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THE LOGISTIC GOMPERTZ DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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ABSTRACT

We have created a continuous three-parameter univariate distribution called Logistic Gompertz distribution. We have illustrated some mathematical and statistical properties of the distribution such as the probability density function, cumulative distribution function and reliability function, quantile function, skewness, and kurtosis measures. The parameters of the proposed distribution are estimated using three well-known methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods. The goodness of fit of the proposed distribution is also evaluated by comparing it with some other existing distributions using a real data set.

Keywords: Logistic distribution, Gompertz distribution, Hazard function, LSE, CVME.

1. INTRODUCTION

The logistic distribution is a single variate continuous distribution and it has been used in many different areas such as logistic regression, logit models and neural networks. It has been also used in the physical sciences, life sciences, sports modeling, and recently in finance as well as insurance. The logistic distribution has thicker tails than a normal distribution so it is more flexible with the underlying data and provides better insight into the likelihood of extreme events.

Let X be a non-negative random variable satisfies the logistic distribution with shape parameter $\vartheta > 0$, and its cumulative distribution function is given by

$$F(y;\theta)=\frac{1}{1+e^{-\theta y}};\quad \theta>0, y\in\Re \tag{1}$$

and its corresponding PDF is

$$f(y;\theta)=\frac{\theta e^{-\theta y}}{\left(1+e^{-\theta y}\right)^2};\quad \theta>0, y\in\Re \tag{2}$$

Tahir et al. (2016) has defined a new generating family of continuous distributions generated from a logistic random variable called the *logistic-X family*. Its density function can be symmetrical, left-skewed, right-skewed and reversed-J shaped, and can have increasing, decreasing, bathtub and upside-down bathtub hazard rates shaped. Mandouh (2018) has introduced Logistic-modified Weibull distribution which is flexible for survival analysis as compared to modified Weibull distribution. Joshi & Kumar (2020) have introduced the Lindley exponential power distribution having a more flexible hazard rate function. Mansoor et al. (2019) have introduced a three-parameter extension of the exponential distribution which contains as sub-models the exponential, logistic-exponential and Marshall-Olkin exponential distributions. The distribution is very flexible and its associated density function can be decreasing or unimodal. Chaudhary & Kumar (2020) have presented the half logistic exponential extension distribution using the parent distribution as exponential extension distribution.

Lan and Leemis (2008) have presented an approach to defining the logistic compounded model and introduced the logistic–exponential survival distribution. This has several useful probabilistic properties for lifetime modeling. Unlike most distributions in the bathtub and upside-down bathtub classes, the logistic–exponential distribution exhibit closed-form density, hazard, cumulative hazard, and survival functions. The survival function of the logistic–exponential distribution is

$$S(x;\lambda)=\frac{1}{1+\left(e^{\lambda x}-1\right)^{\alpha}};\quad \alpha>0,\lambda>0,x\ge 0 \tag{3}$$

In this study, we have taken the Gompertz distribution as a parent distribution which is one of the classical probability distribution that represents survival function based on laws of mortality. This distribution performs a considerable role in modeling human mortality and analyzing actuarial tables. The Gompertz distribution was first introduced by (Gompertz, 1824). It has been used as a growth model and also used to fit the tumor growth. The Gompertz function reduced a significant collection of data in life tables into a single function. It is based on the assumption that the mortality rate decreases exponentially as a person ages. The resulting Gompertz function is for the number of individuals living at a given age as a function of age. Applications and an extensive survey of the Gompertz distribution can be found in (Ahuja & Nash, 1967). Cooray and Ananda (2010) have introduced the Gompertz-sinh family and it was used to analyze the survival data with highly negatively skewed distribution. El-Gohary et al. (2013) have presented a flexible called the generalized Gompertz distribution it has increasing or constant or decreasing or bathtub curve failure rate depending upon the shape parameter. Ieren et al. (2019) have introduced a three-parameter power Gompertz distribution using a power transformation approach.

Using the same approach used by (Lan & Leemis, 2008) we have introduced the new distribution called Logistic Gompertz (LGZ) distribution. The main aim of this study is to introduce a more flexible distribution by inserting just one extra parameter to the Gompertz distribution to attain a better fit for the lifetime data sets. We have discussed some distributional properties and its applicability. The different sections of the proposed study are arranged as follows. In Section 2 we present the Logistic Gompertz (LGZ) distribution and its various mathematical and statistical properties. We have made use of three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises estimation (CVME) methods. For the maximum likelihood (ML) estimate, we have constructed the asymptotic confidence intervals using the observed information matrix are presented in Section 3. In Section 4, a real data set has been analyzed to explore the applications and capability of the proposed distribution. In this section, we present the estimated value of the parameters and log-likelihood, AIC, BIC and AICC criterion for ML, LSE, and CVME also the goodness of fit of the proposed distribution is also evaluated by fitting it in comparison with some other existing distributions using a real data set. Finally, in Section 5 we present some concluding remarks.

2. THE LOGISTIC GOMPERTZ (LGZ) DISTRIBUTION

The Gompertz distribution was first introduced by (Gompertz, 1824). Let X be a random variable follows the Gompertz distribution with parameters β and λ if its cumulative distribution function can be written as,

$$G(x) = 1 - \exp\left\{\frac{\lambda}{\beta}\left(1 - \exp\left(\beta x\right)\right)\right\} \ ; \beta > 0, \lambda > 0, x > 0 \tag{4}$$

and its corresponding probability density function can be expressed as,

$$g(x) = \lambda \exp\left\{\beta x + \frac{\lambda}{\beta}\left(1 - \exp\left(\beta x\right)\right)\right\}; \beta > 0, \lambda > 0, x > 0 \tag{5}$$

Using the same approach used by (Lan & Leemis, 2008) we have defined the new distribution called logistic Gompertz distribution. In this study, we have taken the Gompertz distribution as a baseline distribution.

Let X be a non-negative random variable with positive shape parameters α and β and a positive scale parameter λ then CDF of logistic Gompertz distribution can be defined as

$$F\left(x\right) = 1 - \frac{1}{1 + \left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x} - 1\right)\right\} - 1\right]^{\alpha}}; \ (\alpha, \beta, \lambda) > 0, \ x > 0 \tag{6}$$

The PDF of logistic Gompertz distribution is

$$f\left(x\right) = \frac{\alpha \lambda e^{\beta x} \exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x} - 1\right)\right\}\left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x} - 1\right)\right\} - 1\right]^{\alpha - 1}}{\left\{1 + \left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x} - 1\right)\right\} - 1\right]^{\alpha}\right\}^{2}}; \ (\alpha, \beta, \lambda) > 0, \ x > 0 \tag{7}$$

This CDF function is similar to the log-logistic CDF function with the second term of the denominator being changed in its base to Gompertz function, and hence we named it logistic Gompertz distribution.

Reliability function

The reliability function of LGZ distribution is

$$R(x)=1-F(x)$$

$$=\frac{1}{1+\left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x}-1\right)\right\}-1\right]^{\alpha}};\ (\alpha,\beta,\lambda)>0,\ x>0 \tag{8}$$

Hazard function

The failure rate function of LGZ distribution can be defined as,

$$h(x)=\frac{f(x)}{R(x)}=\frac{\alpha\lambda e^{\beta x}\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x}-1\right)\right\}\left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x}-1\right)\right\}-1\right]^{\alpha-1}}{\left\{1+\left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x}-1\right)\right\}-1\right]^{\alpha}\right\}};\ (\alpha,\beta,\lambda)>0,\ x>0 \tag{9}$$

In Fig 1, we have displayed the plots of the PDF and hazard rate function of LGZ distribution for different values of α, and λ.

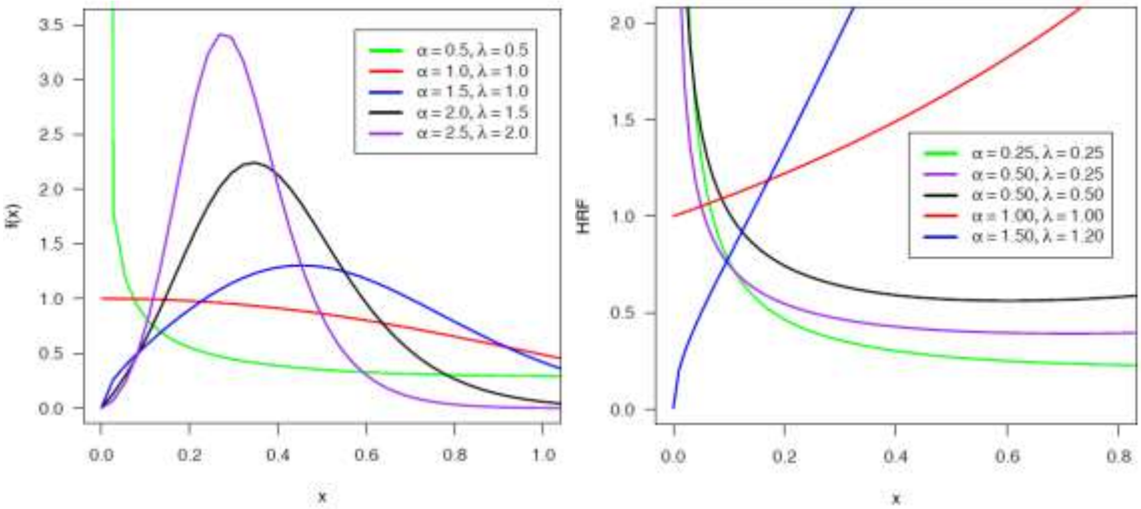

Figure 1. Plots of PDF (left panel) and hazard function (right panel) for different values of α, and λ.

Quantile function

The Quantile function of Logistic Gompertz distribution can be expressed as

$$Q(u)=\frac{1}{\beta}\ln\left[1+\frac{\beta}{\lambda}\ln\left\{\left(\frac{u}{1-u}\right)^{1/\alpha}+1\right\}\right];\quad 0<u<1 \tag{10}$$

Skewness and Kurtosis:

The measures of Skewness based on quantiles is Bowley's coefficient of skewness and it can be expressed as

$$Skewness = \frac{Q(0.75)+Q(0.25)-2Q(0.5)}{Q(0.75)-Q(0.25)} \text{ and} \tag{11}$$

The coefficient of kurtosis based on octiles which was defined by (Moors, 1988) is

$$K_u(M) = \frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}, \tag{12}$$

3. METHODS OF ESTIMATION

In this section, the parameters of the proposed distribution are estimated by applying some well-known estimation methods which are as follows

3.1 Maximum Likelihood Estimates

For the estimation of the parameter, the maximum likelihood method is the most commonly used method (Casella & Berger, 1990). Let, $x_1, x_2, ..., x_n$ is a random sample from $LGZ(\alpha,\beta,\lambda)$ and the likelihood function, $L(\alpha,\beta,\lambda)$ is given by,

$$L(\psi; x_1, x_2...x_n) = f(x_1, x_2,...x_n / \psi) = \prod_{i=1}^{n} f(x_i / \psi)$$

$$L(\alpha,\beta,\lambda) = \alpha\lambda \prod_{i=1}^{n} \frac{e^{\beta x_i} \exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x_i}-1\right)\right\}\left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x_i}-1\right)\right\}-1\right]^{\alpha-1}}{\left\{1+\left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x_i}-1\right)\right\}-1\right]^{\alpha}\right\}^2}; \ (\alpha,\beta,\lambda)>0, \ x>0$$

Now log-likelihood density is

$$\begin{aligned}\ell(\alpha,\beta,\lambda \mid \underline{x}) = n\ln(\alpha\lambda) + \beta\sum_{i=1}^{n} x_i + (\alpha-1)\ln\sum_{i=1}^{n}\left[\exp\left\{\frac{\lambda}{\beta}\left(\mathrm{e}^{\beta x_i}-1\right)\right\}-1\right] \\ + \frac{\lambda}{\beta}\sum_{i=1}^{n}\left(\mathrm{e}^{\beta x_i}-1\right) - 2\sum_{i=1}^{n}\ln\left\{1+\left[\frac{\lambda}{\beta}\left(\mathrm{e}^{\beta x_i}-1\right)-1\right]^{\alpha}\right\}\end{aligned} \tag{13}$$

Differentiating (13) with respect to α, β and λ we get,

$$\frac{\partial l}{\partial \alpha} = \frac{1}{\alpha} + \sum_{i=1}^{n}\ln A(x_i) - 2\sum_{i=1}^{n}\frac{\ln[A(x_i)][A(x_i)]^{\alpha}}{1+[A(x_i)]^{\alpha}}$$

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} x_i + \frac{\lambda}{\beta^2}\left\{(\alpha-1)\sum_{i=1}^{n}\frac{\{A(x_i)+1\}}{A(x_i)} + 1 - 2\alpha\sum_{i=1}^{n}\frac{[A(x_i)+1][A(x_i)]^{\alpha-1}}{1+[A(x_i)]^{\alpha}}\right\}\sum_{i=1}^{n}\left\{(\beta x_i - 1)\mathrm{e}^{\beta x_i}+1\right\}$$

$$\frac{\partial l}{\partial \lambda} = \frac{1}{\lambda} + \frac{(\alpha-1)}{\beta}\sum_{i=1}^{n}\frac{\{A(x_i)+1\}\left(\mathrm{e}^{\beta x_i}-1\right)}{A(x_i)} + \frac{1}{\beta}\sum_{i=1}^{n}\left(\mathrm{e}^{\beta x_i}-1\right) - 2\frac{\alpha}{\beta}\sum_{i=1}^{n}\frac{\left(\mathrm{e}^{\beta x_i}-1\right)[A(x_i)+1][A(x_i)]^{\alpha-1}}{1+[A(x_i)]^{\alpha}}$$

Where $A(x_i) = \left[\exp\left\{\frac{\lambda}{\beta}\left(\mathrm{e}^{\beta x_i}-1\right)\right\}-1\right]$

Equating the above three non-linear equations to zero and solving simultaneously for α, β and λ, we get the maximum likelihood estimate $\hat{\alpha},\ \hat{\beta}\ and\ \hat{\lambda}$ of the parameters α, β and λ. By using computer software like R, Matlab, Mathematica etc for maximization of (13) we can obtain the estimated value of α, β and λ. For the confidence interval estimation of α, β and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α, β and λ can be obtained as,

$$C=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Where

$$C_{11}=\frac{\partial^2 l}{\partial\alpha^2},\ C_{12}=\frac{\partial^2 l}{\partial\alpha\partial\beta},\ C_{13}=\frac{\partial^2 l}{\partial\alpha\lambda}$$

$$C_{21}=\frac{\partial^2 l}{\partial\beta\partial\alpha},\ C_{22}=\frac{\partial^2 l}{\partial\beta^2},\ C_{23}=\frac{\partial^2 l}{\partial\beta\partial\lambda}$$

$$C_{31}=\frac{\partial^2 l}{\partial\lambda\partial\alpha},\ C_{32}=\frac{\partial^2 l}{\partial\beta\partial\lambda},\ C_{33}=\frac{\partial^2 l}{\partial\lambda^2}$$

Let $\Omega=(\alpha,\beta,\lambda)$ denote the parameter space and the corresponding MLE Ω of $\hat{\Omega}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$ as, then $\left(\hat{\Omega}-\Omega\right)\rightarrow N_3\left[0,\left(C(\Omega)\right)^{-1}\right]$ where $C(\Omega)$ is the Fisher's information matrix. Using the Newton-Raphson algorithm to maximize the likelihood creates the observed information matrix and hence the variance-covariance matrix is obtained as,

$$\left[C(\Omega)\right]^{-1}=\begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{cov}(\hat{\alpha},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\beta}) & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\lambda}) & \text{cov}(\hat{\beta},\hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix} \tag{14}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β and λ can be constructed as,

$\hat{\alpha}\pm Z_{\alpha/2}SE(\hat{\alpha})$, $\hat{\beta}\pm Z_{\alpha/2}SE(\hat{\beta})$ and, $\hat{\lambda}\pm Z_{\alpha/2}SE(\hat{\lambda})$

where $Z_{\alpha/2}$ is the upper percentile of standard normal variate

3.2. Method of Least-Square Estimation (LSE)

The ordinary least square estimators and weighted least square estimators are proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. Here we have applied the same course of action for the LGZ distribution. The least-square estimators of the unknown parameters α, β and λ of LG distribution can be obtained by minimizing

$$W\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]^2 \tag{15}$$

with respect to unknown parameters α, β and λ.

Consider $F(X_i)$ denotes the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ where $\{X_1, X_2, \ldots, X_n\}$ is a random sample of size n from a distribution function *F*(.). The least-square estimators of α, β and λ say $\hat{\alpha}, \hat{\beta}$ *and* $\hat{\lambda}$ respectively, can be obtained by minimizing

$$W(X;\alpha,\beta,\lambda) = \sum_{i=1}^{n}\left[1-\left\{1+\left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x_i}-1\right)\right\}-1\right]^{\alpha}\right\}^{-1}-\frac{i}{n+1}\right]^2; x \ge 0, (\alpha,\beta,\lambda) > 0 \quad (16)$$

with respect to α, β and λ.

Differentiating (16) with respect to α, β and λ we get,

$$\frac{\partial W}{\partial \alpha} = -2\sum_{i=1}^{n}\left[1-\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^{-1}-\frac{i}{n+1}\right]\frac{\left[U_i(x)-1\right]^{\alpha}\ln\left[U_i(x)-1\right]}{\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial W}{\partial \beta} = -2\frac{\alpha\lambda}{\beta^2}\sum_{i=1}^{n}\left[1-\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^{-1}-\frac{i}{n+1}\right]\frac{\left[U_i(x)-1\right]^{\alpha-1}U_i(x)\{(\beta x_i-1)e^{\beta x_i}+1\}}{\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial W}{\partial \lambda} = -2\frac{\alpha}{\beta}\sum_{i=1}^{n}\left[1-\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^{-1}-\frac{i}{n+1}\right]\frac{\left[U_i(x)-1\right]^{\alpha-1}U_i(x)\{e^{\beta x_i}-1\}}{\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^2}$$

Where $U_i(x) = \exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x_i}-1\right)\right\}$

Similarly, the weighted least square estimators can be obtained by minimizing

$$W(X;\alpha,\beta,\lambda) = \sum_{i=1}^{n} w_i\left[F(X_{(i)})-\frac{i}{n+1}\right]^2$$

with respect to α, β and λ. The weights w_i are $w_i = \frac{1}{Var(X_{(i)})} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$

Hence, the weighted least square estimators of α, β and λ respectively can be obtained by minimizing,

$$W(X;\alpha,\beta,\lambda) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^{-1}-\frac{i}{n+1}\right]^2 \quad (17)$$

with respect to α, β and λ.

3.3. Method of Cramer-Von-Mises estimation (CVME)

The CVME estimators of α, β and λ are obtained by minimizing the function

$$B(X;\alpha,\beta,\lambda) = \frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\mid\alpha,\beta,\lambda\right)-\frac{2i-1}{2n}\right]^2$$

$$= \frac{1}{12n} + \sum_{i=1}^{n}\left[1 - \left\{1 + \left[\exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x_i} - 1\right)\right\} - 1\right]^{\alpha}\right\}^{-1} - \frac{2i-1}{2n}\right]^2 \qquad (18)$$

Differentiating (18) with respect to α, β and λ we get,

$$\frac{\partial B}{\partial \alpha} = -2\sum_{i=1}^{n}\left[1 - \left\{1 + \left[U_i(x) - 1\right]^{\alpha}\right\}^{-1} - \frac{2i-1}{2n}\right]\frac{\left[U_i(x)-1\right]^{\alpha}\ln\left[U_i(x)-1\right]}{\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial B}{\partial \beta} = -2\frac{\alpha\lambda}{\beta^2}\sum_{i=1}^{n}\left[1 - \left\{1 + \left[U_i(x) - 1\right]^{\alpha}\right\}^{-1} - \frac{2i-1}{2n}\right]\frac{\left[U_i(x)-1\right]^{\alpha-1}U_i(x)\{(\beta x_i - 1)e^{\beta x_i} + 1\}}{\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial B}{\partial \lambda} = -2\frac{\alpha}{\beta}\sum_{i=1}^{n}\left[1 - \left\{1 + \left[U_i(x) - 1\right]^{\alpha}\right\}^{-1} - \frac{2i-1}{2n}\right]\frac{\left[U_i(x)-1\right]^{\alpha-1}U_i(x)\{e^{\beta x_i} - 1\}}{\left\{1+\left[U_i(x)-1\right]^{\alpha}\right\}^2}$$

Where $U_i(x) = \exp\left\{\frac{\lambda}{\beta}\left(e^{\beta x_i} - 1\right)\right\}$

Solving $\frac{\partial B}{\partial \alpha} = 0, \frac{\partial B}{\partial \beta} = 0 \text{ and } \frac{\partial W}{\partial \lambda} = 0$ simultaneously we will get the CVM estimators.

4. ILLUSTRATION WITH A REAL DATASET

In this section, we illustrate the applicability of the LGZ model using a real dataset used by former researchers. We have taken 100 observations on breaking the stress of carbon fibers (in Gba) used by (Nichols & Padgett, 2006).

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

The MLEs are calculated directly by using the optim() function in R software (R Core Team, 2020) and (Ming, 2019) by maximizing the likelihood function (13). We have obtained $\hat{\alpha}$ = 2.0938, $\hat{\beta}$ = 0.3039 and $\hat{\lambda}$ = 0.1776 and the corresponding Log-Likelihood value is -141.2612. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α, β and λ.

Table 1: MLEs, SE and 95% confidence interval of parameters

Parameter	MLE	SE	95% ACI
alpha	2.0938	0.4141	(1.2821, 2.9054)
beta	0.3039	0.1530	(0.0041, 0.6037)
lambda	0.1776	0.04371	(0.0920, 0.2633)

We have displayed the graph of the profile log-likelihood function of α, β and λ in Figure 2 and observed that the MLEs are unique.

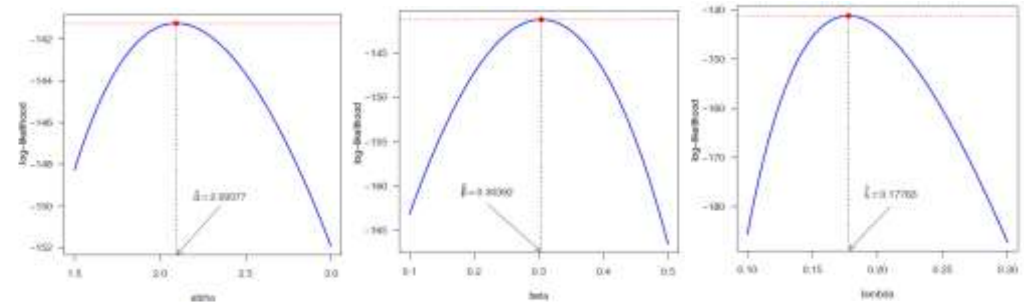

Figure 2. Graph of profile log-likelihood function of α, β and λ.

In Figure 3 we have presented the P-P plot (empirical distribution function against theoretical distribution function) and Q-Q plot (empirical quantile against theoretical quantile).

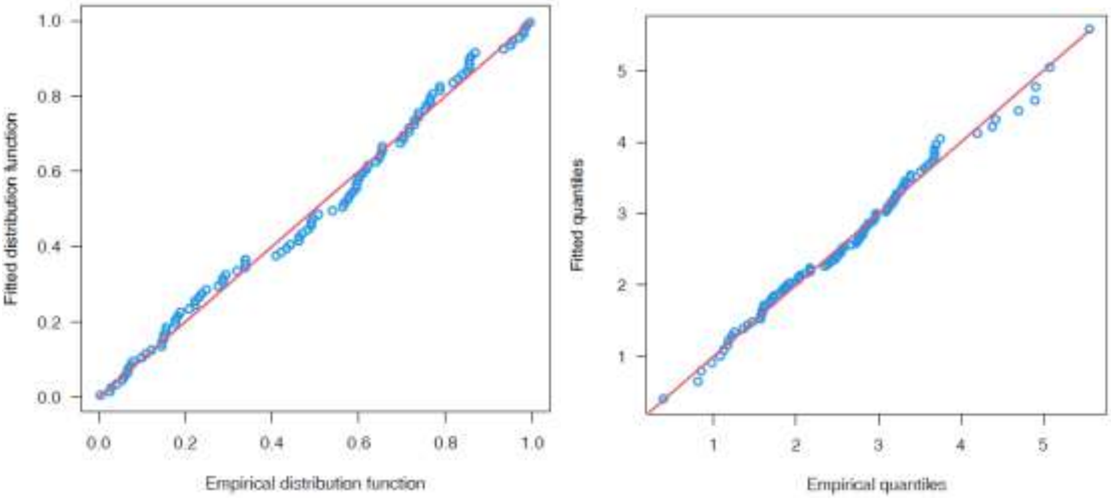

Figure 3. The P-P plot (left panel) and Q-Q plot (right panel) of LG distribution

By using MLE, LSE and CVME methods we estimate the parameter of L-R distribution. For the goodness of fit purpose we use negative log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike Information Criterion (AICC) and Hannan-Quinn information criterion (HQIC) statistic to select the best model among selected models. The expressions to calculate AIC, BIC, AICC and HQIC are listed below:

a) $AIC = -2l(\hat{\theta}) + 2k$

b) $BIC = -2l(\hat{\theta}) + k\log(n)$

c) $AICC = AIC + \dfrac{2k(k+1)}{n-k-1}$

d) $HQIC = -2l(\hat{\theta}) + 2k\log\left[\log(n)\right]$

where k is the number of parameters and n is the size of the sample in the model under consideration. Further, to evaluate the fits of the LG distribution with some selected distributions we have taken the Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistic. These statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. These statistics are calculated as

$$KS = \max_{1\le i\le n}\left(d_i - \frac{i-1}{n}, \frac{i}{n} - d_i\right)$$

$$W = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln d_i + \ln\left(1-d_{n+1-i}\right)\right]$$

$$A^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{(2i-1)}{2n} - d_i\right]^2$$

Where $d_i = CDF(x_i)$; the x_i's being the ordered observations.

In Table 2 we have displayed the estimated value of the parameters of Logistic Gompertz distribution using MLE, LSE and CVME method and their corresponding negative log-likelihood, AIC, BIC, AICC and HQIC information criteria.

Table 2: Estimated parameters, log-likelihood, AIC, BIC, AICC and HQIC

Method of Estimation	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-LL	AIC	BIC	AICC	HQIC
MLE	2.0938	0.3039	0.1776	141.2612	288.5225	296.3380	288.7725	291.6855
LSE	1.3120	0.7533	0.0844	146.5249	299.0498	306.8653	299.2998	302.2129
CVME	1.3156	0.7675	0.0822	147.0686	300.1372	307.9527	300.3872	303.3003

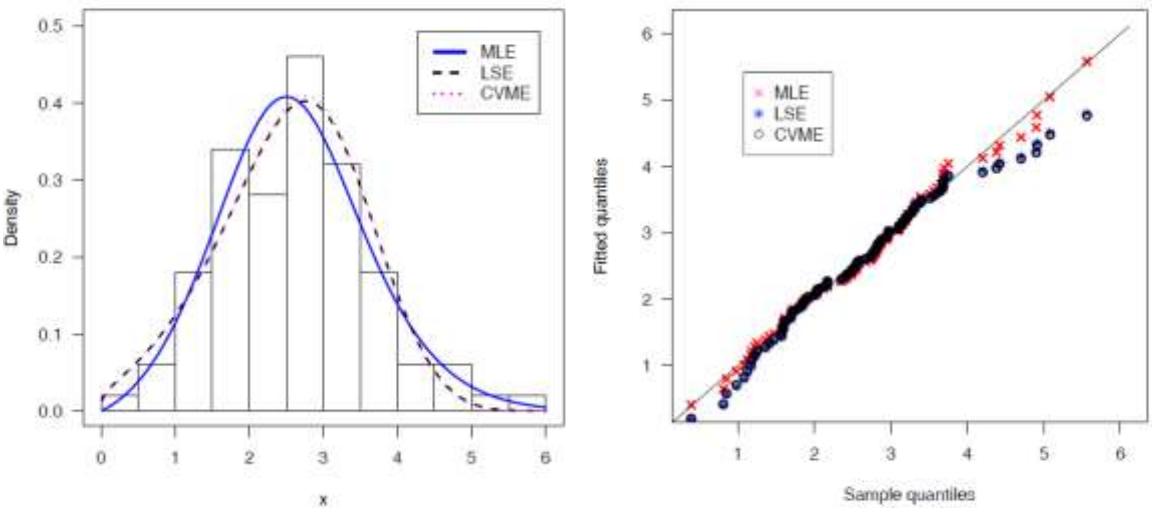

Figure 4. The Histogram and the density function of fitted distributions (left panel) and Q-Q plot of estimation methods MLE, LSE and CVME.

Table 3: The KS, AD and CVM statistic with p-value

Method of Estimation	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
MLE	0.0630(0.8220)	0.0648(0.7848)	0.3828(0.8653)
LSE	0.0473(0.9786)	0.0445(0.9097)	0.7451(0.5217)
CVME	0.0493(0.9685)	0.0439(0.9132)	0.7857(0.4911)

To illustrate the goodness of fit of the Lindley inverse exponential distribution, we have taken some well-known distribution for comparison purpose which are listed blew,

4.1. **Generalized Exponential Extension (GEE) distribution:**

The probability density function of GEE introduced by (Lemonte, 2013) having an upside-down bathtub-shaped hazard function distribution with parameters α , β and λ is

$$f_{GEE}\left(x;\alpha,\beta,\lambda\right)=\alpha\beta\lambda\left(1+\lambda x\right)^{\alpha-1} exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}$$
$$\left[1-exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}\right]^{\beta-1} \quad ;\ x\geq 0.$$

4.2. **Logistic-Exponential (LE) distribution**

The density of logistic-exponential (LE) distribution given by (Lan & Leemis, 2008) with shape parameter α and scale parameter λ is

$$f_{LE}(x)=\frac{\lambda\,\alpha\,e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1}}{\left\{1+\left(e^{\lambda x}-1\right)^{\alpha}\right\}^{2}} \quad ;x\geq 0,\,\alpha>0,\,\lambda>0.$$

4.3. **Generalized Exponential (GE) distribution**

The probability density function of generalized exponential distribution (Gupta & Kundu, 1999)

$$f_{GE}\left(x;\alpha,\lambda\right)=\alpha\,\lambda\,e^{-\lambda\,x}\left\{1-e^{-\lambda\,x}\right\}^{\alpha-1};\left(\alpha,\lambda\right)>0,x>0\cdot$$

4.4. **Exponential Power (EP) Distribution**

The probability density function Exponential power (EP) distribution (Smith & Bain, 1975) is

$$f_{EP}(x)=\alpha\,\lambda^{\alpha}\,x^{\alpha-1}\,e^{(\lambda\,x)^{\alpha}}\,\exp\left\{1-e^{(\lambda\,x)^{\alpha}}\right\} \quad ;(\alpha,\,\lambda)>0,\quad x\geq 0\cdot$$

where α and λ are the shape and scale parameters, respectively.

4.5. **Gompertz distribution (GZ)**

The probability density function of Gompertz distribution (Murthy et al., 2003) with parameters α and θ is

$$f_{GZ}\left(x\right)=\theta\;e^{\alpha\,x}\,exp\left\{\frac{\theta}{\alpha}\left(1-e^{\alpha\,x}\right)\right\} \quad ;x\geq 0,\theta>0,-\infty<\alpha<\infty.$$

For the judgment of the potentiality of the proposed model, we have presented the value of the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (AICC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4: Log-likelihood (LL), AIC, BIC, AICC and HQIC

Model	-LL	AIC	BIC	AICC	HQIC
LGZ	141.2612	288.5225	296.3380	288.7725	291.6855
GEE	141.3708	288.7416	296.5571	288.9916	291.9047
LE	143.2473	290.4946	295.7049	290.6183	292.6033
EP	145.9589	295.9179	301.1282	296.0391	298.0266
GE	146.1823	296.3646	301.5749	296.4883	298.4733
GZ	149.1250	302.2500	307.4604	302.3737	304.3588

The Histogram and the density function of fitted distributions and Empirical distribution function with the estimated distribution function of LGZ and some selected distributions are presented in Figure 5.

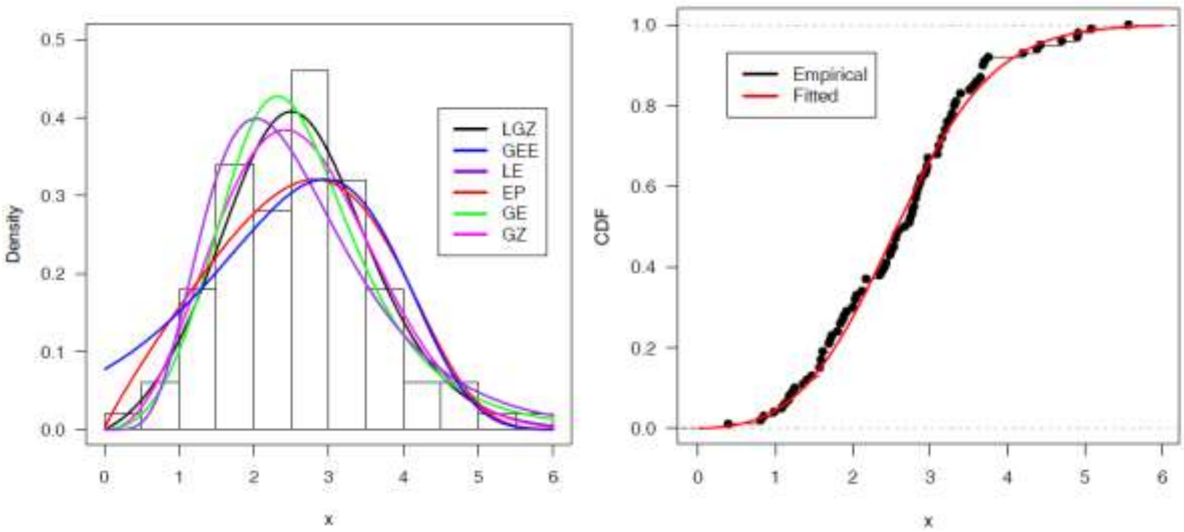

Figure 5. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the LGZ distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 5. It is observed that the LGZ distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the LGZ distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5. The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
LGZ	*0.0630(0.8220)*	0.0648(0.7848)	0.3828(0.8653)
GEE	0.0654(0.7862)	0.0723(0.7385)	0.4202(0.8281)
LE	0.0838(0.4836)	0.1225(0.4860)	0.7042(0.5549)
EP	0.0993(0.2771)	0.1861(0.2963)	1.3081(0.2297)

GE	0.1078(0.1959)	0.2293(0.2174)	1.2250(0.2581)
GZ	0.0962(0.3129)	0.2280(0.2193)	1.7537(0.1261)

5. CONCLUSIONS

In this study, we have introduced a three-parameter univariate continuous distribution named Logistic Gompertz distribution. Some distributional properties of the proposed distribution are presented such as the shapes of the probability density, cumulative density and hazard rate functions, survival function, quantile function, skewness, and kurtosis measures are derived and established and found that the proposed model is flexible and inverted bathtub shaped hazard function. The model parameters are estimated by using three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods and we concluded that the MLEs are quite better than LSE, and CVM. A real data set is considered to explore the applicability and suitability of the proposed distribution and found that the proposed model is quite better than other lifetime model taken into consideration. We hope this model may be an alternative in the field of survival analysis, probability theory and applied statistics.

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