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Some More Properties of One Dimensional Quadratic Family of Mappings $f_c(x) = x^2 - x + c$

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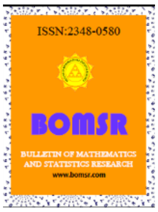

ABSTRACT

While studying the fluctuations in the population of some species, R. M. May and others in 1974 verified the features of the logistic family $F_\mu(x) = \mu x(1 - x)$, where μ is a parameter, and later, it was proved that the logistic mapping undergoes the period doubling phenomenon which is a route to chaos. It has already been proved that the one dimensional family of quadratic mapping $f_c(x) = x^2 - x + c$, where c is a parameter, is also a chaotic mapping. In this paper, we have explored some more properties of the family of quadratic mappings $f_c(x) = x^2 - x + c$in reference to Schwarzian derivative.

Keywords: fixed points, periodic points, critical points, one parameter family of functions, Schwarzian derivative

Mathematics Subject Classification: 37, 37C, 37C05, 37C1.

1. Introduction

The topic of *dynamical systems* and *chaos* have become the most attracting interdisciplinary subject of interest of the researchers all over the world although the subject began in the mid of the sixteenth century with the invention of differential equations by Newton. With the aid of differential equations, the two-body problem of finding the orbit of earth around the sun was solved, but solving the three-body problem is still a big challenge before the scientists. Many mathematicians like Henry Poincare, Van der Pol, Littlewood, Birkhoff, Lorenz, etc. made important contributions in the development of the theory of dynamical systems and chaos. Very strange and unpredictable patterns were found in the one dimensional mappings representing population models of species and, since then, the chaotic behaviour is observed in almost all phenomenon occurring in the nature.

2. A Review of Associated Terminology

In this section, we will recall some concepts related to dynamical systems and chaos. First of all, we will take a look on some definitions of dynamical systems. Many mathematicians have defined dynamical systems considering different approaches which include geometrical and topological ideas. However, we will consider the definitions that are accepted by most of the mathematicians.

2.1 Dynamical System [9]

A dynamical system on the *n*th dimensional Euclidean space $\mathbb{R}^n$ is a continuously differentiable function $\phi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\phi(t, X) = \phi_t(X)$ satisfies

(i) $\phi_0: \mathbb{R}^n \to \mathbb{R}^n$ is the identity map $\phi_0(X_0) = X_0$.

(ii) the composition $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$.

The dynamical system is said to be discrete dynamical system if the time t is measured discretely at equal intervals of time and, if the time is t is continuous, the dynamical system is said to be continuous.

2.2 Trajectory and Orbit[19]

Let $f: \mathbb{R} \to \mathbb{R}$ be a given dynamical system generated by the function f and a be an initial point. It is very important to study what happens to the sequence $\{x_n\}$ defined by $x_0 = a$ and $x_n = f(x_{n-1})$, where $n = 1, 2, 3, ...$ The sequence $\{f^n(a)\}$ of iterations of f computed at a is called the *trajectory* of a and the set of its values is called as the *orbit* of a.

2.3 Fixed Points, Periodic Points and their classification[1, 6, 8]

Let $f: \mathbb{R} \to \mathbb{R}$ be a given dynamical system. A real number $a \in \mathbb{R}$ is said to be a *fixed point* of a function f if $f(a) = a$. It follows that if a is a fixed point of f, then $f^n(a) = a$ for all $n \in \mathbb{Z}^+$. In this case, the orbit of a is the constant sequence $\{a, a, a, \dots\}$. If there exists an $n \in \mathbb{Z}^+$ such that $f^n(a) = a$, then a is called a *periodic point* of f with period n.

Let a be a fixed point of a dynamical system $f: \mathbb{R} \to \mathbb{R}$.

(1) We say that a is an attracting fixed point or a sink of f if there is some neighbourhood of a such that all points in this neighbourhood are attracted towards a. In other words, a is a sink if there exists an epsilon neighbourhood $N_\epsilon(a) = \{x \in S: |x - a| < \epsilon\}$ such that $\lim_{n\to\infty} f^n(x) = a$ for all $x \in N_\epsilon(a)$.

(2) We say that a is a repelling fixed point or a source of f if there is some neighbourhood $N_\epsilon(a)$ of a such that each x in $N_\epsilon(a)$ except for a maps outside of $N_\epsilon(a)$. In other words, a is a source if there exists an epsilon neighbourhood such that $|f^n(x) - a| > \epsilon$ for infinitely many values of positive integers n.

Let a be a periodic point of period n of a function f. Then a is said to be an attracting periodic point or a repelling periodic point according as it is an attracting or a repelling fixed point of the n^{th} iterate f^n.

2.4 Hyperbolic Periodic Points[7, 17]

A periodic point a of a mapping f with prime period n is said to be *hyperbolic* if $|(f^n)'(a)| \neq 1$, otherwise a is said to be a *neutral* periodic point.

For example, consider $f(x) = x^2 - x$. This mapping has $x = 2$ as a hyperbolic fixed point whereas $x = 0$ is a non-hyperbolic fixed point.

2.5 Some Results

Now we recall some results that can be used to decide the nature of fixed and periodic points.

2.6 Theorem [5, 18]

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function, where f' be continuous and a be a hyperbolic fixed point of f.

1. If $|f'(a)| < 1$, then a is an attracting fixed point of f.
2. If $|f'(a)| > 1$, then a is an repelling fixed point of f.

2.7 Theorem [5, 18]

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function, where f' be continuous and a be a periodic point of f with period n. Then the periodic orbit of a is attracting or repelling according as $|(f^n)'(a)| < 1$ or $|(f^n)'(a)| > 1$.

The theorems 1 and 2 do not provide any information in order to study the behavior of a non-hyperbolic periodic point. In such a situation, the higher ordered derivatives of the function at the periodic point can be effectively used. Suppose p is an attracting periodic point for which the rate of convergence of an orbit of a seed near p is much slower than normally observed. In this case, we say that the point p is a weakly attracting periodic point. Similarly, if p is repelling with orbits of the seeds near p going away from p much slowly, then we say that p is weakly repelling. The following theorem gives a criterion for the study of the non-hyperbolic fixed points.

2.8 Theorem [5, 18]

Let p be a neutral fixed point of a function f.

(i) If $f''(p) > 0$, then p is weakly attracting from the left and weakly repelling from the right.

(ii) If $f''(p) < 0$, then p is weakly repelling from the left and weakly attracting from the right.

Let p be a neutral fixed point of f with If $f''(p) = 0$.

(iii) If $f'''(p) > 0$ then p is weakly repelling.

(iv) If $f'''(p) < 0$, then p is weakly attracting.

If this theorem again fails to give any information, then it can be extended to higher order derivatives. Moreover, with slight changes, the theorem can be applied for periodic points also.

2.9 Basin of Attraction[15, 16]

Let a be a fixed point of a function f. When all the iterates near the point a of the function f converge to the point a, we say that the point a is an attracting fixed point. The collection of all such points around a is called as the basin of attraction of a. It follows that the basin of attraction is an open set. Formally, we define the basin of attraction of an attracting fixed point as follows.

Let a be an attracting fixed point of a function f.The basin of attraction of a is the set

$B_a = \{x: f^n(x) \to a \text{ as } n \to \infty\}$.

The basin of attraction of an attracting periodic orbit is the set of all points which approach the given attracting periodic orbit.

For example, consider the function f defined by $f(x) = x^2$. The point 0 is a fixed point of f. It can be observed that $\lim_{n\to\infty} f^n(x) = 0$ for $|x| < 1$. On the other hand, for $|x| > 1$, $|f^n(x)| \geq 1$. Hence the basin of attraction B_0 of the fixed point $x = 0$ is the open interval $(-1, 1)$.

3 Bifurcation Diagram and Schwarzian Derivative[10, 11]

A one parametric family of mappings is said to have bifurcations at a point of the number and the nature of the periodic or the fixed points changes at that point as the parameter value passes through that point. The point where the bifurcation occurs is called as the bifurcation point. The occurrence of bifurcation points in a one parameter family of mappings is an indication of change in the dynamical properties of the mapping. A bifurcation diagram is an important tool for the study of the parameterized families as the number of iterates becomes very high. In order to plot a bifurcation diagram, we consider the values of the parameter μ along the horizontal axis and the higher iterates of the variable *x* along the vertical axis. Thus we plot the set of all points of the form $(\mu, f_\mu^n(x))$, where n is generally greater than 200.

3.1 The Schwarzian Derivative

An important result regarding the maximum number of attracting cycles was proved by the American mathematician David Singer in 1978.. Before stating this result, we will take a review of the associated concepts.

3.2 Definition: Let f be function defined on an interval I whose third derivative f''' is continuous on I. Then the Schwarzian derivative of f at a point x, denoted $(Sf)(x)$, is defined by $(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$.

3.3 Theorem: (Singer's Theorem) Let f be a function defined on a closed interval I such that $f(I) \subseteq I$. Suppose that the Schwarzian derivative of f is negative over I and f has n critical points. Then the function f has at most $n + 2$ attracting cycles.

4 Dynamical Properties of the Quadratic Family $f_c(x) = x^2 - x + c$

The study of the dynamical properties of the quadratic family of mapping $f_c(x) = x^2 - x + c$, where c is a parameter taking real values, has been done by Kulkarni P. R. and Borkar V. C.[12] by varying the parameter values. In the referred paper, the authors have obtained the number of the fixed and periodic points of the family $f_c(x) = x^2 - x + c$ and studied in detail the nature of these fixed and periodic points. We will add some points in the properties of this mapping.

Using the concept of topological conjugacy, Kulkarni P. R. and Borkar V. C.[14] have proved that the family of mappings is chaotic.

Also, the existence of the period three cycle[15] guarantees the chaotic behavior of the family of mappings. The chaos in the behavior of a one parameter family of mappings can be noted by observing the bifurcation diagram only however, it becomes very difficult to find out at what value of the parameter, the periodic orbit of a particular period comes in to picture of the bifurcation diagram.

We have obtained the value of c for which the family of mappings $f_c(x) = x^2 - x + c$ has a period-3 orbit and thus proved that it is chaotic.

4.1 Theorem: Every function in the quadratic family of mappings $f_c(x) = x^2 - x + c$ has at most one attracting cycle.

Proof: For $c > 1$, the line $y = x$ does not intersect the curve $f_c(x) = x^2 - x + c$, hence the fixed point for f_c do not exist and all the orbits have a tendency to move towards infinity. For $c = 1$, solving for x, the equation $f(x) = x$, we get just one fixed point at $x = 1$. As $f_c'(1) = 1$, the point $x = 1$ happens to be a neutral fixed point and in this case, the theorem 3 can be applied. Here $f_c''(1) > 0$ implies that $x = 1$ is weakly attracting from the left and weakly repelling from the right.

For the initial value $x_0 = -0.1$, the values of the iterations $f_c^n(x) = x_n$ of the function $f_c(x)$ are as follows. Note that $x_7 = 1.2576049060\text{E} + 000$ means $x_7 = 1.2576049060 \times 10^{+000}$.

$x_0 = -0.1$	$x_7 = 1.2576049060\text{E} + 000$	$x_8 = 1.3239651936\text{E} + 000$
$x_9 = 1.4289186403\text{E} + 000$	$x_{10} = 1.6128898403\text{E} + 000$	$x_{11} = 1.9885237966\text{E} + 000$
$x_{12} = 2.9657030932\text{E} + 000$	$x_{13} = 6.8296917437\text{E} + 000$	$x_{14} = 4.0814997570\text{E} + 001$
$x_{15} = 1.6260490291\text{E} + 003$	$x_{16} = 2.6424103960\text{E} + 006$	$x_{17} = 6.9823300584\text{E} + 012$
$x_{18} = 4.8752933044\text{E} + 025$	$x_{19} = 2.3768484804\text{E} + 051$	$x_{20} = 5.6494086988\text{E} + 102$
$x_{21} = 3.1915818646\text{E} + 205$	$x_{22} = \text{Infinity}$	$x_{23} = \text{Infinity}$

Here we observe that the iteration values diverge to infinity.

For the initial value $x_0 = -0.09$, the values of the iterations are as follows.

$x_0 = -0.09$	$x_7 = 1.2047230009\text{E} + 000$	$x_8 = 1.2466345080\text{E} + 000$
$x_9 = 1.3074630886\text{E} + 000$	$x_{10} = 1.4019966394\text{E} + 000$	$x_{11} = 1.5635979376\text{E} + 000$
$x_{12} = 1.8812405728\text{E} + 000$	$x_{13} = 2.6578255198\text{E} + 000$	$x_{14} = 5.4062109741\text{E} + 000$
$x_{15} = 2.4820906122\text{E} + 001$	$x_{16} = 5.9225647461\text{E} + 002$	$x_{17} = 3.5017647524\text{E} + 005$
$x_{18} = 1.2262321364\text{E} + 011$	$x_{19} = 1.5036452522\text{E} + 022$	$x_{20} = 2.2609490445\text{E} + 044$
$x_{21} = 5.1118905820\text{E} + 088$	$x_{22} = 2.6131425322\text{E} + 177$	$x_{23} = \text{Infinity}$

In this case also, the iteration values diverge to infinity.

For the initial value $x_0 = 0.001$, the values of the iterations are as follows.

$x_0 = 0.001$	$x_7 = 1.0010070422\text{E} + 000$	$x_8 = 1.0010080564\text{E} + 000$
$x_9 = 1.0010090725\text{E} + 000$	$x_{10} = 1.0010100908\text{E} + 000$	$x_{11} = 1.0010111111\text{E} + 000$
$x_{12} = 1.0010121334\text{E} + 000$	$x_{13} = 1.0010131578\text{E} + 000$	$x_{14} = 1.0010141843\text{E} + 000$
$x_{15} = 1.0010152129\text{E} + 000$	$x_{16} = 1.0010162435\text{E} + 000$	$x_{17} = 1.0010172763\text{E} + 000$
$x_{18} = 1.0010183111\text{E} + 000$	$x_{19} = 1.0010193481\text{E} + 000$	$x_{20} = 1.0010203872\text{E} + 000$
$x_{21} = 1.0010214283\text{E} + 000$	$x_{22} = 1.0010224717\text{E} + 000$	$x_{23} = 1.0010235171\text{E} + 000$
$x_{24} = 1.0010245647\text{E} + 000$	$x_{25} = 1.0010256144\text{E} + 000$	$x_{26} = 1.0010266663\text{E} + 000$
$x_{27} = 1.0010277204\text{E} + 000$	$x_{28} = 1.0010287766\text{E} + 000$	$x_{29} = 1.0010298350\text{E} + 000$

$x_{30} = 1.0010308955\text{E} + 000$

Continuing in this way, we observe that the iterations converge to the fixed point 1.

The tendency of the orbits in the above cases can be observed by means of the orbit diagram[4] as shown in Figure 1.

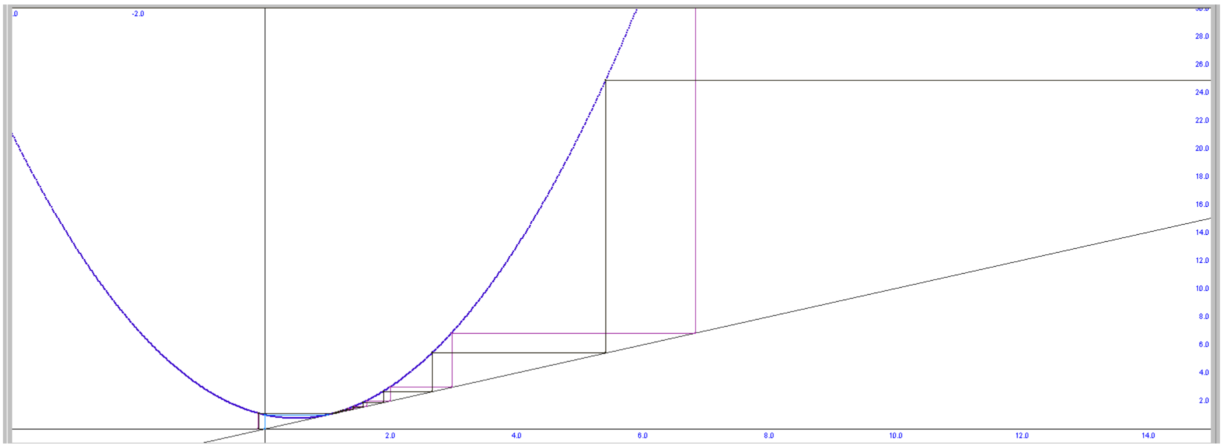

Figure 1: The orbit diagram

In this case also, the iteration values diverge to infinity.

It can be verified that the basin of attraction in this case is the open interval $(0,\ 1)$ as can be observed form the above numerical data and the orbit diagram.

For the case $0 < c < 1$, there are two fixed points $1 + \sqrt{1-c}$ and $1 - \sqrt{1-c}$. Using theorems 1 and 2, it can be verified that $1 + \sqrt{1-c}$ is a repelling fixed point and $1 - \sqrt{1-c}$ is an attracting fixed point. When *c* falls down through 0, we get two fixed points and an attracting periodic 2-cycle for the range $-\frac{1}{2} < c < 0$. As *c* assumes a value less that –1/2, there are two fixed points, there is period 2-cycle, which loses its stability and an attracting period 4-cycle appears. In this manner, we come across a period doubling bifurcation.

When *c* assumes again smaller values, the period 4-cycle loses its stability and a periodic 8-cycle is born; again for smaller values of the parameter c, this periodic 8-cycle becomes unstable and an attracting period 16-cycle comes into picture and so on. Thus a period doubling is observed, which is an indication of the chaotic nature of the family of mappings.

For the family $f_c(x) = x^2 - x + c$, all the interesting dynamics occur in the interval $-2 \leq c \leq 1$. Hence to obtain the bifurcation diagram for this family, we divide the parameter range [–2, 2] into a number of specified subdivisions and for each parameter value in this subdivision, the orbits are computed and plotted using the initial condition $x_0 = 0$ as shown in the Figure 2.

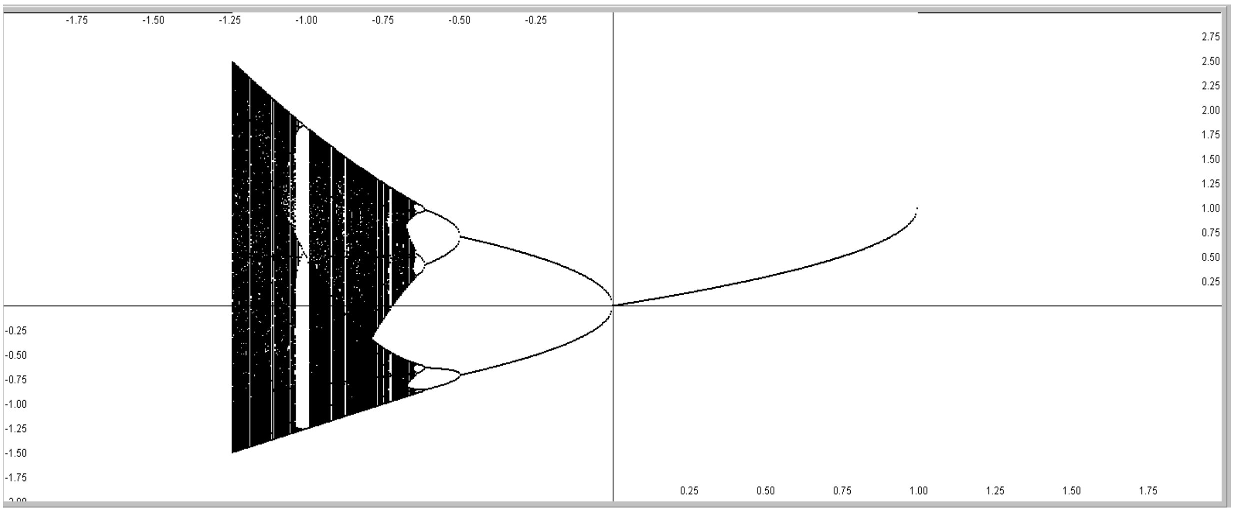

Figure 2: Bifurcation diagram

One can observe the repeated period doublings in this diagram. The first period doubling is observed at $c = 0$ and then as c decreases through 0, we come across successive period doubling bifurcations which proved that the mapping is chaotic.

Now we will obtain the number of attracting cycles for the family of mappings $f_c(x) = x^2 - x + c$. If we observe carefully the zoom-in picture of the bifurcation diagram as shown in Figure 3, we notice six horizontal curves that dominate the window. These six curves can represent six attracting fixed points, three attracting 2-cycles, two attracting 3-cycles, or an attracting 6-cycle.

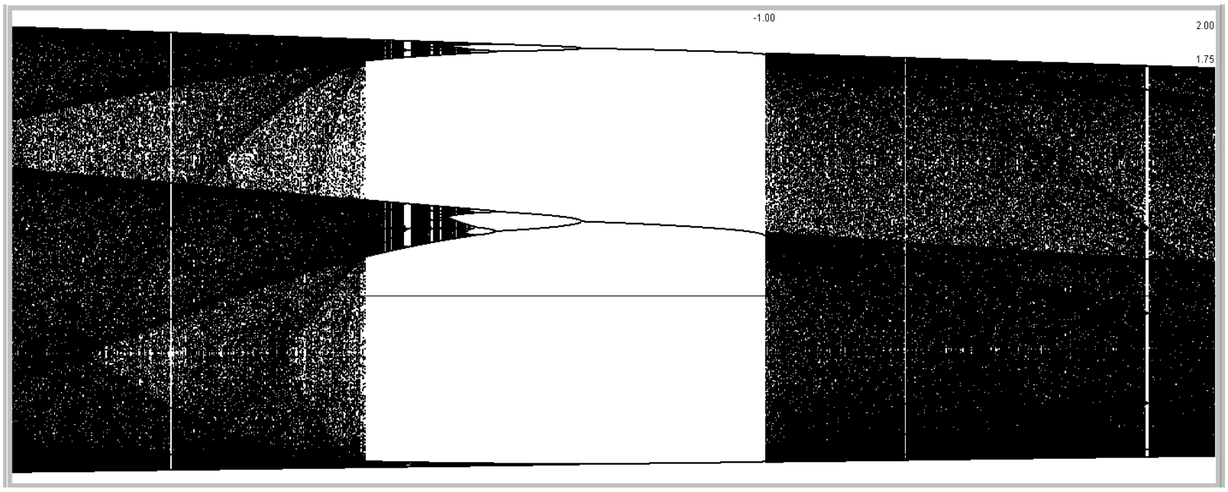

Figure 3: Zoom-in of the bifurcation diagram

Using Singer's theorem, now we will complete the proof of the theorem. The Schwarzian derivative of the family $f_c(x) = x^2 - x + c$ is given by

$$(Sf_c)(x) = \frac{f_c'''(x)}{f_c'(x)} - \frac{3}{2}\left[\frac{f_c''(x)}{f_c'(x)}\right]^2$$

$$= \frac{0}{2x-1} - \frac{3}{2}\left[\frac{2}{2x-1}\right]^2 < 0$$

Assume that $-1.25 < c < 1$. Since $f_c(x) = x^2 - x + c$ has a unique critical point at $x = \frac{1}{2}$. Hence by Singer's theorem, there can be at most three attracting cycles, each of which is associated with the intervals of the form $(0,\ A)$, $(A,\ B)$ and $(B,\ 1)$, where $0 < A < B < 1$. Since $x\ =\ 1$ is weakly attracting from the left and weakly repelling from the right, neither of the open intervals $0,\ A)$ and $(B,\ 1)$ appears as a basin of attraction for the cycles of f_c. This proves that f_c has at most one attracting cycle. This completes the proof of the theorem.

5. CONCLISION:

By proving that every function in the quadratic family of mappings $f_c(x)\ =\ x^2 -\ x\ +\ c$ has at most one attracting cycle, we have proved that the six horizontal curves in the window in Figure 3 represent an attracting 6-cycle.

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