Vol.9.Issue.3.2021 (July-Sept)
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BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

LAPLACE TRANSFORMS TECHNIQUES ON EQUATION OF ADVECTION-DIFFUSION IN ONE-DIMENSION WITH SEMI-INFINITE MEDIUM TO FIND ANALYTICAL SOLUTION

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DOI:10.33329/bomsr.9.3.86

ABSTRACT

In this research article, Advection-diffusion equation (ADE) in one-dimension is proposed with coefficient of variables for the three kinds of dispersion problems: dispersion through heterogeneous medium, momentary reliant on continuous flow and dispersal along continuous flow through heterogeneous medium. The analytical solution is achieved by applying the Laplace Transformation Techniques (LTT). Three boundary conditions Dirichlet, Neumann, Cauchy are used in semi-infinite medium. Furthermore, two new transformations of time t and space x variables are presented. The coefficients of variable in equation of advection-diffusion are compact in coefficients of constant. Effect of inhomogeneity parameters and all the possible combination of dispersion dependency are presented with the help of graphs.

Keywords: Advection, Diffusion, Error Function, Laplace transformation techniques

1. Introduction:

Advection-diffusion equation (ADE) is mixture of the advection and dispersal equations define natural phenomena where physical quantities are moved within a physical structure due to two process of advection and diffusion. Because of the increasing superficial and sub-surface hydro environs humiliation and smog, the equation of advection-diffusion has been equally important in soil science, chemical engineering, groundwatehydrology, environmental sciences, petroleum engineering, civil engineering & mathematical modelers for the explanation of comparable processes. It has been essential to first understand the physical, chemical, and biological procedures for controlling the movement of solutes in ground water in order to better ground water resource management. [1] has scientifically estimated the longitudinal dispersion coefficient for one-dimensional flow within a given range of perameters, and developed the mathematical models for understanding and prediction of solute transport phenomenon in an aquifer. [2] Have presented detailed analysis of the testing of the diffusion coefficient's dependence on dimensionless number. [3] has provided the corporal foundation for estimating soil-water flow in hydro-logical simulations, primarily that via the unsaturated soil constituency that acts as a boundary between the surrounding and the groundwater region. [4] Have used mothed of Green function to solve analytically for solutes in a two sheet semi-infinite medium with constant boundary and initial conditions. [5] Have applied the Laplace transformation power series techniques (LTPST) to describe advection–diffusion equation in cylinder-shaped coordinates in an outward convergent flow field. [6] Has examined restrictions of analytical result for parabolic type PDE's with variables coefficients being the function of the space (x) parameter. [7] Have solved linear fractional ADE by applying finite element method. [8] Have used integral Laplace transformation methods to find an exact solution to equation of advection-diffusion for one-dimension. [9] Have presnted linear advection-diffusion equatin for three dispersion prolems along uniform and plus type input sources solve analytically applying Laplace techniques. [10] Have introduced the two advection-diffusion models (one-dimensional) despersion cofficient and flow velocity. In present work, the methodical solutions of one-dimensional advection-diffusion equation (ADE) are presented in semi-infinite medium. The permeable medium is assumed semi-infinite along transversal direction. One-dimensional ADE are found for three cases: (i) Dispersion through heterogeneous media, (ii) Time dependent along steady and uniform flow and (iii) Dispersal along unvarying flow through inhomogeneous standard. New variables of time and space are established to reduce the coefficients of variables of the ADE into constant coefficients. The analytical solutions are obtaining for Dirichlet ,Cauchy and Neumann boundary conditions with the help of The Laplace Transformation Techniques (LTT). The Laplace transformation techniques are wieldy used because of being easier than other techniques and analytical solutions achieved by applying Laplace techniques are more accurate in validating the numerical results in terms of the precision and stability.

2. Mathematical formulation of analytical solutions

The one-dimensional advection-diffusion equation is derive in the basis of Law of conservation of mass by applying Law of Fick's and is written as:

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x}-U(x,t)C\right), \tag{1}$$

where C, D, U, x and t respectively solute concentration, dispersion coefficient, uniform velocity, position and time.

Let us considered the dispersion coefficient and uniform velocity in Eq. (1) as:

$$D(x,t)=D_0\psi(x,t) \quad \text{and} \quad U(x,t)=U_0\phi(x,t).$$

Let us established the new space variable by applying the transformation

$$X = -\int \frac{dx}{\psi(x,t)} \quad \text{and} \quad \frac{dX}{dx} = -\frac{1}{\psi(x,t)} \tag{2}$$

The above transformation is applied in Eq. (1), then

$$\psi(x,t)\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial X^2} + U_0 \frac{\partial}{\partial X}(\phi(x,t)C). \tag{3}$$

Now the Eq.(1) is analytically resolved for three dispersion problems, respectively [8].

2.1. Dispersion Through Inhomogeneous Medium

The velocity of flow along a medium varies according to its inhomogeneity. It is considered about a change of increasing nature.

Let us considered the expression for velocity at the origin $x=0$ of the domains be U_0, which increases to $U_0(1+b)$ at $x=L$, where $b<1$ implies that the velocity change is of a small order, which is a essential condition for the parameter of velocity in the ADE (Kumar *et al*, 2012).

Thus, the formulation for the velocity at any point x, $U(x)=U_0(1+ax)$, where $a=b/L$ and a is parameter of inhomogeneity and the dimension of a is less than 1.0.

$$\psi(x,t) = (1+ax)^2 \quad \text{and} \quad \phi(x,t) = (1+ax), \tag{4}$$

using the (2) and the partial differential equation (3) become as

$$\frac{\partial C}{\partial t} = a^2 X^2 D_0 \frac{\partial^2 C}{\partial X^2} + aXU_0 \frac{\partial C}{\partial X} - aU_0 C. \tag{5}$$

The medium is assumed the solute free at initially, so $C(x,0)=0$. (6)

2.1.1 Uniform Input Nature

Boundary conditions are

$$C(0,t) = C_0 \tag{7}$$

And $$\frac{\partial C(\infty,t)}{\partial x} = 0. \tag{8}$$

Using a transformation

$$Z = -\ln X = \ln\left[a(1+ax)\right]. \tag{9}$$

PDE (5) reduced to the ADE as:

$$\frac{\partial C}{\partial t} = a^2 D_0 \frac{\partial^2 C}{\partial Z^2} - \omega_0 \frac{\partial C}{\partial Z} - aU_0 C, \tag{10}$$

where $\omega_0 = aU_0 - a^2 D_0$.

Now, convert all conditions in the term of Z

$$C(Z,0) = 0, \tag{11}$$

$$C(\ln a,t)=0 \tag{12}$$

And $$\frac{\partial C(\infty,t)}{\partial Z}=0. \tag{13}$$

So, the analytical solution of ADE (10) is modified from Bear (1972, p. 630) and written as:

$$C(x,t)=\frac{C_0}{2}\left[(\beta)^{-1}erfc(\beta_1)+(\beta)^{\lambda}erfc(\beta_2)\right], \tag{14}$$

where

$$\beta=(1+ax),\ \beta_1=\frac{\ln(\beta)}{2a\sqrt{D_0t}}-\gamma\sqrt{t},\ \beta_2=\frac{\ln(\beta)}{2a\sqrt{D_0t}}+\gamma\sqrt{t},$$

$$\omega_0=au_0-a^2D_0\ ,\ \gamma=\sqrt{\frac{\omega_0^2}{4a^2D_0}+aU_0}\ \text{and}\ \lambda=\frac{U_0}{aD_0}.$$

2.1.2 Input Increasing Nature

The input point source may not remain constant as human activities increase, but it may increase with time. A mixed type nonhomogeneous condition expresses this premise as:

$$-D(x,t)\frac{\partial C(0,t)}{\partial x}+\alpha U(x,t)C(0,t)=U_0C_0, \tag{15}$$

where α is a real constant. For (4) the above expression is written in the term of Z as

$$-aD_0\frac{\partial C(\ln a,t)}{\partial Z}+\alpha U_0C(\ln a,t)=U_0C_0. \tag{16}$$

Now, the ADE (17), initial condition (18), 2nd type of boundary condition (20) and input condition (23) in (Z,t) region compare with the Laplace transformation techniques is written, for $\alpha\neq 1$, as

$$C(x,t)=\frac{C_0U_0}{2\sqrt{D_0}}\left[\frac{1}{\gamma+\eta}(\beta)^{-1}erfc(\beta_1)-\frac{1}{\gamma-\eta}(\beta)^{\lambda}erfc(\beta_2)\right.$$
$$\left.+\frac{2\eta}{\gamma^2-\eta^2}(\beta)^{\lambda_1}\times\exp\left\{-(\gamma^2-\eta^2)t\right\}\times erfc(\beta_3)\right], \tag{17}$$

where

$$\eta=\frac{\alpha U_0}{\sqrt{D_0}}-\frac{\omega_0}{2a\sqrt{D_0}}\ ,\ \lambda_1=\frac{\gamma}{a\sqrt{D_0}}+\frac{\omega_0}{2a^2D_0}\ \text{and}\ \beta_3=\frac{\ln(\beta)}{2a\sqrt{D_0t}}+\eta\sqrt{t}.$$

If $\alpha=1$, then preceding solution is not find if $\gamma=\eta$

$$C(x,t)=\frac{C_0U_0}{\sqrt{D_0}}\left[\frac{1}{4\gamma}(\beta)^{-1}erfc(\beta_1)-\frac{1}{4\gamma}(\beta)^{\lambda}\times\left\{1+\frac{2\gamma}{a\sqrt{D_0}}\ln(\beta)+4\gamma^2t\right\}\right.$$
$$\left.\times erfc(\beta_2)+\sqrt{\frac{t}{\pi}}\times\exp\left\{\frac{\omega_0}{2a^2D_0}\ln(\beta)-\frac{1}{4a^2D_0t}\left\{\ln(\beta)\right\}^2-\gamma^2t\right\}\right]. \tag{18}$$

2.2. Time Dependent Dispersion Along Uniform Flow

Time-dependent solute dispersion along uniform flow with uniform and increasing nature is considered in partially-infinite homogeneous and initially solute-free medium.

$$\psi(x,t)=\psi(mt) \quad \text{and} \quad \phi(x,t)=1, \tag{19}$$

where m is a coefficient with the inverse dimension of t [11].

From Eq. (4)

$$X=-\frac{x}{\psi(mt)}. \tag{20}$$

[12] introduce a new Time-scale T such that

$$T=\int_0^t \frac{dt}{\psi(mt)} \tag{21}$$

$$\frac{\partial C}{\partial T}=D_0\frac{\partial^2 C}{\partial X^2}+U_0\frac{\partial C}{\partial X}. \tag{22}$$

2.2.1 Uniform Input Nature

Now, using transformation

$$Z=-X=\frac{x}{\psi(x,t)}.$$

The partial differential equation (22) reduced to the advection-diffusion equation

$$\frac{\partial C}{\partial T}=D_0\frac{\partial^2 C}{\partial Z^2}+U_0\frac{\partial C}{\partial Z}. \tag{23}$$

Now, convert conditions (6)-(8) in the term of Z

$$C(Z,0)=0, \tag{24}$$

$$C(0,T)=C_0 \tag{25}$$

And $$\frac{\partial C(\infty,T)}{\partial Z}=0. \tag{26}$$

By applying Laplace transformation techniques and the analytical solution is modified form [13] written as:

$$C(x,t)=\frac{C_0}{2}\left[erfc(\omega_1)+\exp\left(\frac{U_o x}{D_0\psi(mt)}\right)\times erfc(\omega_2)\right], \tag{27}$$

where

$$\omega_1=\frac{x/\psi(mt)-U_oT}{2\sqrt{D_0T}}, \qquad \omega_2=\frac{x/\psi(mt)+U_oT}{2\sqrt{D_0T}}$$

and T is expressed in the expression of $\psi(mt)$ using (21).

2.2.2 Input Increasing Nature

The condition can be specify the input increasing nature at $x=0$

$$-D(x,t)\frac{\partial C(0,t)}{\partial x}+U(x,t)C(0,t)=U_0C_0, \tag{28}$$

now, for (4) the above equation convert in the term of variable Z

$$-D_0\frac{\partial C(0,T)}{\partial Z}+U_0C(0,T)=U_0C_0. \tag{29}$$

Thus, the desired analytical solution is modified by applying the Laplace transformation techniques from [13] and is written as:

$$C(x,t)=\frac{C_0}{2}\left[2U_0\sqrt{\frac{T}{\pi D_0}}\exp\left\{-\frac{x^2/\psi(mt)^2}{4D_0T}-\frac{U_0^2T}{4D_0}+\frac{U_ox}{2D_0\psi(mt)}\right\}\right.$$

$$\left.+erfc(\omega_1)-\left\{1+\frac{U_ox}{D_0\psi(mt)}+\frac{U_0^2T}{4D_0}\right\}\times\exp\left(\frac{U_ox}{D_0\psi(mt)}\right)erfc(\omega_2)\right]. \tag{30}$$

2.3. Dispersion Along Unvarying Flow from Non-homogeneous Medium

To suppose the concentional distribution behavior along uniform flow through non-homogeneous medium of unvarying input nature and that input accumulative nature along uniform flow, we consider following expressions

$$\psi(x,t)=\psi(mt)(1+ax)^2 \quad \text{and} \quad \phi(x,t)=(1+ax), \tag{31}$$

where m is the coefficient of variable time and its dimension is inverse of t.

From (2) becomes as

$$X=\frac{1}{\psi(mt)a(1+ax)} \tag{32}$$

[12] Established a new variable of time T

$$T=\int\psi(mt)dt, \tag{33}$$

$$\frac{\partial C}{\partial T}=a^2X^2D_0\frac{\partial^2 C}{\partial X^2}+\omega_0aX\frac{\partial C}{\partial X}-\omega_0C, \tag{34}$$

where

$$\omega_0=\frac{U_0}{\psi(mt)}.$$

It may note that the Eq.(34) similar as the Eq.(5) with the similar conditions, so the solution for uniform input nature and input increasing nature is same as the (14) and (17), respectively.

3. Results and discussion

The concentration values (C/C_0) of dispersion problems are discussed in an limited longitudinal area $0 \le x(\text{km}) \le 1$ of the semi-infinite media, respectively. In initial problem the analytical solutions of (14) and (17) are find for the input data $C_0 = 1.0,\ D_0 = 0.71$ and $U_0 = 0.60$ at $t(\text{years}) = 0.1, 0.3, 0.5, 0.7 \text{ and } 1.0$. It may be noted that the dimensions of dispersion coefficient (D_0) and uniform flow (U_0) are (km/year) and $(\text{km}^2/\text{year})$ respectively. The different parameters of inhomogeneity are used to find the solutions of (14) and (17). It is denoted by "a" and is less then 1.0, where $a = b/L$. Its various values represent media with varying degrees of heterogeneity. The inhomogeneity parameter assigned $a = 1.0\ \text{km}^{-1}$ in (4) then the value of uniform velocity varies from U_0 to $2U_0$ and dispersion coefficient varies from D_0 to $4D_0$.

In figure 1 & 2, the concentration behaviour for inhomogeneity parameter $a = 1.0\ \text{km}^{-1}$ described the analytical solution of (14) and (17) through the heterogeneous medium. These figures shows the concentration values (C/C_0) decreases with different values of times (t) against increasing the position (x). In figure 3 & 4, the effects of different parameters of inhomogeneity for the analytical solutions of (21) and (24). For a small change in inhomogeneity parameter, a significant variation in concentration pattern can be observed. Concentration remains smaller for higher inhomogeneity parameters values and concentrations values higher for smaller inhomogeneity parameters values.

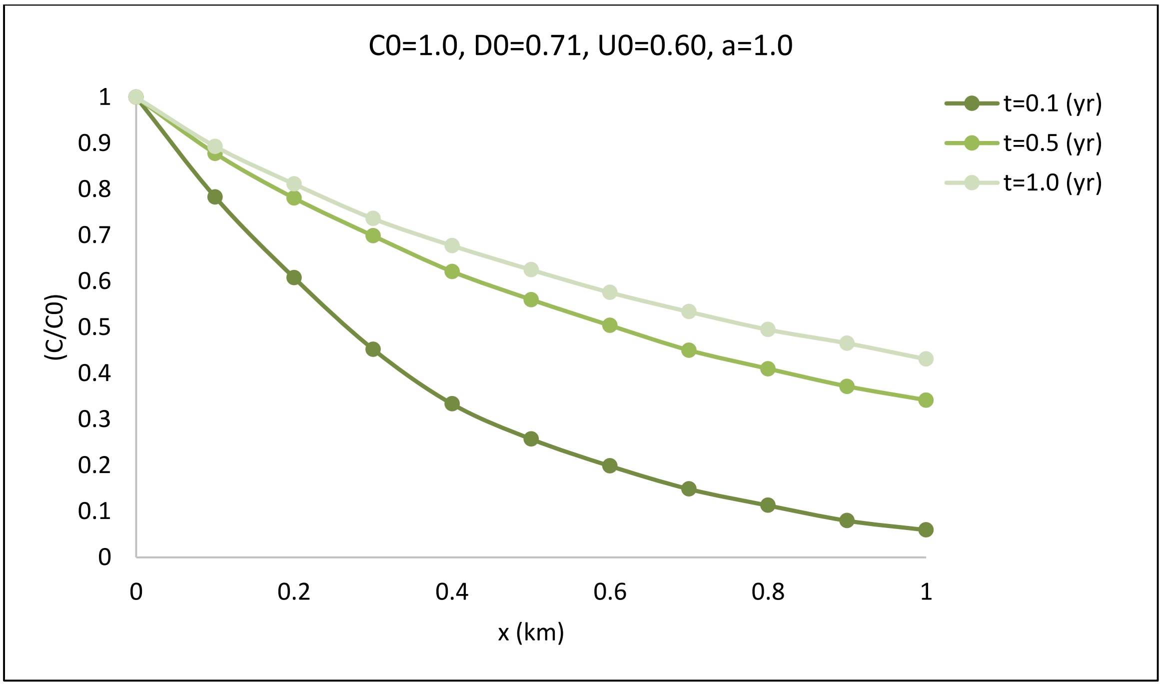

Figure 1: Dispersion through inhomogeneous medium with uniform input nature described by (14), for $a = 1.0\ \text{km}^{-1}$.

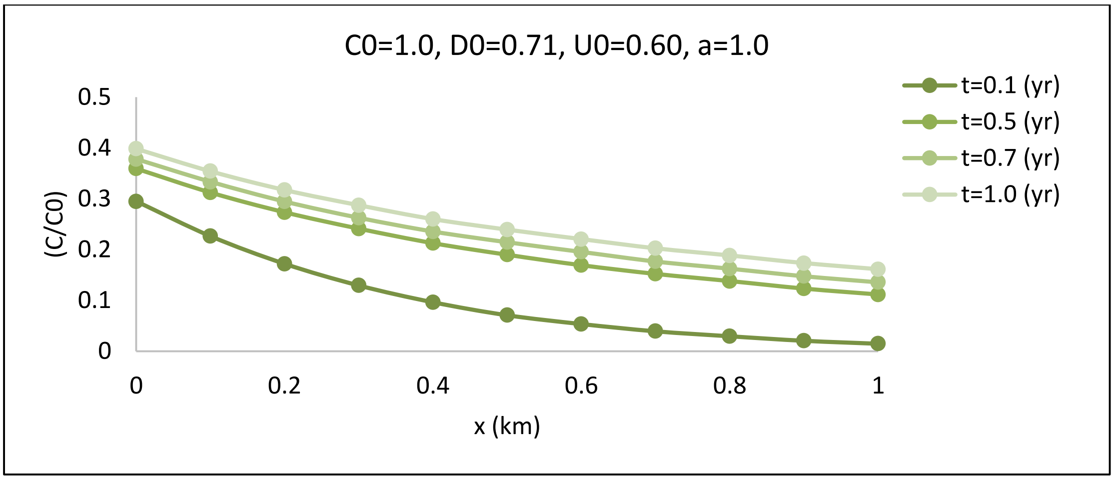

Figure 2: Dispersion through inhomogeneous medium with increasing nature described by (17), for $a = 1.0\ \text{km}^{-1}$.

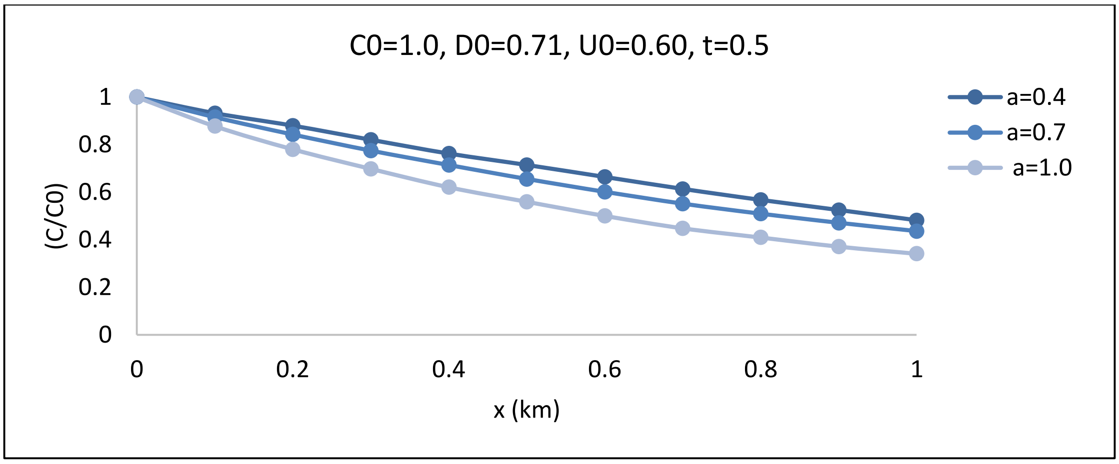

Figure 3: Compression of solutions (14) at $t = 0.5$ (yr) for three values of a .

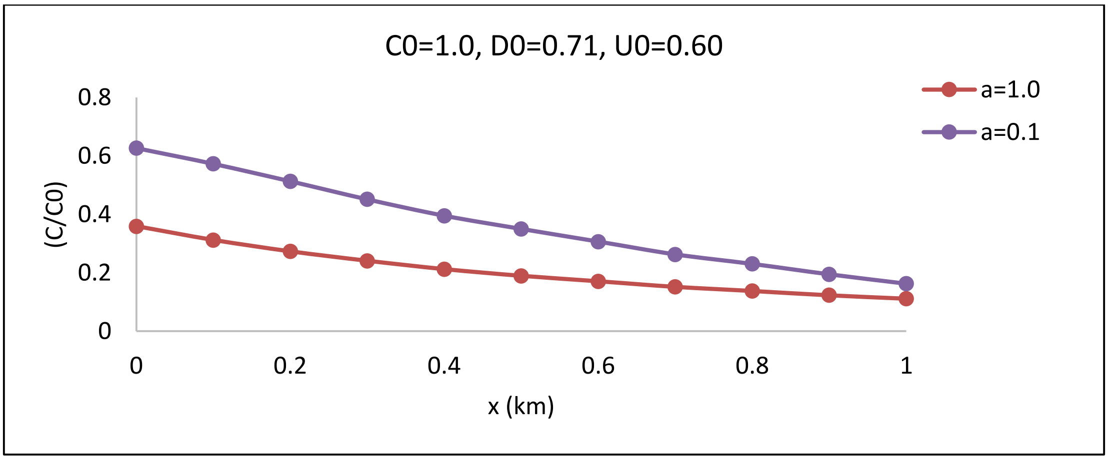

Figure 4: Compression of two inhomogeneity parameters $a = 1.0\ (\text{km})^{-1}$ and $a = 0.1\ (\text{km})^{-1}$ for (17) at $t = 0.5$.

In second problem three form of $\psi(mt)$ are considered exponentially increasing flow, exponentially decreasing flow and uniform flow to solve the (27) and (30) along with uniform point source and increasing nature. The concentration values (C/C_0) are analyzed in slimier for various input data

$m = 0.1\,(\text{year})^{-1}$ $D_0 = 1.71$ and $U_0 = 1.60$ except $C_0 = 1.0$ at different $t(\text{year}) = 0.05, 0.20, 0.35 \text{ and } 0.5$. The expression $\psi(mt)$ and variable T are achieved from (27a), assumed in both solutions for exponentially decelerating flow are

$$\psi(mt) = \exp(-mt)$$

and $$T = \int_0^t \frac{1}{\exp(-mt)}\,dt,$$

$$T = \frac{1}{m}\left[\exp(mt) - 1\right], \text{ respectively.}$$

Similarly, find the values for exponentially accelerating flow and uniform flow.

In figure 5 & 6, the exponentially increasing flow for the analytical solutions of (27) and (30). These figures shows that the concentration values (C/C_0) decreases with different times against increasing the position (x). In figure 7 & 8, the comparison between three different functions of $\psi(mt)$ (exponentially increasing, exponentially decreasing and uniform flow) for the analytical solutions of (27) and (30). These figures illustrates that the concentration values (C/C_0) decreases with different functions of $\psi(mt)$ against increasing the position (x).

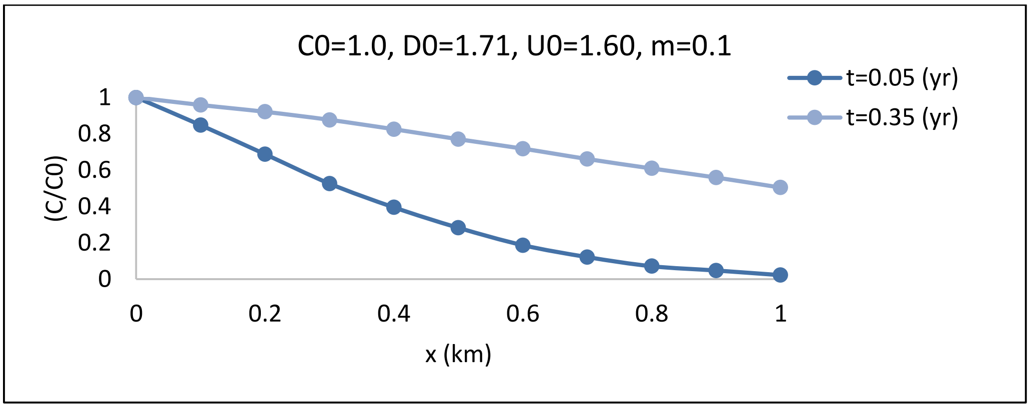

Figure 5: Exponentially accelerating flow for uniform input nature described by (27).

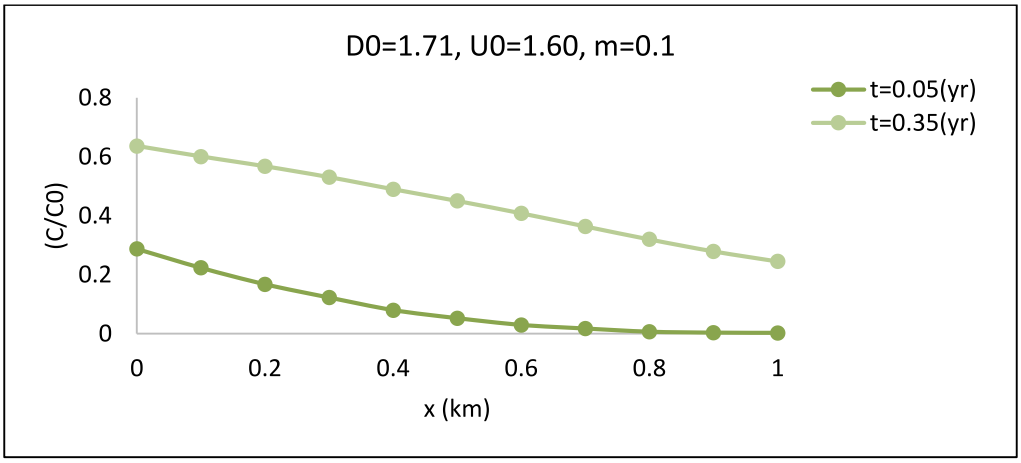

Figure 6: exponentially accelerating flow for increasing nature described by (30).

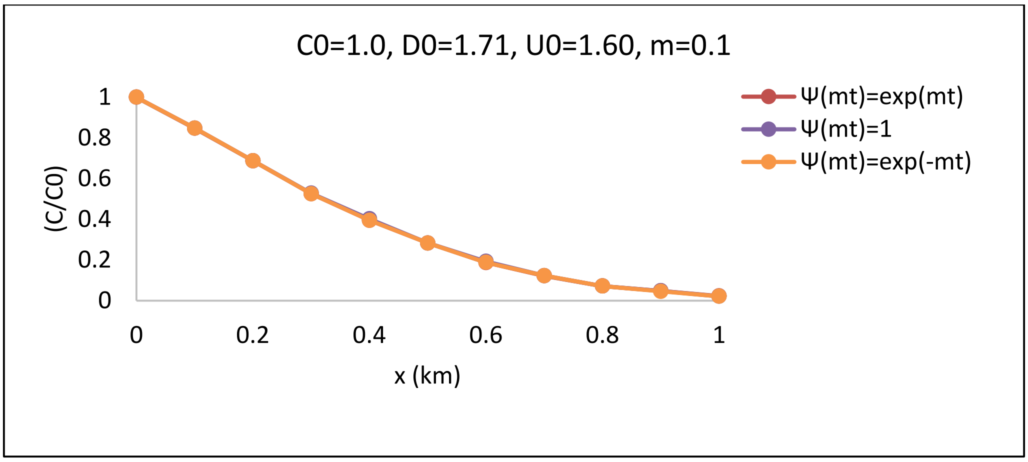

Figure 7: comparison the solution of (27), at $t = 0.05$ (year) for: (1) exponentially accelerating flow (2) uniform flow and (3) exponentially decelerating flow.

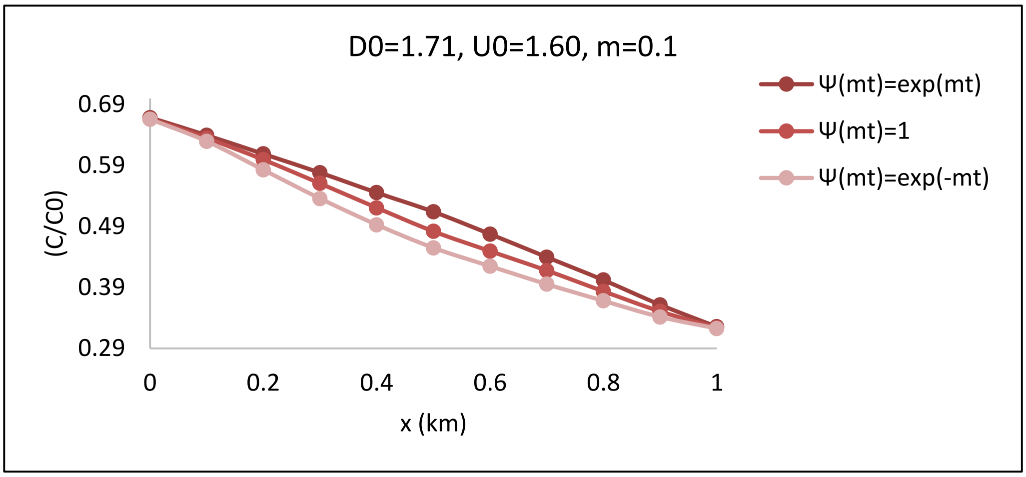

Figure 8: Comparison the solution of (30), at $t = 0.5$ (year) for: (1) exponentially accelerating flow (2) uniform flow and (3) exponentially decelerating flow.

Conclusion

Analytically solutions of one-dimensional equations of advection-diffusion in one-dimensional have been presented in semi-infinite medium. The permeable medium has been considered semi-infinite along longitudinal direction. we have discussed the numerous graphical representation for the results of three dispersal problems. In first case the analytical solution of uniform point source has been associated with analytically results increasing nature. Also different parameters of inhomogeneity have been compared with each other. In second case three different forms of $\psi(mt)$ (exponentially accelerating, exponentially decelerating and uniform) have been considered. The analytical solutions for all forms of $\psi(mt)$ have been compared.

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