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FEW IDENTITIES OF ROGERS-RAMANUJAN TYPE OF MODULO 17 AND 19

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ABSTRACT

The present paper is based on the transformation theory of the basic Hyper geometric series and its generalizations. We derive some identities of Rogers-Ramanujan type related to modulo 17 and 19 analytically as an application of some known transformations of basic Hyper geometric series.

Keywords: Rogers-Ramanujan Identity, Basic Hyper geometric Series, Jacobi's Triple Product Identity, Bailey Pair etc.

INTRODUCTION

The most famous of the "Series = product" Identities are:

For |q|<1,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\, n\neq 0,\pm 2 \text{ (mod 5)}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\, n\neq 0,\pm 1 \text{(mod 5)}$$

$$\text{where } (q;q)_n=(1-q)(1-q^2)..(1-q^n),$$

which are known as the celebrated original Rogers-Ramanujan Identity. These two identities have motivated extensive research over the past hundred years. There are two aspects of these identities:

One analytical aspect and the other is combinatorial aspect. In this paper we study only the analytical aspect.

Definitions: For |q|<1, the q-shifted factorial is defined by

$$(a;q)_0 = 1$$

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ for n≥1}$$

and $(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k)$.

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,....,a_m;q)_n = (a_1;q)_n(a_2;q)_n\dots(a_m;q)_n$$

$$(a_1,a_2,....,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\dots(a_m;q)_\infty.$$

The Basic Hyper geometric Series is

$$ {}_{p+1}\phi_{p+r}\left(\begin{matrix} a_1,a_2,.......,a_{p+1};q;x \\ b_1,b_2,.......,b_{p+r}\end{matrix}\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\dots(a_{p+1};q)_n x^n(-1)^{nr}q^{\frac{n(n-1)r}{2}}}{(q;q)_n(b_1:q)_n(b_2;q)_n\dots(b_{p+r};q)_n} \tag{1.1}$$

The series ${}_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For r=0 it converges only when |x|<1.

We also define the infinite product as

$$\prod\left(\begin{matrix} a_1, & a_2,\dots & a_r;q \\ b_1, & b_2,\dots & b_r\end{matrix}\right) = \prod_{j=0}^{\infty}\frac{(1-a_1q^j)(1-a_2q^j)(1-a_3q^j)....(1-a_rq^j)}{(1-b_1q^j)(1-b_2q^j)(1-b_3q^j)....(1-b_rq^j)}$$

Further, the well-poised series

$$ {}_{p+3}\phi_{p+2}\left(\begin{matrix} a\ ,q\sqrt{a},-q\sqrt{a},b_1,b_2.......,b_p;q;z \\ \sqrt{a},-\sqrt{a},\frac{aq}{b_1},\frac{aq}{b_2},.......,\frac{aq}{b_p}\end{matrix}\right)$$

is generally abbreviated as

$$ {}_{p+3}W_{p+2}$$

The q-analogue of Saalchütz Theorem is

$$ {}_3\phi_2\left(\begin{matrix} e\ ,f\ ,q^{-n};q \\ \frac{aq}{c},\frac{cefq^{-n}}{a}\end{matrix}\right) = \frac{\left(\frac{aq}{ec}\right)_n\left(\frac{aq}{cf}\right)_n}{\left(\frac{aq}{c}\right)_n\left(\frac{aq}{cef}\right)_n} \tag{1.2}$$

We require the following **Jacobi's Triple Product Identity** (see [5],2.2.10 , 2.2.11)

$(zq^{1/2}, z^{-1}q^{1/2}, q; q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n^2/2}$, and its corollary

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in} = \sum_{n=0}^{\infty}(-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in}.(1-q^{(2n+1)i})$$
$$= \prod_{n=0}^{\infty}\left(1-q^{(2k+1)(n+1)}\right)\left(1-q^{(2k+1)n+i}\right)\left(1-q^{(2k+1)(n+1)-i}\right) \tag{1.3}$$

Bailey Lemma: The following lemma is due to Bailey:

If L is a non-negative integer, then

$$(aq;q)_\infty \sum_{n=0}^{\infty} a^n q^{n^2-Ln}.\beta_n = \sum_{j=0}^{L}\frac{(q^{-L},q)_{j(-a)^j q^{\frac{j(j+1)}{2}}}}{(q;q)_j}.\sum_{n=0}^{\infty} a^n q^{n^2-Ln+2nj}.\alpha_n$$

where $\beta_n = \sum_{k=0}^{n}\frac{\alpha_k}{(q;q)_{n-k}(aq;q)_{n+k}}$ (For proof of this lemma, see [2], 2.15)

We begin by introducing the following transformations:

$$_{10}W_9\left(a;b,x,-x,y,-y,z,-z;q;\frac{-a^3q^3}{bx^2y^2z^2}\right)=\prod\begin{bmatrix}a^2q^2,\frac{a^2q^2}{x^2y^2},\frac{a^2q^2}{x^2z^2},\frac{a^2q^2}{y^2z^2} & ;q^2\\ \frac{a^2q^2}{x^2},\frac{a^2q^2}{y^2},\frac{a^2q^2}{z^2}, & \frac{a^2q^2}{x^2y^2z^2}\end{bmatrix}.$$

$$_5\Phi_4\begin{pmatrix}x^2,y^2,z^2,\frac{-aq}{b},\frac{-aq^2}{b}; & q^2;q^2\\ \frac{x^2y^2z^2}{a^2},\frac{a^2q^2}{b^2},-aq, & aq^2\end{pmatrix}+\prod\begin{bmatrix}x^2,y^2,z^2,\frac{a^4q^4}{b^2x^2y^2z^2} & ;q^2\\ \frac{a^2q^2}{x^2},\frac{a^2q^2}{y^2},\frac{a^2q^2}{z^2}, & \frac{x^2y^2z^2}{a^2q^2}\end{bmatrix}$$

$$.\prod\begin{bmatrix}aq,\ \frac{-a^3q^3}{x^2y^2z^2},\ \frac{a^2q^2}{x^2z^2}; & q\\ \frac{aq}{b}, & \frac{-a^3q^3}{bx^2y^2z^2}\end{bmatrix}.$$

$$_5\Phi_4\begin{pmatrix}\frac{a^2q^2}{x^2y^2},\frac{a^2q^2}{x^2z^2},\frac{a^2q^2}{y^2z^2}, & \frac{-a^3q^3}{bx^2y^2z^2},\frac{-a^3q^4}{bx^2y^2z^2};q^2;q^2\\ \frac{a^2q^4}{x^2y^2z^2},\frac{a^4q^4}{b^2x^2y^2z^2},\frac{-a^3q^3}{x^2y^2z^2}, & \frac{-a^3q^3}{x^2y^2z^2}\end{pmatrix} \tag{2.1}$$

Proof of (2.1): (See [3], Transformation (3.1), page 12]

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a^2;q^6)_{n+2r}\,(x;q^2)_{n+3r}\,(y;q^2)_{n+3r}(\frac{-aq^3}{b};q^3)_{2r}}{(q^6;q^6)_r(q^2;q^2)_n(-aq^3;q^3)_{2r}\,(a^2;q^2)_{2n+6r}(\frac{a^2q^6}{b^2};q^6)_r}.\frac{a^{2n+8r}q^{6r(r-p+1)+2n-2pn}}{(xy)^{n+3r}}$$

$$=\prod\begin{bmatrix}\frac{a^2q^2}{x} & \frac{a^2q^2}{y};q^2\\ a^2q^2 & \frac{a^2q^2}{xy}\end{bmatrix}.\sum_{j=0}^{p}\frac{(q^{-2p};q^2)_j q^{2j}}{(q^2;q^2)_j(\frac{xy}{a^2};q^2)_j}.$$

$$.\sum_{n=0}^{\infty}\frac{(x;q^2)_{3n+j}(y;q^2)_{3n+j}(a;q^3)_n(1-aq^{6n})(b;q^3)_n a^{9n}q^{3n(3n-2p+2)}}{(q^3;q^3)_n(1-a)(\frac{a^2q^2}{x};q^2)_{3n}(\frac{a^2q^2}{y};q^2)_{3n}(\frac{aq^3}{b};q^3)_n b^n(xy)^{3n}} \tag{2.2}$$

Proof of (2.2): Replacing q by q^3 and then setting $x=q^{-n}, y=q^{1-n}$, $z=q^{2-n}$ in (2.1), we get

$$\sum_{r=0}^{n/3}\frac{(a;q^3)_r(1-aq^{6r})(b;q^3)_r a^{3r}q^{9r^2}}{(q^3;q^3)_r(1-a)(\frac{aq^3}{b};q^3)_r(a^2q^2;q^2)_{n+3r}(q^2;q^2)_{n-3r}b^r}$$

$$=\frac{1}{(a^2;q^2)_{2n}}.\quad\sum_{r=0}^{n/3}\frac{(a^2;q^6)_{n-r}(-\frac{aq^3}{b};q^3)_{2r}a^{2r}q^{6r^2}}{(q^6;q^6)_r(-aq^3;q^3)_{2r}(\frac{a^2q^6}{b^2};q^6)_r(q^2;q^2)_{n-3r}} \tag{2.3}$$

Now, in Bailey Transformation [9], setting

$$u_s = \frac{1}{(q^2;q^2)_s},\ v_s = \frac{1}{(a^2q^2;q^2)_s},\ \alpha_{3s+1} = \alpha_{3s+2} = 0,\ \alpha_{3s} = \frac{(a;q^3)_s(1-aq^{6s})(b;q^3)_s a^{3s} q^{9s^2}}{(q^3;q^3)_s(1-a)(\frac{aq^3}{b};q^3)_s},$$

$\delta_s = (x;q^2)_s(y;q^2)_s(\frac{a^2q^{2-2p}}{xy})^s$ and evaluating $<\beta_n>, <\gamma_n>$ by (2.3) and the formula [4]

$${}_2\phi_1\begin{pmatrix} a, b; q; \frac{ec}{ab} \\ e \end{pmatrix} = \prod\begin{pmatrix} \frac{e}{a}, \frac{e}{b}; q \\ e, \frac{e}{ab} \end{pmatrix} 3\Phi2\begin{pmatrix} a, b, c; q; q \\ \frac{abq}{e}, 0 \end{pmatrix} \tag{2.4}$$

(where either a, b or c is of the form q^{-p}. In case only c of the form q^{-p} then (2.4) is valid only if $|\frac{ec}{ab}| < 1$), we get (2.2)

3. ***Identities related to modulo 19:***

Taking $b, x, y \to \infty$ in the transformation (2.1) and then replacing q by $q^{1/3}$, it reduces to

$$(a^2q^{2/3};q^{2/3})_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a^2;q^2)_{n+2r}\, a^{2n+8r}\, q^{(\frac{2}{3}n^2+4nr+8r^2-\frac{2}{3}np-2rp)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(a^2;q^{2/3})_{2n+6r}(-aq;q)_{2r}}$$

$$=\sum_{j=0}^{p} \frac{(q^{-2p/3};q^{2/3})_j(-1)^j a^{2j} q^{j(j+1)/3}}{(q^{2/3};q^{2/3})_j} \cdot \sum_{n=0}^{\infty} \frac{(a;q)_n(1-aq^{2n})(-1)^n a^{9n} q^{\frac{19}{2}n^2-\frac{n}{2}+4nj-2np}}{(q;q)_n(1-a)} \tag{3.1}$$

Now setting $a = 1, p = 0, 1$ in (3.1) we obtain the following identities-

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{n+2r-1}\, q^{(\frac{2}{3}n^2+4nr+8r^2)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r-1}(-q;q)_{2r}}$$

$$= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n(1+q^n)q^{\frac{19n^2-n}{2}}$$

$$= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{19n^2-n}{2}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)},\ n \not\equiv 0,9,10 \pmod{19} \tag{3.2}$$

and

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{n+2r-1}\, q^{(\frac{2}{3}n^2+4nr+8r^2-\frac{2}{3}n-2r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r-1}(-q;q)_{2r}}$$

$$= \frac{1}{(q;q)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n(1+q^n)q^{\frac{19n^2-5n}{2}} + \sum_{n=-\infty}^{\infty}(-1)^n(1+q^n)q^{\frac{19n^2+3n}{2}}\right]$$

$$= \frac{1}{(q;q)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{19n^2-5n}{2}} + \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{19n^2-3n}{2}}\right]$$

$$= \prod_{n=1}^{\infty}\frac{1}{(1-q^n)} + \prod_{n=1}^{\infty}\frac{1}{(1-q^n)} \tag{3.3}$$

where $n \not\equiv 0,7,12 \pmod{19}$, $n \not\equiv 0,8,11 \pmod{19}$

Again, setting $a = q,\ p = 0, 1$ in (3.1) we get,

$$\frac{(q^{8/3};q^{2/3})_\infty(1-q)}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{n+2r}\, q^{(\frac{2}{3}n^2+4nr+8r^2+2n+8r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^2;q^{2/3})_{2n+6r}(-q^2;q)_{2r}}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \, (1-q^{2n+1}) q^{\frac{19n^2+17n}{2}}$$

$$= \frac{1}{(q;q)_\infty} \left[\sum_{n=0}^{\infty} (-1)^n \, q^{\frac{19n^2+17n}{2}} + \sum_{n=-1}^{\infty} (-1)^n \, q^{\frac{19n^2+17n}{2}}\right]$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{19n^2+17n}{2}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \, , n \not\equiv 0,1,18 \ (\text{mod } 19) \qquad (3.4)$$

and,

$$\frac{(q^{8/3};q^{2/3})_\infty (1-q)}{(q;q)_\infty} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{n+2r} \, q^{(\frac{2}{3}n^2+4nr+8r^2+\frac{4}{3}n+6r)}}{(q^2;q^2)_r (q^{2/3};q^{2/3})_n (q^2;q^{2/3})_{2n+6r} \, (-q^2;q)_{2r}}$$

$$= \frac{1}{(q;q)_\infty} \left[\sum_{n=0}^{\infty} (-1)^n \, (1-q^{2n+1}) q^{\frac{19n^2+13n}{2}} + \sum_{n=0}^{\infty} (-1)^n \, (1-q^{2n+1}) q^{\frac{19n^2+21n+4}{2}}\right]$$

$$= \frac{1}{(q;q)_\infty} \left[\sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{19n^2+13n}{2}} \quad - \quad q.\sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{19n^2+17n}{2}}\right]$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \qquad - \qquad q.\ \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \qquad (3.5)$$

$where\ n \not\equiv 0,3,16$ (mod 19) , $\qquad n \not\equiv 0,1,18$ (mod 19)

Also, using (3.4) in (3.5), we get another identity viz,

$$\frac{(q^{8/3};q^{2/3})_\infty (1-q)}{(q;q)_\infty} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{n+2r} \, (1+q^{\frac{2}{3}n+2r+1}) \, q^{(\frac{2}{3}n^2+4nr+8r^2+\frac{4}{3}n+6r)}}{(q^2;q^2)_r (q^{2/3};q^{2/3})_n (q^2;q^{2/3})_{2n+6r} \, (-q^2;q)_{2r}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \, , n \not\equiv 0,3,16 \ (\text{mod } 19) \qquad (3.6)$$

4. ***Identities related to modulo 17:***

Taking $x, y \to \infty,\ b \to 0$ in the transformation (2.2) and then replacing q by $q^{1/3}$, it reduces to

$$(a^2q^{2/3};q^{2/3})_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a^2;q^2)_{n+2r} \, a^{2n+8r} \, q^{(\frac{2}{3}n^2+4nr+9r^2-\frac{2}{3}np\ -\ 2rp)}}{(q^2;q^2)_r (q^{2/3};q^{2/3})_n (a^2;q^{2/3})_{2n+6r} \, (-aq;q)_{2r}}$$

$$= \sum_{j=0}^{p} \frac{(q^{-2p/3};q^{2/3})_j (-1)^j a^{2j} q^{j(j+1)/3}}{(q^{2/3};q^{2/3})_j} \cdot \sum_{n=0}^{\infty} \frac{(a;q)_n (1-aq^{2n}) (-1)^n a^{8n} \, q^{\frac{17}{2}n^2-\frac{n}{2}+4nj-2np}}{(q;q)_n (1-a)} \qquad (4.1)$$

Setting $a = 1,\ p = 0,$ 1, 2 in (4.1) we obtain the following identities-

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{n+2r-1} \, (-1)^r \, q^{(\frac{2}{3}n^2+4nr+9r^2)}}{(q^2;q^2)_r (q^{2/3};q^{2/3})_n (q^{2/3};q^{2/3})_{2n+6r-1} \, (-q;q)_{2r}}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \, (1+q^n) q^{\frac{17n^2-n}{2}}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{17n^2-n}{2}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \, , n \not\equiv 0,\ 8,\ 9 \ (\text{mod } 17) \qquad (4.2)$$

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{n+2r-1}\,(-1)^r\,q^{(\frac{2}{3}n^2+4nr+9r^2-\frac{2n}{3}-2r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r-1}\,(-q;q)_{2r}}$$

$$=\frac{1}{(q;q)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n\,(1+q^n)q^{\frac{17n^2-5n}{2}}+\sum_{n=0}^{\infty}(-1)^n\,(1+q^n)q^{\frac{17n^2+3n}{2}}\right]$$

$$=\frac{1}{(q;q)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2-5n}{2}}+\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2-3n}{2}}\right]$$

$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\quad+\quad\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\qquad(4.3)$$

$where\ n\not\equiv 0,6,11$ (mod 17) , $n\not\equiv 0,7,10$ (mod 17)

and

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{n+2r-1}\,(-1)^r\,q^{(\frac{2}{3}n^2+4nr+9r^2-\frac{4n}{3}-4r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r-1}\,(-q;q)_{2r}}$$

$$=\frac{1}{(q;q)_\infty}\Big[\sum_{n=0}^{\infty}(-1)^n\,(1+q^n)q^{\frac{17n^2-9n}{2}}+(1+q^{\frac{-2}{3}})\sum_{n=0}^{\infty}(-1)^n\,(1+q^n)q^{\frac{17n^2-n}{2}}$$
$$+\sum_{n=0}^{\infty}(-1)^n\,(1+q^n)q^{\frac{17n^2+7n}{2}}\Big]$$

$$=\frac{1}{(q;q)_\infty}\Big[\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2-9n}{2}}+(1+q^{\frac{-2}{3}})\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2-n}{2}}$$
$$+\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2-7n}{2}}\Big]$$

$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\quad+\quad(1+q^{\frac{-2}{3}})\,.\,\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\quad+\quad\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\qquad(4.4)$$

$where\ n\not\equiv 0,4,13$ (mod 17), $n\not\equiv 0,8,9$ (mod 17), $n\not\equiv 0,5,12$ (mod 17)

Also, Setting $a=q,\ p=0,1$ in (4.1) we obtain the following identities-

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{n+2r-1}\,(-1)^r\,q^{(\frac{2}{3}n^2+4nr+9r^2+2n+8r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r+2}\,(-q;q)_{2r+1}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\,(1-q^{2n+1})q^{\frac{17n^2+15n}{2}}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2+15n}{2}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\ ,n\ \not\equiv 0,1,16\ (\text{mod }17)\qquad(4.5)$$

and,

$$\frac{(q^{8/3};q^{2/3})_\infty\left(1-q^{\frac{2}{3}}\right)\left(1-q^{\frac{4}{3}}\right)(1-q^2)}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{n+2r-1}\,(-1)^r\,q^{(\frac{2}{3}n^2+4nr+9r^2+\frac{4}{3}n+6r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r+2}\,(-q;q)_{2r+1}}$$

$$=\frac{1}{(q;q)_\infty}\Big[\sum_{n=0}^{\infty}(-1)^n\,q^{\frac{17n^2+11n}{2}}+\sum_{n=0}^{\infty}(-1)^n\,q^{\frac{17n^2-11n}{2}}-\sum_{n=0}^{\infty}(-1)^n\,q^{\frac{17n^2+15n+2}{2}}$$
$$-\quad\sum_{n=0}^{\infty}(-1)^n\,q^{\frac{17n^2-15n+2}{2}}\Big]$$

which implies

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{n+2r-1}\,(-1)^r\,q^{(\frac{2}{3}n^2+4nr+9r^2+\frac{4}{3}n+6r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r+2}\,(-q;q)_{2r+1}}$$

$$=\frac{1}{(q;q)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2+11n}{2}}-q.\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{17n^2+15n}{2}}\right]$$

$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\qquad-\;q\;.\;\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\tag{4.6}$$

$where\ n\not\equiv 0,3,14$ (mod 17) $\qquad n\not\equiv 0,1,16$ (mod 17)

Now, using (4.5) in (4.6), we get,

$$\frac{(q^{2/3};q^{2/3})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{n+2r-1}(-1)^r(1+q^{\frac{2n}{3}+2r+1})\;q^{(\frac{2}{3}n^2+4nr+9r^2+\frac{4}{3}n+6r)}}{(q^2;q^2)_r(q^{2/3};q^{2/3})_n(q^{2/3};q^{2/3})_{2n+6r+2}\,(-q;q)_{2r+1}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)},\ n\not\equiv 0,3,14\ (\text{mod }17)\tag{4.7}$$

References

[1]. R. P. Agarwal and A. Verma, Generalised Basic Hypergeomeric Series with unconnected bases. *Proc. Cambridge philos. Soc.* 63(1967), 727-734.

[2]. A. Verma and V. K. Jain, Transformation between Basic Hypergeomeric Series on different bases and identities of Rogers-Ramanujan Type, J*. math. Anal. Appl.* 76 (1980), 230-269.

[3]. A. Verma and V. K. Jain, Transformation of Non-terminating Basic Hypergeomeric Series, Their Contour Integrals and Applications to Rogers-Ramanujan Identities, J*. math. Anal. Appl.87 (1982), 9-44.*

[4]. A. Verma, Certain Summation formulae for Basic Hypergeomeric Series, Canad. Math. Bull. 20 (1977) 369-375.

[5]. G.E. Andrews, "Encyclopedia of Mathematics and its application",(*Ed: Gian- Carlo Rota (ed.)2, The Theory of partitions, Addison Wesley co.,*

[6]. V. K. Jain, Certain transformation of Basic Hypergeometric Series and their applications. Pacific J. Math., in press.

[7]. L. J. Slater, *Further Identities of Rogers-Ramanujan Type. Proc.London Math*. Soc.54 (1952) 147-167.

[8]. S.F.U.Ali Ahmed et. al.*, International Journal of Modern Sciences and Engineering Technology* (IJMSET) ISSN 2349-3755; Volume 1, Issue 6, 2014, pp.98-106

[9]. W. N. Bailey, Some Identities in Combinatory Analysis*, Proc. London Math. Soc.* (2) 49 (1947), 421-425.