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# DETERMINATION OF MOMENTS OF BINOMIAL MIXTURES USING RECURSIVE RELATIONS

**Andrea Otwande**
School of Physical and Applied Sciences, Kirinyaga University, P.O Box 143-10300, Kerugoya, Kenya
Email: aotwande@kyu.ac.ke
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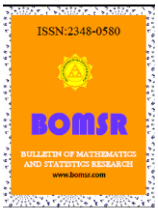


**ABSTRACT**
Mixed distributions have proved to be very important in modeling data whose distributions are heterogeneous. However determining their moments can be cumbersome due to the fact that their constituent distributions are different. This challenge can be addressed using the recursive relations which can be constructed through integration by parts technique. These recursive equations are useful in insurance industry where they are applied in calculation of total aggregated claims.
Keywords: Mixtures; integrations by parts; recursive relations; moments

## 1.0 INTRODUCTION

The process of obtaining probability distribution functions of binomial mixtures can be cumbersome given that the mixture is a combination of binomial and beta distributions which are discrete and continuous respectively. However their moments can be obtained using their recursive relations. A recursive relation is a mathematical equation which is based on a sequence such that the terms of the sequences generated depend on the immediately previous term. The recursive relations have been discussed in the literature by various researchers. Katz(1965) derived a recursive equation of the form

$$\frac{f(x+1)}{f(x)} = \frac{a+bx}{1+a} \qquad \text{where } x = 0,1,2,\ldots$$

Which could be used to generate numerous probability distributions such as Poisson distribution, negative binomial distribution, binomial distribution etc. Panjer (1981) later obtained a recursive relation of the form

$$f(x) = \left(a + \frac{b}{x}\right) f(x-1)$$

which he applied to obtain Poisson distribution, binomial distribution, negative distribution and geometric distribution using probability generating function techniques. Jewel and Sundt (1981) used Panjers recursive model of

$$xf(x) = (ax + b)f(x-1)$$

to obtain Poisson distribution, geometric distribution using iterative technique. Several other researchers such as Hesselagers(1984), Wang(1984) etc followed the same pattern of deriving recursive models and used them to obtain many distributions and mixed distributions. However there were no beta-binomial distribution mixture obtained. This paper aims to formulate beta-binomial mixture in expectation form, express beta-binomial mixture in recursive form, use the recursive relations constructed to obtain their moments which include mean and variance using integration by parts technique.

## 2.0 METHOD

### 2.1 Classical Beta-Binomial distribution

The classical beta-binomial distribution is defined as

$$g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)} \quad 0 < p < 1 \quad a, b > 0 \qquad [1]$$

Classical binomial distribution is given as

$$m(p) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0,1,2,\dots \qquad [2]$$

Beta-binomial distribution is constructed as

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)} dp \qquad [3]$$

$$= \binom{n}{x} \frac{1}{B(a,b)} \int_0^1 \binom{n}{x} p^{x+a-1}(1-p)^{n-x+b-1} dp$$

Let $I_x = \frac{f(x)B(a,b)}{\binom{n}{x}} = \int_0^1 p^{x+a-1}(1-p)^{n-x+b-1} dp$

Let $u = (1-p)^{n-x+b-1}, \quad du = -1(n-x+b-1)(1-p)^{n-x+b-2}\, dp$

$dv = p^{x+a-1}, \quad v = \frac{p^{x+a}}{x+a}$

$$= \frac{p^{x+a}}{x+a}(1-p)^{n-x+b-1}\Big|_0^1 + \int_0^1 \frac{p^{x+a}}{x+a}(n-x+b-1)(1-p)^{n-x-2} dp$$

$$= \frac{(n-x+b-1)B(a,b)}{x+a} \int_0^1 \frac{p^{x+a}(1-p)^{n-x+b-2}}{B(a,b)} dp$$

$$= \frac{n-x+b-1}{x+a} I_{x+1}$$

$$= \frac{(n-x+b-1)f(x+1)\, B(a,b)}{(x+a) \quad \binom{n}{x}}$$

$$\frac{f(x)(n-x)!x!}{n!} = \frac{(n-x+b-1)(n-x-1)!(x+1)!}{(x+a)} f(x+1)$$

$$= [(n-x)(x+a)]f(x) = [(x+1)(n-x+b-1)]f(x+1)$$

$$[(x+a)(n-x)]f(x) = [(n+b)(x+1) - (x+1)^2]f(x+1) \qquad [4]$$

Summing the equation [4] over n we get

$$n\sum_{x=0}^{n}[(n+b)(x+1) - (x+1)^2]f(x+1) = \sum\nolimits_{x=0}^{n-1}[na + (n-a)x - x^2]f(x)$$

$$(n+b)\sum_{x=0}^{n}(x+1)f(x+1) - \sum_{x=0}^{n}(x+1)^2 f(x+1) = na\sum_{x=0}^{n-1}f(x) + (n-a)\sum_{x=0}^{n-1}xf(x) - \sum_{x=0}^{n-1}x^2 f(x)$$

But

$$\sum_{x=0}^{n}xf(x) = \sum_{x=0}^{n}(x+1)f(x+1) = M_1$$

$$\sum_{x=0}^{n}x^2 f(x) = \sum_{x=0}^{n}(x+1)^2 f(x+1) = M_2$$

Therefore

$$(n+b)M_1 - M_2 = na + (n-a)M_1 - M_2$$

$$[(n+b) - (n-a)]M_1 = na$$

$$(b+a)M_1 = na$$

$$M_1 = \frac{na}{a+b}$$

Thus

$$E(X) = \frac{na}{a+b}.$$

To obtain the second moment we multiply equation [4] by $(x+1)$ and sum the result over $n$.

$$\sum_{x=0}^{n}(x+1)\,[(x+a)(n-x)]f(x) = \sum_{x=0}^{n}(x+1)\,[(n+b)(x+1) - (x+1)^2]f(x+1)$$

$$(n+b)\sum_{x=0}^{n}(x+1)^2 f(x+1) - \sum_{x=0}^{n}(x+1)^3 f(x+1)$$
$$= na\sum_{x=0}^{n}(x+1)f(x) + (n-a)\sum_{x=0}^{n}x(x+1)f(x) - \sum_{x=0}^{n}x^2(x+1)f(x)$$

$$(n+b)M_2 - M_3 = naM_1 + na + (n-a)M_2 + (n-a)M_1 - M_3 - M_2$$
$$[(n+b)+1-(n-a)]M_2 = [na+n-a]M_1 + na$$
$$M_2 = \left[\frac{na+n-a}{a+b+1}\right]\left[\frac{na}{a+b}\right] + na$$
$$M_2 = \frac{na(na+n+b)}{(a+b)(a+b+1)}$$

To get the variance we have

$$Var(X) = M_2 - (M_1)^2$$
$$= \frac{na(na+n+b)}{(a+b)(a+b+1)} - \left(\frac{na}{a+b}\right)^2$$

Therefore

$$Var(X) = \frac{nab(n+a+b)}{(a+b)^2(a+b+1)}$$

**2.2 Special cases of beta-binomial mixture**

2.2.1 ***Uniform Binomial distribution***

Uniform distribution also called rectangular distribution (Johnson, Katz and Kemp (1992) pg 272-274)

Is a special case of beta distribution obtained when the parameters $a = b = 1$

Thus

$$g(p) = \begin{cases} 1 & 0 < p < 1 \\ 0 & \text{otherwise} \end{cases}$$

The Uniform-Binomial distribution is

$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p)dp$ where $g(p)$ is the Uniform distribution.

Therefore

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} dp \qquad [5]$$

let

$$I_x = \frac{f(x)}{\binom{n}{x}} = \int_0^1 p^x (1-p)^{n-x} dp$$

Using integration by parts

Let $u = (1-p)^{n-x}$, $du = -1(n-x)(1-p)^{n-x-1}dp$

and

$$dv = p^x, \qquad v = \frac{p^{x+1}}{x+1}$$

$$= (1-p)^{n-x}\frac{p^{x+1}}{x+1}\Big|_0^1 + \int_0^1 \frac{p^{x+1}}{x+1}(n-x)(1-p)^{n-x-1}dp$$

$$\frac{f(x)}{\binom{n}{x}} = \frac{n-x}{x+1}\int_0^1 p^{x+1}(1-p)^{n-x-1}dp$$

$$= \frac{n-x}{x+1}I_{x+1}$$

Therefore

$$\frac{f(x)}{\binom{n}{x}} = \frac{n-x}{x+1}\frac{f(x+1)}{\binom{n}{x+1}}$$

$$f(x) = \frac{n-x}{x+1} - \frac{n!\,(n-x-1)!\,(x+1)!}{x!\,n!\,\,(n-x)!}f(x+1)$$

$$[(x+1)(n-x)]f(x) = [(n-x)(x+1)]f(x+1)$$

$$[n+nx-x-x^2]f(x) = [(n+1-x-1)(x+1)]f(x+1)$$

$$[n+(n-1)x-x^2]f(x) = [(n+1)(x+1)-(x+1)^2]f(x+1) \qquad [6]$$

The first moment is obtained by summing the equation [6] over n. Thus

$$\sum_{x=0}^{n}[(n+1)(x+1)-(x+1)^2]f(x+1) = \sum_{x=0}^{n}[n+(n-1)x-x^2]f(x)$$

$$(n+1)\sum_{x=0}^{n}(x+1)f(x+1) - \sum_{x=0}^{n}(x+1)^2f(x+1) = n + (n-1)\sum_{x=0}^{n}xf(x) - \sum_{x=0}^{n}x^2f(x)$$

$$(n+1)M_1 - M_2 = n + (n-1)M_1 - M_2$$

$$(n+1-n+1)M_1 = n$$

$$M_1 = \frac{n}{2}$$

$$E[X] = \frac{n}{2}$$

To determine the second moment we multiply equation [6] by $(x+1)$ and sum the result over n.

$$\sum_{x=0}^{n}[(n+1)(x+1)-(x+1)^2](x+1)f(x+1) = \sum_{x=0}^{n}[n+(n-1)x-x^2](x+1)f(x)$$

$$(n+1)\sum_{x=0}^{n}(x+1)^2f(x+1) - \sum_{x=0}^{n}(x+1)^3f(x+1)$$

$$= n\sum_{x=0}^{n}(x+1)f(x) + (n-1)\sum_{x=0}^{n}(x+1)f(x) - \sum_{x=0}^{n}x^2(x+1)f(x)$$

$$(n+1)M_2 - M_3 = nM_1 + n + (n-1)M_2 + (n-1)M_1 - M_3 - M_2$$

$$\{(n+1)+1-n+1\}M_2 = (n+n-1)M_1 + n$$

$$3M_2 = (2n-1)\frac{n}{2} + n$$

$$M_2 = \frac{n}{6}(2n+1)$$

$$Var(X) = M_2 - M_1^2$$

$$= \frac{n}{6}(2n+1) - \frac{n^2}{4}$$

$$= \frac{n}{12}(n+2)$$

### *2.2.2 Power function-Binomial Distribution.*

Power function is a special case of beta distribution when the parameter $b = 1$.

Thus we obtain

$$g(p) = \frac{P^{a-1}(a)!}{(a-1)!}$$

$$g(p) = \begin{cases} aP^{a-1} & 0 < P < 1 \quad a > 0 \\ 0 & \text{otherwise} \end{cases}$$

The mixed distribution becomes

$$f(x) = \int_0^1 \binom{n}{x} P^x (1-P)^{n-x} aP^{a-1} dp$$

$$f(x) = \binom{n}{x} a \int_0^1 P^{x+a-1} (1-P)^{n-x} dp \qquad [7]$$

Let $u = (1-P)^{n-x}$ $\quad du = -(n-x)(1-P)^{n-x-1}$

$$dv = P^{x+a-1} \qquad v = \frac{P^{x+a}}{x+a}$$

$$|_x = (1-P)^{n-x} \frac{P^{x+a}}{x+a}\Big|_0^1 + \frac{n-x}{x+a}\int_0^1 P^{x+a}(1-P)^{nx-1} dp$$

$$= \frac{n-x}{x+a} |_{x+1}$$

$$\frac{f(x)}{\binom{n}{x}a} = \frac{n-x}{x+a}\frac{f(x+1)}{\binom{n}{x+1}a}$$

$$f(x) = \frac{n-x}{x+a}\frac{n!\,(n-x-1)!\,(x+1)!\,a}{x!\,n!\,(n-x)!\,a} f(x+1)$$

$$[(x+a)(n-x)]f(x) = [(n-x)(x+1)]f(x+1)$$

$$[na + x(n-a) - x^2]f(x) = [(n+1)(x+1) - (x+1)^2]f(x+1) \qquad [8]$$

To get the first moment we have

$$\sum_{x=0}^{n}[(n+1)(x+1)-(x+1)^2]f(x+1) = \sum_{x=0}^{n}[na+(n-a)x-x^2]f(x)$$

$$(n+1)\sum_{x=0}^{n}(x+1)f(x+1)-\sum_{x=0}^{n}(x+1)^2f(x+1) = na+(n-a)\sum_{x=0}^{n}xf(x)-\sum_{x=0}^{n}x^2f(x)$$

$$(n+1)M_1 - M_2 = na + (n-a)M_1 - M_2$$

$$(n+1-n+a)M_1 = na$$

$$M_1 = \frac{na}{a+1}$$

Therefore

$$E(X) = \frac{na}{a+1}$$

The second moment is obtained by multiplying equation [8] by $(x+1)$ and sum over n.

$$\sum_{x=0}^{n}[(n+1)(x+1)-(x+1)^2](x+1)f(x+1) = \sum_{x=0}^{n}[na+(n-a)x-x^2](x+1)f(x)$$

$$(n+1)\sum_{x=0}^{n}(x+1)^2 f(x+1)-\sum_{x=0}^{n}(x+1)^3 f(x+1)$$

$$= na\sum_{x=0}^{n}(x+1)f(x)+(n-a)\sum_{x=0}^{n}x(x+1)f(x)-\sum_{x=0}^{n}x^2 (x+1)f(x)$$

$$(n+1)M_2 - M_3 = naM_1 + na + (n-a)[M_1+M_2] - [M_3 - M_2]$$

$$(n+1)M_2 - M_3 = naM_1 + na + (n-a)M_1 + (n-a)M_2 - M_3 - M_2$$

$$[n+1-n+a+1]M_2 = (na+n-a)M_1 + na$$

$$(2+a)M_2 = (na+n-a)\frac{na}{a+1} + na$$

$$M_2 = \frac{na(na+n+1)}{(2+a)(a+1)}$$

$$Var(X) = \frac{na(na+n+1)}{(2+a)(a+1)} - \left(\frac{na}{a+1}\right)^2$$

$$= \frac{(a+1)(n^2a^2+an^2+na)-(na)^2(2+a)}{(2+a)(a+1)^2}$$

$$Var(X) = \frac{na(n+a+1)}{(2+a)(a+1)^2}$$

***2.2.3. Arc-sine-Binomial distribution***

Given a beta distribution whose probability density function is given as

$$g(p) = \frac{P^{a-1}(1-P)^{b-1}}{B(a,b)} \qquad 0 < P < 1 \qquad a,b > 0$$

$$\text{Let } a = b = \frac{1}{2}$$

Substituting in the above function we have

$$g(p) = \frac{P^{-\frac{1}{2}}(1-P)^{-\frac{1}{2}}}{B\left(\frac{1}{2},\frac{1}{2}\right)}$$

$$g(p) = \begin{cases} \dfrac{1}{\pi\sqrt{P(1-P)}} & 0 < P < 1 \quad a, b > 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \pi$ which is

an Arc-sine distribution with $0 < P < 1 \quad a, b > 0$

The Arc-sine –Binomial distribution is

$$f(x) = \binom{n}{x}\frac{1}{\pi}\int_0^1 P^x(1-P)^{n-x}P^{-\frac{1}{2}}(1-P)^{-\frac{1}{2}}dp$$

$$f(x) = \binom{n}{x}\frac{1}{\pi}\int_0^1 P^{x-\frac{1}{2}}(1-P)^{n-x-\frac{1}{2}}dp \qquad [10]$$

To get the recursive form, let

$$\text{I}_x = \frac{f(x)\pi}{\binom{n}{x}} = \int_0^1 P^{x-\frac{1}{2}}(1-P)^{n-x-\frac{1}{2}}dp.$$

Let $u = (1-P)^{n-x-\frac{1}{2}}$, $\quad du = -1\left(n-x-\frac{1}{2}\right)(1-P)^{n-x-\frac{3}{2}}$

$$dv = P^{x-\frac{1}{2}}, \qquad v = \frac{P^{x+\frac{1}{2}}}{x+\frac{1}{2}}$$

$$= (1-P)^{n-x-\frac{1}{2}}\frac{P^{x+\frac{1}{2}}}{x+\frac{1}{2}}\Bigg|_0^1 + \int_0^1\left(\frac{n-x-\frac{1}{2}}{x+\frac{1}{2}}\right)P^{x-\frac{1}{2}}(1-P)^{n-x-\frac{3}{2}}dp$$

$$= \left(\frac{n-x-\frac{1}{2}}{x+\frac{1}{2}}\right)\text{I}_{x+1}$$

$$\frac{f(x)\pi}{\binom{n}{x}} = \left(\frac{n-x-\frac{1}{2}}{x+\frac{1}{2}}\right)\frac{f(x+1)\pi}{\binom{n}{x+1}}$$

$$\frac{(n-x)!\,x!}{n!}f(x) = \frac{\left(n-x-\frac{1}{2}\right)(n-x-1)!\,(X+1)!}{\left(x+\frac{1}{2}\right)n!}f(x+1)$$

$$\left[(n-x)\left(x+\frac{1}{2}\right)\right]f(x) = \left[\left(n-x-\frac{1}{2}\right)(x+1)\right]f(x+1)$$

$$\left[\frac{n}{2}+\left(n-\frac{1}{2}\right)x-x^2\right]f(x) = \left[\left(n+\frac{1}{2}\right)(x+1)-(X+1)^2\right]f(x+1) \qquad [11]$$

The first moment of Arc-sine-Binomial distribution is obtained when we sum the equation [11] over n.

$$\sum_{x=0}^{n}\left[\left(n+\frac{1}{2}\right)(x+1)-(x+1)^2\right]f(x+1)=\sum_{x=0}^{n}\left[\frac{n}{2}+\left(n-\frac{1}{2}\right)x-x^2\right]f(x)$$

$$\left(n+\frac{1}{2}\right)\sum_{x=0}^{n}(x+1)f(x+1)-\sum_{x=0}^{n}(x+1)^2 f(x+1)=\frac{n}{2}+\left(n-\frac{1}{2}\right)\sum_{x=0}^{n}xf(x)-\sum_{x=0}^{n}x^2 f(x)$$

$$\left(n+\frac{1}{2}\right)M_1-M_2=\frac{n}{2}+\left(n-\frac{1}{2}\right)M_1-M_2$$

$$\left(n+\frac{1}{2}-n+\frac{1}{2}\right)M_1=\frac{n}{2}$$

$$M_1=\frac{n}{2}$$

$$E(X)=\frac{n}{2}$$

The second moment becomes

$$\sum_{x=0}^{n}\left[\left(n+\frac{1}{2}\right)(x+1)-(x+1)^2\right](x+1)f(x+1)=\sum_{x=0}^{n}\left[\frac{n}{2}+\left(n-\frac{1}{2}\right)x-x^2\right](x+1)f(x)$$

$$\left(n+\frac{1}{2}\right)\sum_{x=0}^{n}(x+1)^2 f(x+1)-\sum_{x=0}^{n}(x+1)^3 f(x+1)=$$

$$\frac{n}{2}\sum_{x=0}^{n}xf(x)+\frac{n}{2}\sum_{x=0}^{n}f(x)+\left(n-\frac{1}{2}\right)\sum_{x=0}^{n}x^2 f(x)+\left(n-\frac{1}{2}\right)\sum_{x=0}^{n}xf(x)-\sum_{x=0}^{n}x^3 f(x)-\sum_{x=0}^{n}x^2 f(x)$$

$$\left(n+\frac{1}{2}\right)M_2-\left(n-\frac{1}{2}\right)M_2+M_2=\frac{n}{2}+\frac{n}{2}M_1+\left(n-\frac{1}{2}\right)M_1$$

$$\left(n+\frac{1}{2}-n+\frac{1}{2}+1\right)M_2=\frac{n}{2}+\left(\frac{n}{2}+\frac{2n}{2}-\frac{1}{2}\right)M_1$$

$$M_2=\frac{n}{4}\left(\frac{3n+1}{2}\right)$$

Therefore

$$\begin{aligned}Var(X)&=\frac{n}{4}\left(\frac{3n+1}{2}\right)-\frac{n^2}{4}\\&=\frac{n}{8}(n+1)\end{aligned}$$

**3.0 CONCLUSION**

Application of Beta-binomial mixed distribution in modelling statistical data whose distribution cannot be fitted to Binomial distribution alone is a milestone in statistical modelling. Equally important is the method that can be used to determine moments of the mixed distribution so constructed. Using integration by parts technique to derive recursive relations seem to be more motivating and straight forward to apply as demonstrated in this paper. It is also important to note that recursive relations are widely applied in insurance industry to determine total aggregate claims. Thus the focus on techniques of the construction of recursive equations and their properties is an achievement in making Beta-binomial mixed distribution more applicable.

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