-Ulam stability of following Cauchy type additive functional equation

$$f\left(\frac{1}{K}\sum_{i=1}^{k} x_{k+1} + \sum_{i=1}^{k} x_i\right) = \sum_{i=1}^{k} f\left(\frac{x_{k+1}}{k}\right) + \sum_{k=1}^{k} f(x_i) \qquad (1.1)$$

and

when $\boldsymbol{X}$ is a additive group and Y is a non-Archimedean Banach space we solve and prove the Hyers-Ulam stability of following quadratic type functional equation

$$f\left(\frac{1}{K}\sum_{i=1}^{k} x_{k+1} + \sum_{i=1}^{k} x_i\right) + f\left(\frac{1}{K}\sum_{i=1}^{k} x_{k+1} - \sum_{i=1}^{k} x_i\right) = 2\sum_{i=1}^{k} f\left(\frac{x_{k+1}}{k}\right) + 2\sum_{i=1}^{k} f\ (x_i) \quad (1.2)$$

Note: k be a fixed integer with k ≥ 2:

The study of the functional equation stability originated from a question of S.M. Ulam [34], concerning the stability of group homomorphisms. Let $(\mathbb{G},*)$; be a group and let $(\mathbb{G}', o, d)$ be a metric group with metric $d\,(.,.)$. Geven ϵ > 0, does there exist a $\delta > 0$ such that if $f : \mathbb{G} \to \mathbb{G}'$ satisfies.

$$d\big(f(x*y), f(x) o\, f(y)\big) < \delta$$

for all $x;\ y\ \in\ \mathbb{G}$ then there is a homomorphism $h : \mathbb{G} \to \mathbb{G}'$ with

$$d\big(f(x), h(x)\big) <\ \epsilon$$

for all $x\,\epsilon\,\mathbb{G}$? ?, if the answer, is affirmative, we would say that equation of homomophism $h(x*y) = h(y), o, h(y)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality di_er from those of the given function equation? Hyers[18] gave a first affirmative answes the question of Ulam as follows:

Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies inequality,

$$\|f(x+y) - f(x) - f(y)\| \leq\ \epsilon$$

for all $x;\ y\ \in\ \mathbb{E}_1$ where $\epsilon\ \geq 0$ is a constant. Then the limit $T(x) =\ \log_{n\to\infty} 2^{-n}\, f(\,2^n x)$ exists for each $y\ \in\ E_1$and T is the unique additive mapping,

$$\|f(x) - T(x)\| \leq\ \epsilon, \forall_x \in\ \mathbb{E}_1$$

Also if for each x the functional $t\ \to\ f(xt)$from $\mathbb{R}\ to\ \mathbb{E}_2$is continuous on $\mathbb{R}$: If f continuous at a single point of $\mathbb{E}_1$; then T is continuous everywhere in $\mathbb{E}_1$

Next Th. M. Rassias [29] provided a generalization of Hyers' Theorem as a special case. Suppose $\mathbb{E}$ and $\mathbb{E}'$ is normed space with $\mathbb{E}'$ a complete normed space, $f : \mathbb{E}\ \to\ \mathbb{E}'$ is a mapping such that for each fixed $x\ \in\ E$ the mapping $t\ \to\ f\ (xt)$is continuous on $\mathbb{R}$.

Assume that there exist $\epsilon >\ 0$ and $p\ \in\ [0;\ 1]$ such that,

$$\|f(x+y) - f(x) - f(y)\| \leq\ \epsilon, (\|x\|^p + \|y\|^p), \forall_{x,y} \in\ \mathbb{E}$$

Then ther exists a unique linear L: $\mathbb{E}\ \to\ \mathbb{E}'$ satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{1 - 2^{1-p}} \|x\|^p x \in\ \mathbb{E}$$

The case of the existence of a unique additive mapping had been obtained by Aoki [1], as it is recently noticed by Lech Maligranda. However, Aoki [1] had claimed the existence of a unique linear mapping, that is not true because he did not allow the mapping f to satisfy some continuity assumption. Th. M. Rassis[29], who independently introduced the unbouned di erence was the first to prove that there exists a unique linear mapping T satisfying

$$\|f(x)-T(x)\| \le \frac{\epsilon}{1-2^{1-p}}\|x\|^p x \in \mathbb{E}$$

In 1990, Th. M. Rassias [31] during the 27th International Symposium on Functional Equation asked the question whether such a theorem can also be proved for $p \geq 1$: In 1991, Z. Gajda [15] following the same approach as in Th. M. Rassias [31], gave an affirmative solution to this question for $p > 1$: It was proved by Gajda [15], as well as by Th. M. Rassias and P. Semrl [32] that one can not prove a Th. M. Rassias type therem when $p = 1$: In 1994, P. Gavruta [17] provided a further generalization of Th. M. Rassias theorem in which he replaced the bouned $\epsilon(\|x\|^p + \|x\|^p)$ by a general control function $\psi(x,y)$ for the existence of a unique linear mapping. In [12], Czerwik proved the generalizaed Hyers-Ulam stability of the quadratic functional equation. Borelli and Forti [10] generalizaed stability the result as follows [19]:

Let $\boldsymbol{G}$ be an $Abelian$ group, and $\boldsymbol{X}$ a Banach space. Assume that a mapping $\boldsymbol{f} : \boldsymbol{G} \to \boldsymbol{X}$ satisfies the functional inequality

$$\|f(x+y)+f(x-y)-2(x)-2f(y)\| \le \varphi(x,y), \forall_{x,y} \in \mathbb{G}$$

and $\varphi: \mathbb{G} \times \mathbb{G} \to [0,\infty]$ is function such that

$$\psi(x,y)=\sum_{i=0}^{\infty}\varphi(2'x,2'y)<\infty$$

$\forall_{x,y} \in \mathbb{G}$. Then there exists a unique quadratic mapping Q: $\mathbb{G} \to \mathbb{X}$ with th eproperties

$$\|f(x)+Q(x)\| \le \psi(x,x)\ \forall_{x,y} \in \mathbb{G}$$

Here, we cannot fail to notice that S-M. Jung [19] dealt with stability problem for the quadratic function equation of pexider type

$$f_1(x+y)+f_2(x-y)=f_3(x)-f_4(y)$$

In addition, the conditional stability of quadratic equation and stability of the quadratic mappings in Banach modules were stdied by M. S. Mosilehian [22] and C. Park [27]. next In 2007 Mohammad Sal Moslehian, Themistocles M. Rassias [21] proved the generalized Hyers-Ulam stability of Cauchy additive functional equation and quadratic func- tional equation. Recently, in [3-6, 21] the authors studied the Hyers-Ulam stability for the following functional equations

$$f(x+y)=f(x)+f(y) \quad 1.3$$

and

$$f(x+y)+f(x-y)=2f(x)+2f(y) \quad 1.4$$

Next

$$f\left(\frac{x+y}{2}+z\right)=f\left(\frac{x+y}{2}\right)+f(z) \quad 1.5$$

and

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)=2f\left(\frac{x+y}{2}\right)+2f(z) \quad 1.6$$

in non-Archimedean spaces. So that we solve and proved the Hyers-Ulam type stability for functional equation (1.1) and (1.2) ie the functional equations with $2k$-variables. Under suitable assumptions on spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$, we will prove that the mappings satisfying the functional equations (1.1) or (1.2).

Thus, the results in this paper are generalization of those in [3-6, 21] for functional equations with $2k$-variables.

The paper is organized as follows:

In section preliminaries we remind some basic notations in [3-6, 11, 20] such as Non Archimedean field, Non-Archimedean normed space and Non-Archimedean Banach space.

Section 3 we prove the generalized Hyers-Ulam stability of the Cauchy type additive functional equation (1.1) when $\boldsymbol{G}$ is an additive semigroup and $\boldsymbol{X}$ Non-Archimedean Banach space.

Section 4 we prove the generalized Hyers-Ulam stability of the quadratic type functional equation (1.2) when G is an additive group and $\boldsymbol{X}$ Non-Archimedean Banach space.

2. Preliminaries

2.1. Non-Archimedean normed and Banach spaces. In this subscetion we recall some basic notations [11, 20] such as Non-Archimedean fields, Non-Archimedean normed spaces and Non-Archimedean normed spaces.

A valuation is a function | . | from a field $\mathbb{K}$ into k $[0;1)$ such that 0 is the unique element having the 0 valuation,

$$|r| = 0 \leftrightarrow r = 0$$

$$|rs| = |r|\,|s|\,\forall_r, s \in \mathbb{K}$$

And the triangle inequality holds, ie.,

$$|r+s| \le |r| + |s|\,\forall_r, s \in \mathbb{K}$$

A field K is called a valued filed if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inaquality. If the tri triangle inequality is replaced by

$$|r+s| \le \max\{|r|, |s|\}\,\forall_r, s \in \mathbb{K}$$

then the function is | . | called a norm-Archimedean valuational, and filed. Clearly $|\,1\,| = |-1| = 1\ and\ |n| \le 1\ \forall_n \in N$. A trivial expamle of a non- Archimedean valuation is the function talking everything except for 0 into 1 and |0| = 0 this paper, we assume that the base field is a non-Archimedean filed, hence call it simply a filed.

Definition 2.1 Let be a vector space over a filed **K** with a non-archimedean | . |. A function $\|.\|: X \to [0, \infty)$ is said a non-archimedean norm if it satisifies the following conditions

$$(1)\ \|x\| = 0\ if\ and\ only\ if\ x = 0;$$

$$(2)\ \|r, x\| = |r|\,\|x\|(r \in \mathbb{K}, x \in X);$$

$$(3)\ \|x+y\| \le \max\{\,\|x\|, \|y\|\}\,x, y\ \in X\ hold.$$

Then $(X, \|.\|)$called a norm archimedean norm space .Due to the facti that

$$\|x_n + x_m\| \le \max\{\|x_{j+1} + x_j\|: m \le j \le n-1\}\ (n > m)$$

Definition 2.2. Let $\{x_n\}$, be a sequence in a norm -Archimedean normed space **X**.

(1) A sequence $\{x_n\}_{n=1}^{\infty}$in a non -Archimedean space is a Cauchy sequence if the $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero

(2) The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer **N** and $x\ \in\ X$ such that

$$\|x_n + x\| \leq \epsilon. \forall n \geq N,$$

for all $n, m \geq N$. Then the point $x \in X$ is called the limit of sequence x_n, which is denoted by $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in $\boldsymbol{X}$ convergent, then the norm -Archimedean normed space $\boldsymbol{X}$ is called a norm -Archimedean Bnanch space.

2.2. Solutions of the inequalities. The functional equation

$$f\ (x + y) = f\,(x) + f\,(y)$$

is called the Cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an $additive\ mapping$.

The functional equation

$$f\ (x + y) + f\ (x - y) = 2f\ (x) + 2f\ (y)$$

is called the quadratic equation. In particular, every solution of the quadratic equation is said to be a $quadratic\ mapping$.

MAIN RESULTS

3. STABILITY of THE CAUCHY TYPE ADDITIVE FUNCTIONAL EQUATION

In this section, assume that $\boldsymbol{X}$ is an additive semigroup and $\boldsymbol{Y}$ is a complete non- Archimedean space.

Theorem 3.1. Let $L : X^{2k} \to [0, \infty)$ be a functional such that

$$\lim_{n\to\infty} \frac{L:(2k)^n x_1,(2k))^n x_2 \ldots\ldots (2k)^n x_{2k})}{|2k|^2} \tag{3.1}$$

For all$x_1, x_2, \ldots \ldots x_{2k} \in X\ and\ let\ each\ x \in X\ then\ limit$

$$\phi(x) = \lim_{n\to\infty} \max\{\frac{L:(2k)^j x,(2k))^j x \ldots\ldots (2k)^j x)}{|2k|^j}\}\ 0 \leq j < n \tag{3.2}$$

Exists, suppose that $f: X \to Y\ be\ a\ mapping\ satisfying$

$$\left\| f\left(\sum_{i=1}^{k} x_i + \frac{1}{K}\sum_{i=1}^{k} x_{k+i}\right) - \sum_{i=1}^{k} f\ (x_i) - \sum_{k=1}^{k} f\ \left(\frac{x_{k+i}}{k}\right)\ \right\| \leq L(x_1, x_2 \ldots \ldots x_{2k}) \tag{3.3}$$

then let there exists an additive mapping $T : X \to Y$ such that

$$\|\ f(x) - T(x)\| \leq \frac{1}{|2k|}\phi(x) \tag{3.4}$$

for all $x \in X$. Moreover, if

$$\lim_{p\to\infty}\lim_{n\to\infty}\ \max\{\frac{L:(2k)^j x,(2k))^j x \ldots\ldots (2k)^j x)}{|2k|^j};\ p \leq j < n + p\} = 0 \tag{3.5}$$

then T is the unique additive mapping satisfying (3.4)

$Proof$. Putting $xi = x$ and $xi + 1 = kx$ for all $i = 1;\ 2; ::: ;\ k\ in$ (3.3), we get

$$\|\ f(2kx) - 2kf(x)\| \leq L(x, x, \ldots..x) \tag{3.6}$$

for all $x \in X$. Replacing x by $(2k)^{n-1}x$ in (3.6), we obtain

$$\left\|\frac{f(2k)^n x}{(2k)^n} - \frac{f((2k)^{n-1}x)}{(2k)^{n-1}}\right\| \leq \frac{L((2k)^{n-1}x,(2k)^{n-1}x, \ldots..(2k)^{n-1}x)}{|2k|^n} \tag{3.7}$$

It following from (3.1) and (3.7) that the sequence $\{\frac{f((2k)^n x)}{(2k)^n}\}$ is Cauchy sequence.

Since Y is complete, we conlude that $\{\frac{f((2k)^n x)}{(2k)^n}\}$ is convergent. Set

$$\mathrm{T(x)} = \lim_{n\to\infty} \frac{f((2k)^n x)}{(2k)^n}$$

Using induction one can show that

$$\left\|\frac{f(2k)^n x}{(2k)^n} - f(x)\right\| \le \frac{1}{|2k|}\max\left\{\frac{L((2k)^p x, (2k)^p x, \dots..(2k)^p x)}{|2k|^p}; 0 \le p < n\right\} \quad (3.8)$$

for all $n \in \mathbb{N}$ and all $x \in X$. By taking n to approach infinity in (3.8), and using (3.2), one obtains (3.4). Replacing x_i and x_{k+i} by $(2k)^n x_i$ and $2k)^n x_{k+1}$, respectively, in (3.3)

$$\left\|\frac{1}{(2k)^n} f\left(\sum_{i=1}^{k}(2k)^n x_i + \frac{1}{K}\sum_{i=1}^{k}(2k)^n x_{k+i}\right) - \frac{1}{(2k)^n}\sum_{i=1}^{k} f\ ((2k)^n x_i) - \frac{1}{(2k)^n}\sum_{k=1}^{k} f\left(\frac{(2k)^n x_{k+i}}{k}\right)\right\| \le \frac{L((2k)^n x_1, (2k)^n x_2 \dots\dots (2k)^n x_{2k})}{|2k|^n}$$

Taking the limit as n → ∞ and using (3.1) we get

$$f\left(\sum_{j=1}^{k} x_j + \frac{1}{n}\sum_{i=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f\ (x_j) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) \quad (3.9)$$

$for\ all\ x1, x2, \dots\dots x_{2k} \in X\ To\ prove\ the\ uniqueness\ property\ of\ T, let\ P: X \to \boldsymbol{Y}\ be\ another\ function\ satisfying\ (3.4). Then$

$$\begin{aligned}\| T(x) - P(x)\| &= \lim_{n\to\infty}|2k|^{-n}\| T((2k)^n x) - P((2k)^n x)\| \\ &\le \lim_{n\to\infty}|2k|^{-n}\max\{\| T((2k)^n x) - f((2k)^n x)\|, \ \| f((2k)^n x) - P((2k)^n x)\|\} \\ &\le \frac{1}{|2k|}\ \lim_{p\to\infty}\lim_{n\to\infty}\ \max\{\frac{L: \left((2k)^j x, (2k))^j x \dots\dots..(2k)^j x\right)}{|2k|^j}; p \le j \\ &< n + p\} = 0\end{aligned}$$

for all $x \in X$. Therefore $T = P$, and the proof is complete.

Corollary 3.2. Let $\beta: [0, \infty) \to [0, \infty)$ be mapping satisfying

$$\beta(|2k|) \le \beta(|2k|)\beta(t)(t \ge 0)\ and\ \ \beta(|2k|) < |2k|$$

Let $\delta > 0, \boldsymbol{X}$ be a normed space and $f: X \to Y$ fulfill the inequality

$$\left\|f\left(\sum_{i=1}^{k} x_i + \frac{1}{K}\sum_{i=1}^{k} x_{k+i}\right) - \sum_{i=1}^{k} f\ (x_i) - \sum_{k=1}^{k} f\left(\frac{x_{k+i}}{k}\right)\right\| < \beta\left(\|x_i\| + \frac{1}{k}\sum_{i=1}^{k}\|x_{k+i}\|\right) \quad (3.10)$$

for all x_1, x_2 ,......x_{2k} ∈ X. Then exists a additive mapping T : X → Y such that

$$\| f(x) - T(x)\| \le \frac{2}{|2k|}\delta\beta(\|x\|)$$

for all x ∈ X

Proof. Defining $L: x^{2k} \to [0, \infty)\ by$

$$L(x_1, x_2, \dots., x_{2k}) := \delta(\sum_{i=1}^{k}\beta\|x_i\| + \frac{1}{k}\sum_{i=1}^{k}\beta\|x_{k+i}\|$$

then we have

$$\lim_{n\to\infty} \frac{L((2k)^n x_1,(2k)^n x_2 \ldots\ldots (2k)^n x_{2k})}{|2k|^n} \le \lim_{n\to\infty} \left(\frac{\beta(|2k|)}{|2k|}\right)^2 L(x_1, x_2, \ldots., x_{2k}) \quad (3.11)$$

$$\phi(x) = \lim_{n\to\infty} \max\{\frac{L:(2k)^j x,(2k))^j x \ldots\ldots (2k)^j x)}{|2k|^j};\ 0 \le j < n\} = L(x, x, \ldots x) \quad (3.12)$$

$$\lim_{p\to\infty}\lim_{n\to\infty} \max\{\frac{L:\left((2k)^j x,(2k)\right)^j x \ldots \ldots. (2k)^j x)}{|2k|^j};\ p \le j < n + p = 0$$

$$= \lim_{n\to\infty} \frac{((2k)^n x_1, (2k)^n x_2 \ldots \ldots (2k)^n x_{2k})}{|2k|^n} = 0$$

Applying Theorem (3.1) we conclude the required result.

4. Stability of the quadratic type functional equation

In this section, assume that $\boldsymbol{G}$ is an additive group and $\boldsymbol{X}$ is a complete non-Archimedean space.

Theorem 4.1. Let $L : G^{2k} \to [0, \infty)$ be a functional such that

$$\lim_{n\to\infty} \frac{L:(2k)^n x_1,(2k))^n x_2 \ldots\ldots (2k)^n x_{2k})}{|4k|^2} \quad (4.1)$$

For all$x_1, x_2, \ldots \ldots x_{2k} \in G$ *and let each* $x \in G$ *then limit*

$$\phi(x) = \lim_{n\to\infty} \max\{\frac{L:(2k)^j x,(2k))^j x \ldots\ldots (2k)^j x)}{|4k|^j}\ 0 \le j < n\} \quad (4.2)$$

Exists, suppose that $f: G \to Y$ *be a mapping satisfying* $f(0) = 0$ and

$$\left\| f\left(\sum_{i=1}^k x_i + \frac{1}{K}\sum_{i=1}^k x_{k+i}\right) + f\left(\frac{1}{k}\sum_{i=1}^k x_{k+i} - \sum_{i=1}^k x_i\right) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f\left(\frac{x_{k+i}}{k}\right) \right\| \le \delta\left(\sum_{i=1}^k \gamma\ \|x_i\| + \frac{1}{k}\sum_{i=1}^k \gamma\ \|x_{k+i}\|\right) \quad (4.3)$$

then there exists a quadratic mapping T : G → X such that

$$\| f(x) - T(x)\| \le \frac{1}{|4k|}(\delta\gamma(\|x\|)^2 \quad (4.4)$$

for all $\boldsymbol{x} \in \boldsymbol{G}$, moreover, if

$$\lim_{p\to\infty}\lim_{n\to\infty} \max\{\frac{L:(2k)^j x,(2k))^j x \ldots\ldots (2k)^j x)}{|4k|^j};\ p \le j < n + p\} = 0 \quad (4.5)$$

then $\boldsymbol{T}$ is the unique quadratic mapping satisfying (4.4)

Proof. Putting $x_i = x$ and $x_{+1i} = kx$ for all i = 1,2, … …. , k in (4.3), we get

$$\| f(2kx) - 4kf(x)\| \le L(x, x,, x) \quad (4.6)$$

for all $\boldsymbol{x} \in \boldsymbol{G}$. *Replacing* x *by* $(2k)^{n-1}x$ *in* (4.6), *we obtain*

$$\left\|\frac{f(2k)^n x}{(4k)^n} - \frac{f((2k)^{n-1}x)}{(4k)^{n-1}}\right\| \le \frac{L((2k)^{n-1}x,(2k)^{n-1}x, \ldots.(2k)^{n-1}x)}{|4k|^n} \quad (4.7)$$

It following from (4.1) and (4.7) that the sequence $\{\frac{f((2k)^n x)}{(4k)^n}\}$ is Cauchy sequence.

Since $\boldsymbol{X}$ is complete, we conlude that $\{\frac{f((4k)^n x)}{(4k)^n}\}$ is convergent. Set

$$\mathrm{T(x)} = \lim_{n\to\infty} \frac{f((2k)^n x)}{(4k)^n}$$

Using induction one can show that

$$\left\|\frac{f(2k)^n x}{(4k)^n}-f(x)\right\| \leq \frac{1}{|4k|}\max\left\{\frac{L((2k)^p x,(2k)^p x, \ldots..(2k)^p x)}{|4k|^p}; 0 \leq p < n\right\} \quad (4.8)$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (4.8), and using (4.2),

one obtains (4.4). Replacing x_i and x_{k+i} by $(2k)^n x_i$ and $(2k)^n x_{k+1}$, respectively, in (4.3)

$$\left\|\frac{1}{(4k)^n} f\left(\sum_{i=1}^{k}(2k)^n x_i+\frac{1}{K}\sum_{i=1}^{k}(2k)^n x_{k+i}\right)-\frac{1}{(4k)^n}\sum_{i=1}^{k} f\ ((2k)^n x_i)\right.$$

$$\left.-\frac{1}{(4k)^n}\sum_{k=1}^{k} f\ \left(\frac{(2k)^n x_{k+i}}{k}\right)\right\| \leq \frac{L((2k)^n x_1,(2k)^n x_2 \ldots \ldots (2k)^n x_{2k})}{|4k|^n}$$

Taking the limit as n → ∞ and using (4.1) we get

$$f\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j}+\sum_{i=1}^{k} x_i+\frac{1}{k}\right)+f\left(\frac{1}{k}\sum_{i=1}^{k} x_{k+j}-\sum_{i=1}^{k} x_i\right)=2\sum_{i=1}^{k} f\ \left(\frac{x_{k+j}}{k}\right)+2\ \sum_{i=1}^{k} x_i \quad (4.9)$$

$for\ all\ x1, x2, \ldots \ldots x_{2k}\ \in\ G.\ To\ prove\ the\ uniqueness\ property\ of\ T, let\ P:\ G$
$\rightarrow\ \boldsymbol{Y}\ be\ another\ function\ satisfying\ (4.4). Then$

$$\| T(x)-P(x)\| = \lim_{n\to\infty}|4k|^{-n}\| T((2k)^n x)-P((2k)^n x)\|$$
$$\leq \lim_{n\to\infty}|4k|^{-n}\max\{\| T((2k)^n x)-f((2k)^n x)\|,\quad \| f((2k)^n x)-P((2k)^n x)\|\}$$
$$\leq \frac{1}{|4k|}\quad \lim_{p\to\infty}\lim_{n\to\infty}\ \max\{\frac{L:\left(2k\right)^j x,(2k))^j x \ldots \ldots..(2k)^j x)}{|4k|^j};\ p \leq j$$
$$< n+p\} = 0$$

for all $x \in G$. Therefore $T = P$, and the proof is complete.

Corollary 4.2. Let $\gamma:[0,\infty) \to [0,\infty)$ be mapping satisfying

$$\gamma(|2k|) \leq \gamma(|2k|)\gamma(t)(t \geq 0)\ and\ \ \gamma(|2k|) < |2k|$$

Let $\delta > 0, \boldsymbol{G}$ be a normed space and be an even mapping satisfying $f: G \to Y$ fulfill f(0)=0 and the inequality

$$\left\|f\left(\frac{1}{k}\sum_{i=1}^{k} x_{k+i}+\sum_{i=1}^{k} x_i\right)+f\left(\frac{1}{k}\sum_{i=1}^{k} x_{k+i}-\sum_{i=1}^{k} x_i\right)-2\sum_{i=1}^{k} f\left(x_i\right)-2\sum_{i=1}^{k} f\left(\frac{x_{k+i}}{k}\right)\right\|$$
$$\leq \delta\Big(\sum_{i=1}^{k}\gamma\left\|x_i\right\|+\frac{1}{k}\sum_{i=1}^{k}\gamma\left\|x_{k+i}\right\|\Big) \quad (4.10)$$

$$\left\|f\left(\frac{1}{k}\sum_{i=1}^{k}x_{k+j}+\sum_{i=1}^{k}x_i\right)+f\left(\frac{1}{k}\sum_{i=1}^{k}x_{k+j}-\sum_{i=1}^{k}x_i\right)-2\sum_{i=1}^{k}f\ (x_i)-2\sum_{i=1}^{k}f\left(\frac{x_{k+j}}{k}\right)\right\|$$
$$<\delta\left(\sum_{i=1}^{k}\gamma\|x_i\|+\frac{1}{k}\sum_{i=1}^{k}\gamma\|x_{k+i}\|\right) \quad (4.10)$$

for all x_1, x_2 ,……x_{2k} ∈ G. Then exists aquadratic mapping T : G → Y such that

$$\|f(x)-T(x)\|\leq\frac{2}{|4k|}\delta\gamma(\|x\|)^{\,2}$$

for all x ∈ G

Proof. Defining $L: G^{2k} \to [0,\infty)\ by$

$$L(x_1,x_2,\dots,x_{2k}) := \delta(\sum_{i=1}^{k}\gamma\|x_i\|+\frac{1}{k}\sum_{i=1}^{k}\gamma\|x_{k+i}\|$$

then we have

$$\lim_{n\to\infty}\frac{L((2k)^n x_1,(2k)^n x_2\dots\dots(2k)^n x_{2k})}{|4k|^n}\leq\lim_{n\to\infty}\left(\frac{\beta(|2k|)}{|2k|}\right)^{2n}L(x_1,x_2,\dots,x_{2k}) \quad (4.11)$$

$$\phi(x)=\lim_{n\to\infty}\max\{\frac{L:(2k)^j x,(2k))^j x\dots\dots\dots(2k)^j x)}{|4k|^j};\ 0\leq j<n\}=L(x,x,\dots x) \quad (4.12)$$

$$\lim_{p\to\infty}\lim_{n\to\infty}\ \max\{\frac{L:\left((2k)^j x,(2k))^j x\dots\dots.(2k)^j x\right)}{|4k|^j};\ p\leq j<n+p\}$$
$$=\lim_{n\to\infty}\ \frac{((2k)^n x_1,(2k)^n x_2\dots\dots(2k)^n x_{2k})}{|4k|^n}=0 \quad (4.13)$$

Applying Theorem (3.1) we conclude the required result.

References

[1] T. Aoki, On the stability of the linear transformation in Banach space, J. Math. Soc. Japan 2(1950), 64-66.

[2] Y:Aribou; S:Kabbaj. Generalized functional in inequalities in Non-Archimedean normed spaces, Applied Mathematics Letters 2, (2018) Pages: 61-66.

[3] Ly V an An. Hyers-Ulam stability of functional inequalities with three variable in Banach spaces and Non-Archemdean Banach spaces International Journal of Mathematical Analysis Vol.13, 2019, no.11. 519-53014), 296-310. .https://doi.org/10.12988/ijma.2019.9954.[4] Ly V an An. Hyers-Ulam stability additive -functional inequalities with three variable in Banachspaces and Non-Archemdean Banach spaces International Journal of Mathematical Analysis Vol.14,2020, no. 5-8. 519-53014), 296-310. .https://doi.org/10.12988/ijma.2020.91169.

[5] Ly V an An. Hyers-Ulam stability quadratic -functional inequalities with three variable in -homogeneous normed spaces International Journal of Mathematic Research. ISSN 0976-5840 Volume12, Number 1 (2020), pp. 47-64 .https://www.irphouse.com.

[6] Ly V an An. generalized hyers-ulam type stability of the 2k variable Additive -functional inequalities and equations in complex banach spaces International Journal of Mathematic Trends and Technology (IJMTT)-Volume 66 Issue 7-July 2020 .http:wwwijmttjournal.org.

[7] A.Bahyrycz, M. Piszczek, Hyers stability of the Jensen function equation, Acta Math. Hungar.,142 (2014),353-365.

[8] Badora, R.: On approximate ring homomorphisms. J Math. Anal. Appl., 276, 589-597 (2000) .

[9] M.Balcerowski, On the functional equations related to a problem of z Boros and Z. Droczy, Acta Math. Hungar.,138 (2013), 329-340.

[10] C. Borelli, G. L. Forti: On the Hyers-Ulam general stability reusult Internat. J. Math. Math. Sci.18 (1995), 229-236.

[11] Y:J:Cho;Choonkil Park; and R:Saadati functional in equalities in Non-Archimedean normed spaces, Applied Mathematics Letters 23(2010), 1238-1242.

[12] S:Czerwik on the stability of the quadratic mapping in normed spaces. Abh. Math. Sem. Univ Ham-burg, 62 (1992), 59-64.

[13] W. Fechner, On some functional inequalities related to the logarithmic mean, Acta Math., Hungar.,128 (2010,)31-45, 303-309.

[14] Attila Gilanyi, On a problem by K. Nikodem, Math. Inequal. Appl., 5 (2002), 707-710.

[15] Z. Gajda: On stability of additive mappings, Internat. J. Math. Sci.,14 (1991), 431-434.

[16] Attila Gilanyi, Eine zur parallelogrammleichung aquivalente ungleichung, Aequations 5 (2002),707-710. https: //doi.org/ 10.7153/mia-05-71 .

[17] Pascus. Gavruta, A generalization of the Hyers-Ulam -Rassias stability of approximately additive mappings, Journal of mathematical Analysis and Aequations 184 (3) (1994), 431-436. https://doi.org/10.1006/jmaa.1994.1211 .

[18] Donald H. Hyers, On the stability of the functional equation, Proceedings of the National Academy of the United States of America, 27 (4) (1941), 222.https://doi.org/10.1073/pnas.27.4.222,

[19] S. M. Jung: Stability of the quadratic equation of Pexider type. Abh. Math. Sem Univ. Hamburg, 70 (2000), 175-190.

[20] Jung Rye Lee, Choonkil Park, and Dong Yun Shin. Additive and quadratic functional in equalities inNon-Archimedean normed spaces, International Journal of Mathematical Analysis, 8 (2014), 1233-1247.

[21] Mohammad Sal Moslehian, Themistocles M Rassias. stability of the functional equations in non Archimedean spaces Applicable analysis and discrete mathematic, 1 (2007) 325-334 vailable electronically at http: // pefmth.etf.bgac.yu.

[22] Mohammad Sal Moslehian, on the orthogonal stability of the Pexiderized quadratic J.Di er equationsAppl., 11 (2005) 999-1004.

[23] L. Maligranda. Tosio Aoki (1910-1989) In International symposium on Banach and function spaces:14/09/2006-17/09/2006, pages 1{23. Yokohama Publishers, 2008.

[24] A. Najati and G. Z. Eskandani. Stability of a mixed additive and cubic functional equation in quasiBanach spaces J. Math. Anal. Appl., 342(2):1318{1331, 2008.

[25] C. Park,Y. Cho, M. Han. Functional inequalities associated with Jordan-von Newman-type additive functional equations , J. Inequality .Appl. ,2007(2007), Article ID 41820, 13 pages.

[26] Choonkil: Park. Additive -functional inequalities, Journal of Nonlinear Science and Appl. 7(2014), 296-310.

[27] C. Park, On the stability of the linear mapping in Banach modules, J. Math. Anal Appl. 275 (2002),711-720. htt://dx.doi.org/10.1016/S0022-247X(02)/00386-144, . .

[28] W. P and J. Schwaiger, A system of two inhomogeneous linear functional equations, Acta Math. Hungar 140 (2013), 377-406.

[29] Th.M Rassias, On the stability of the linear mappings in Banach space, Proc. Amer. Math. So. 72 (1978) 297-300 .

[30] Th.M. Rassias and P.Semrl, on the Hyers-Ulam stability of linear mappings, j.Math. Anal. Appl 173(1993) 325-338.

[31] Th.M. Rassias Problem 16; 2, Report of the 27th International Symp on functional equations AequationMath (1990), 292-293; 309.

[32] Th.M. Rassias, P.Semri: on the behaviour of mappings which do not satisfy Hyers-Ulam stability Proc Amer . Math. Soc., 114 (1992), 989-993.

[33] Jurg Ratz On inequalities assosciated with the jordan-von neumann functional equation, Aequationes matheaticae, 66 (1) (2003), 191-200 https;// doi.org/10-1007/s00010-0032684-8.

[34] S M. ULam. A collection of Mathematical problems, volume 8, Interscience Publishers. New York,1960.

[35] Deng Hua ZHANG and Huai Xin CAO stability of function equations in several variables Acta Mathmatica, English series Feb, 2007, vol. 23, No. 2, pp. 321-326 Published online: Nov. 6, 2006 DOI: 10. 1007/s10114-005-0862-4 https://www. ActaMath.com.