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SOME RESULTS ON FUZZY SEQUENCE IN METRIC SPACE

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ABSTRACT

In this paper, our objective is to introduce the basic notion of fuzzy sequence, its boundedness and convergence and its characterization in metric space. Besides reviewing some of the generalizations of convergence of sequence, fuzzy sequence and its boundedness. We shall also establish and verify some relations between crisp sequence and fuzzy sequence in metric space with various examples. Finally, we shall prove convergence of fuzzy sequence as the concept of convergence of crisp sequence with converse part.

Keywords: Fuzzy sequence, Boundedness, Fuzzy convergence, Metric space, Fuzzy metric space

1. Introduction and Historical Motivation

So far a bulk numbers of research works have been done in fuzzy mathematics. It is a widely used instrument which deals with uncertainty in a computationally suitable method. In 1937, Prof. Max Black introduced the notion of fuzzy sets and later in 1965, Prof. Zadeh [13] further developed the concept of fuzzy sets and fuzzy set operations. Mathematicians have been using the binary logic but people realized the inadequacy of binary logic in many situations. Hence the multiway logic namely fuzzy logic was introduced by Zadeh. Using the notion of fuzzy, Mathematicians introduced and developed the fuzzy algebra, fuzzy topology and more new concepts. In 1968 Prof. C.L. Chang [1] introduced the concept of fuzzy topological spaces. Kramosil and Michalek [7] introduced the notion of fuzzy metric space in 1975 and then in 1994, George and Veeramani [2] modified the notation of fuzzy metric space with the help of continuous t-norm and also develops some of related theorems, properties to the convergence of crisp sequence and fuzzy sequence. Later in 2019 introduced the concept of bounded fuzzy sequence in metric space as an extension concept of bounded crisp sequence by Muthukumari [9].

In recent years, there has been an increasing interest in various mathematical aspects of operations defined on fuzzy sets and fuzzy numbers. As the set of all real numbers can be embedded in the set of all fuzzy numbers, many results in real numbers can be considered as a special case of those fuzzy numbers. However, the set of all fuzzy numbers is partially ordered and most of the theorems known for the sequence of real numbers may not be valid in fuzzy setting. Therefore, fuzzy theory is not a trivial extension of real case. Several results in fuzzy sequence in metric space and fuzzy metric space has been introduced by several authors in different directions.

The fuzzy sets can be used to study the qualitative variables. By using notion of fuzzy theory, Mathematicians developed fuzzy sets, fuzzy logic, fuzzy sequence, fuzzy metric and more convergent concept which is an extension of already existing crisp concept. The theory of Fuzzy sequence plays leading roles in several branches of mathematics to develop many theories with examples and verifications. We study fuzzy sequence, convergence of sequences and fuzzy sequence in metric spaces, boundedness of fuzzy sequence and presents the relation between crisp sequence and fuzzy sequence in metric space. The theory of fuzzy sequence occupies main position in several branches of mathematics, mainly in the field of analysis. This leads to introduce several new concepts in functional analysis and thereby develops its theory in metric space and fuzzy metric space.

2 Preliminaries

Before proceeding with the work, we recall some of the basic notations and definitions that are used in this paper.

Definition 2.1 ([11], [12])

Let $X = \{ x_i : i = 1,2, \ldots, n\}$ be a finite universal set. A fuzzy set F in X is defined as a set of ordered pairs

$$F(x) = \{(x_i, \mu_F(x_i)) : x_i \in X;\ i = 1, 2, \ldots, n\}$$

where, the value $\mu_F(x_i)$ is called the degree of membership of x_i in fuzzy set F, and $\mu_F : X \to [0,1]$ is called membership function .

Definition 2.2 ([3], [8], [12])

A function $t : [0,1] \times [0,1] \to [0,1]$ is said to be t-norm if it satisfies the following conditions:

For all $a, b, c, d \in [0,1]$

a) $t(a,b) = t(b,a)$

b) $t(a,b) \leq t(c,d)$ if $a \leq c$ and $b \leq d$

c) $t(a, t(b,c)) = t(t(a,b), c)$

d) $t(a,1) = a$; $t(a,0) = 0 \quad \forall\, a \in [0,1]$.

If 't' is continuous, then it is called t-norm continuous. It is binary operation on [0,1] and usually denoted by '*' .

In particular, for all $a,\ b \in [0,1]$ the operation $a * b = min(a,b)$ is t-norm.

Definition 2.3

Let *M* be a non empty set and *d* be a distance function defined from $M \times M \rightarrow \mathbb{R}^{+}$, then the pair (M, d) is said to be metric space if it satisfies the following properties. For all $a, b, c \in M$

a) $d(a,b) \geq 0$

b) $d(a,b) = 0 \Leftrightarrow a = b$

c) $d(a,b) = d(b,a)$ and

d) $d(a,b) \leq d(a,c) + d(c,b)$.

In particular, if $S = \mathbb{R}^n$ be a set of n-dimensional real vector space, define a function $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$d(a,b) = \max_{1 \leq j \leq n} |a_j - b_j| \quad \forall\, a, b \in S.$$ Then *(S , d)* is Metric space.

Definition 2.4 ([2],[3],[4],[8])

Let *X* be a non empty set, $F : M \rightarrow [0,1]$ be the fuzzy set in $M \subseteq X^2 \times (0, \infty)$ and '*' be continuous *t*-norm, then a 3-tuples $(X, F, *)$ is called Fuzzy metric space if it satisfies the following conditions:

For all $x, y, z \in X$; $s, t \in (0, \infty)$

a) $F(x,y,t) > 0$

b) $F(x,y,t) = 1 \Leftrightarrow x = y$

c) $F(x,y,t) = F(y,x,t)$

d) $F(x,y,t) * F(y,z,s) \leq F(x, z, t+s)$

e) $F(x,y, .): (0, \infty) \rightarrow [0, 1]$ is continuous.

Here *F* is called Fuzzy metric on X and $F(x,y,t)$ denotes the degree of nearness between *x* and *y* with respect to *t.*

Example 2.4.1

Let (X, d) be a Metric space, $F : M \rightarrow [0,1]$ be the fuzzy set in $M \subseteq X^2 \times (0, \infty)$ and '*' be continuous *t*-norm. Define

$$F(x,y,t) = \frac{t}{t + d(x,y)} \text{ for all } x, y \in X,\ t > 0 : x * y = xy.$$

Then $((X, d), F, *)$ is Fuzzy metric space, (see,[3],[11]).

A Fuzzy metric space induced by metric '*d*', a distance function is called Standard fuzzy metric.

Definition 2.5 [5]

A crisp sequence $\{x_n\}$ in a metric space *(M, d)* is Cauchy sequence if

$$\text{for all } \varepsilon > 0\ \exists\, n_0 \in N : d(x_m, x_n) < \varepsilon\ \forall\, m, n \geq n_0 .$$

It is said to convergent sequence to a point $p \in M$ if

$$\text{for all } \varepsilon > 0\ \exists\, n_0 \in N : d(x_n, p) < \varepsilon .$$

we write $x_n \rightarrow p$. Otherwise $\{x_n\}$ is said to be divergent.

Definition 2.6 ([2] [6] [10])

Let X be a non empty set. A fuzzy set F on $\mathbb{N} \times X$ is called a fuzzy sequence in X

i.e. $F: N \times X \rightarrow [0,1]$ is called a fuzzy sequence in X.

In particular, let a function $F: \mathbb{N} \times \mathbb{N} \rightarrow [0,1]$ defined as

$$F(n,x) = \frac{1}{n+x^2} \quad \forall\, n,\ x \in \mathbb{N}$$

then F is a fuzzy sequence in $\mathbb{N}$.

Definition 2.7 [9]

Let $F: \mathbb{N} \times \mathbb{R} \rightarrow [0,1]$ be a fuzzy sequence then F is said to be Bounded fuzzy sequence if there exists a positive real number M such that

$$F(n,x) = 1 \Rightarrow |x| \leq M,\ \ M > 0 .$$

Example 2.7.1

Let a fuzzy sequence $F: \mathbb{N} \times \mathbb{R} \rightarrow [0,1]$ defined as:

$$F(n,x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} - \{-1,1\} \\ 1 & \text{otherwise} \end{cases}$$

Then F is bounded fuzzy sequence, since, taking $M = 1$,

$$F(n,x) = 1 \Rightarrow x = 1 \text{ or} -1 \Rightarrow |x| = 1 \Rightarrow |x| \leq 1 = M .$$

3 Main Results

In this section, we shall study some results that characterized Fuzzy sequence in a metric space. We first begin with the following theorem.

Theorem 3.1

A crisp sequence $\{x_n\}$ in Metric space (S,d) is bounded iff the corresponding fuzzy sequence F in (S,d) is bounded.

Proof

Let $\{x_n\}$ be a crisp sequence and F be the corresponding fuzzy sequence in metric space (S,d). Define $F: \mathbb{N} \times \mathbb{R} \rightarrow [0,1]$ as:

$$F(n,x) = \begin{cases} 1 & \text{if } x = x_n \\ 0 & \text{otherwise} \end{cases}$$

Since $\{x_n\}$ is bounded on S, so there exists $M > 0$ such that $|x_n| \leq M \ \forall\, n \in \mathbb{N}$.

Now $F(n,x) = 1 \Rightarrow x = x_n \Rightarrow |x| = |x_n| \leq M$.

Hence F is bounded in (S,d).

Conversely, suppose that F is bounded in (S,d) then there exists $M > 0$ such that

$F(n,x) = 1 \Rightarrow |x| \leq M \Rightarrow |x_n| \leq M \ \forall\, n \in \mathbb{N}$.

Hence $\{x_n\}$ is bounded on (S,d).

Now we immediately obtain the following result.

Theorem 3.2 [10]

Every crisp sequence in metric space (S, d) is fuzzy sequence in (S, d).

Proof:

Let $X = \{f : f$ is a crisp sequence in $(S, d)\}$ and

$A = \{F : F$ is a fuzzy sequence in $(S, d)\}$, where *S* be a non empty set.

Define a function

$$h : X \to A \text{ and } f \in X\,,\ f : \mathbb{N} \to S$$

such that

$$F(n, x) = \begin{cases} 1 & \text{if } f(n) = x \\ 0 & \text{otherwise} \end{cases}$$

Suppose $f, g \in X$ such that $f \neq g\ \exists\, n_0 \in \mathbb{N} : f(n_0) \neq g(n_0)$.

Let $f(n_0) = x$ and $g(n_0) = y$

which implies

$$F(n_0, x) = 1 \text{ and } F(n_0, y) = 0 \Rightarrow h(f) \neq h(g).$$

This shows that *h* is injective. Hence every crisp sequence is fuzzy sequence.

Example 3.2.1

The following example shows that every convergent fuzzy sequence in metric space may not be bounded.

Consider the metric space $\mathbb{R}^1$ and define $F: \mathbb{N} \times \mathbb{R} \to [0,1]$ as

$$F(n, x) = \begin{cases} 1 & \text{if } n \leq k \\ 1 & \text{if } n > k, x = p \\ 0 & \text{otherwise} \end{cases}$$

where *k, p* >1 are positive integers. First we show $F \to p$.

Let $\varepsilon > 0$ be given then $\forall\, n \geq n_0$ and choose $n_0 = p + 1$, we have

$F(n, x) = 1 \Rightarrow x = p \Rightarrow |x - p| < \varepsilon \ \ \forall\, n \geq n_0$. Hence $F \to p$

Taking, $n = p - 1$, $F(n, x) = 1\ \forall\, x$, there doesn't exists $M > 0$ such that $|x| \leq M$.

This implies *F* is not bounded.

Example 3.2.2

There is an example which illustrates that every fuzzy sequence may not be crisp sequence.

Consider a sequence $F : \mathbb{N} \times X \to [0, 1]$ by

$$F(n, x) = \frac{1}{n + x^2}\ \forall\, n \in \mathbb{N}, x \in X$$

Then, *F* is a fuzzy sequence for $X = \mathbb{N}$.

But for $X = \mathbb{Z}$ there doesn't exists corresponding crisp sequence.

Theorem 3.3 [10]

For any non empty set *X* a fuzzy sequence *F* on *X* will be crisp sequence if it satisfies the following conditions:

a) For each $n \in \mathbb{N}$ $F(n,x) = 0$ or 1 $\forall\ x \in X$

b) For each $n \in \mathbb{N}\ \exists\ x \in X\ : F(n,x) = 1$.

Definition 3.4 [9]

Let $F: \mathbb{N} \times \mathbb{R} \to [0,1]$ be a fuzzy sequence then *F* is said to Almost Bounded fuzzy sequence if there exists a positive real number *M* and $n \geq n_0$ such that

$$F(n,x) = 1 \Rightarrow |x| \leq M\ ,\ \ n_0 \in \mathbb{N}\ .$$

Example 3.4.1

Consider $\mathbb{R}^1$ as a metric space. We define $F: \mathbb{N} \times \mathbb{R} \to [0,1]$ as

$$F(n,x) = \begin{cases} 1 & \text{if } \ n \leq k \\ 1 & \text{if } n > k,\ x = p\ . \\ 0 & \text{otherwise} \end{cases}$$

Take *M* = p and $p, k = n_0 \in \mathbb{N}$, then for all $n \geq n_0$,

$F(n,x) = 1 \Rightarrow x = p \Rightarrow |x| \leq p = M$. Hence *F* is almost bounded.

The following theorem and example ensure that every bounded fuzzy sequence are almost bounded in metric space but converse may not be true.

Theorem 3.5

Every convergent fuzzy sequence in metric space *(S, d)* is almost bounded but converse may not be true.

Proof:

Let *F* be a fuzzy sequence in metric space *(S, d)*. Let *F* converges to *p*. i.e $F \to p$

then for all $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N}\ : n \geq n_0\ ,\ F(\mathrm{n,x}) = 1 \Rightarrow |x - p| < \varepsilon$

$$\Rightarrow |x| < \varepsilon + |p| = M \quad \text{(say)}$$

That is, there exists *M > 0* and $n \geq n_0 \in \mathbb{N}$, $F(n,x) = 1 \Rightarrow |x| \leq M$.

Hence *F* is almost bounded.

Conversely,

In metric space $\mathbb{R}^1$, we define $F: \mathbb{N} \times \mathbb{R} \to [0,1]$ as

$$F(n,x) = \begin{cases} 1 & \text{if } \ n \leq k \\ 1 & \text{if } \ n > k,\ |x| = p\ . \\ 0 & \text{otherwise} \end{cases}$$

take *M* = *p* and $p,\ k = n_0 \in \mathbb{N}$ then for all $n \geq n_0$, we have $F(n,x) = 1 \Rightarrow |x| = p$.

Hence, *F* is almost bounded.

But *F* is not convergent. Since $F(n,x) = 1 \Rightarrow |x| = p$.

Then for all $\varepsilon > 0\ \nexists\ n_0 \in \mathbb{N}$ such that $n \geq n_0$; $F(\mathrm{n,x}) = 1 \Rightarrow |x - p| < \varepsilon$. i.e $F \nrightarrow p$.

Definition 3.6 ([6], [10])

Let (S, d) be a metric space and *F* be fuzzy sequence on *S* and $\alpha \in (0,1]$, $p \in S$ then we say *F* converges to *p* i.e. $F \rightarrow p$ at level α if

a) For all $n \in \mathbb{N}\ \exists\ x \in S : F(n, x) \geq \alpha$.

b) For any $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N}: d(x,p) < \varepsilon\ \forall\ n \geq n_0$ and $F(n,x) \geq \alpha\ \forall\ x \in S$.

Example 3.6.1

Let, in a metric space $\mathbb{R}^1$, we define $F: \mathbb{N} \times \mathbb{R} \rightarrow [0,1]$ as

$$F(n,x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

we show that $F \rightarrow 0$. Since $\alpha \in (0,1]$ we have

a) For all $n \in \mathbb{N}\ \exists\ x = \frac{1}{n} \in S : F(n, \frac{1}{n}) \geq \alpha$.

b) Let $\varepsilon > 0$ given. Then $\exists\ n_0 \in \mathbb{N} : n_0 > \frac{1}{\varepsilon}\ \forall\ n \geq n_0, F(n,x) \geq \alpha$.

And $d(x,p) = |x - p| = |x - 0| = |x| = \left|\frac{1}{n}\right| < \varepsilon$. i.e $d(x, 0) < \varepsilon$.

The following theorem concludes that the concept of bounded fuzzy sequence is the extension of concept of bounded crisp sequence in metric space.

Theorem 3.7

A crisp sequence *f* in a metric space (*S, d*) converges to 1 if and only if its corresponding fuzzy sequence *F* converges to 1.

Proof:

Let *F* be the corresponding Fuzzy sequence of a crisp sequence *f* in a metric space (*S d*).

Consider $f \rightarrow 1$, then for every $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that

$$d(x_n, 1) < \varepsilon\ \forall\ n \geq n_0 .$$

For all $x \in S$, $n \in \mathbb{N}$, let in a metric space $\mathbb{R}^1$, we define $F: \mathbb{N} \times \mathbb{R} \rightarrow [0,1]$ as

$$F(n,x) = \begin{cases} 1 & \text{if } x = x_n \\ 0 & \text{otherwise} \end{cases}$$

i.e, For any $x \in S$, $n \in \mathbb{N}, F(n,x) = 1 \geq \alpha \in (0,1]$.

Also, for all $n \geq n_0, F(n,x) \geq \alpha \Rightarrow n \geq n_0, F(n,x) = 1 \Rightarrow x = x_n$

Therefore, $d(x,1) = d(x_n, 1) < \varepsilon\ \forall\ n \geq n_0$. Hence $F \rightarrow 1$.

Conversely

Let $F \rightarrow 1 \geq \alpha \in (0,1]$, then for every $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N} : n \geq n_0$

$$F(n,x) \geq \alpha \Rightarrow d(x, 1) < \varepsilon$$

Since, $F(n,x) \geq \alpha \Rightarrow F(n,x) = 1 \Rightarrow x = x_n$

Therefore, $d(x, 1) < \varepsilon \Rightarrow d(x_n, 1) < \varepsilon$.

Hence, for any $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N} : d(x_n, 1) < \varepsilon\ \forall\ n \geq n_0$ implies $f \rightarrow 1$.

Conclusion

In this paper, we have studied the basic notion of fuzzy sequence and its convergence in metric space with some of its generalizations. We have also established some relations between crisp sequence and fuzzy sequence. In fact, this work extends the many other authors existing literature and can be used for further research work in Fuzzy theory in Metric space.

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