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GENERALIZED HYERS-ULAM-RASSIAS TYPE STABILITY OF THE $2k$-VARIABLE QUADRATIC FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES AND BANACH SPACES

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ABSTRACT

In this paper we use the direct method to proved two the generalized quadratic functional inequalities with 2k-variables and their Hyers-Ulam-Rassias stability. First are investigated in Banach spaces and the last are investigated in non-Archimedean Banach spaces. We will show that the solutions of the inequalities are quadratic map- pings. These are the main results of this paper.

Mathematics subject classification: Primary 46S10, 47H10, 39B62, 39B72, 39B52, 12J25.

Keywords: functional equation, quadratic functional inequality, quadratic β–functional inequalities, Banach space, non-Archimedian Banach space, Hyers-Ulam-Rassias stability.

Let X_1 and X_2 be a normed spaces on the same field $\mathbb{K}$, and F : $X_1 \to X_2$. We use the notation ||•|| for all the norm on both X_1 and X_2. In this paper, we investisgate some quadratic functional inequality when X_1 and X_2 is a Banach spaces or X_1 is a non-Archimedean normed space and X_2 is a non-Archimedean Banach space.

In fact, when X_1 and X_2 is Banach spaces we solve and prove the Hyers-Ulam-Rassias type stability of forllowing quadratic functional inequality.

$$\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$
$$\leq\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2} \quad (1.1)$$

and when X_1 is a non-Archimedean normed space and X_2 is a non-Archimedean Banach spaces we solve and prove the Hyers-Ulam stability of forllowing quadratic functional inequality.

$$\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$
$$\leq\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2} \quad (1.2)$$

The study of the functional equation stability originated from a question of S.M. Ulam [22], concerning the stability of group homomorphisms. Let $(G,*)$ be a group and let (G',o,d) be a metric group with metric $d(\bullet,\bullet)$. Geven $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$d\big(f(x*y),f(x)o\,f(y)\big)<\delta$$

for all $x,y \in G$ then there is a homomorphism $h : G \to G'$ with

$$d\big(f(x),f(x)\big)<\epsilon$$

for all $x \in G$?, if the answer, is affirmative, we would say that equation of homomophism $h\big(x*y)=h(y)o\,h(y)\big)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation?

The stability of quadratic functional equation was proved by Skof [21] for mappings $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space . Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation:

$$f\ (x+y)+f\ (x+y)-2f\ (x)-2f\ (y)$$

is called the quadratic functional equation.

The functional equation:

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f\ (x)-\frac{1}{2}f\ (y)=0$$

is called a Jensen type the quadratic functional equation. The first work on the stability problem for functional equations in non-Archimedean spaces was started by Moslehian and Rassias [16]. Moslehian and Sadeghi [15] investigated the stability of cubi functional equations in non-Archimedean normed space.

In [10] Gila´ny showed that is if satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \leq \|f(xy)\|$$

Then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)-f(xy^{-1}) \tag{1.3}$$

Seen [20]. Gilanyi [11] and Fechner [8] proved the Hyers-Ulam-Rassia stability of the functional inequality.

Choonkil Park [18] obtained the solutions of the quadratic functional inequality. Recently, in [2, 14, 18] the authors studied the Hyers-Ulam-Rassia stability for the following functional inequalities in Banach space and non-Archimedean Banach space:

$$\|f(x+y)+f(x+y)-2f(x)-2f(y)\| \leq \left\|f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right\| \tag{1.4}$$

and

$$\left\|f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y)\right\| \leq \|f(x+y)+f(x-y)-2f(x)-2f(y)\| \tag{1.5}$$

Next

$$\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right\| \leq \left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+f\left(\frac{x-y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\| \tag{1.6}$$

And

$$\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+f\left(\frac{x-y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\| \leq \left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right\| \tag{1.7}$$

In this paper, we solve and proved the Hyers-Ulam-Rassias type stability for two quadratic functional inequalities (1.1)-(1.2), ie the quadratic functional inequalities with $2k-$ variables . Under suitable assumptions on spaces X_1 and X_2, we will prove that the mappings satisfying the quadratic functional inequatilies (1.1) or (1.2). Thus, the results in this paper are generalization of those in [2, 14, 18] for functional inequatilies with $2k-$ variables.

The paper is organized as followns:

In section preliminaries we remind some basic notations in [6, 14] such as We only redefine the solution definition of the quadratic equation function.

Section 3: is devoted to prove the Hyers-Ulam stability of the quadratic functional in equalities (1.1) when we assume that X_1 and X_2 is a Banach spaces.

Section 4: is devoted to prove the Hyers-Ulam stability of the quadratic functional inequalities (1.2) when X_1 is a non-Archimedean normed space and X_2 is a non-Archimedean Banach space.

2. PRELIMINARIES

2.1. non-Archimedean normed spaces. In this subsection we recall some basic notations from [14, 18] such as non-Archimedean fields, non-Archimedean normed spaces and non-Archimedean Banach spaces.

A valuation is a function $|.|$ from a field $\mathbb{K}$ into [0, ∞) such that 0 is the unique element having the 0 valuation,

$$|r| = 0 \Leftrightarrow r = 0,$$

$$|rs| = |r||s|, \forall r, s \in \mathbb{K}$$

and the triangle inequality holds, ie;

$$|r + s| \leq |r| + |s|, \forall r, s \in \mathbb{K}$$

A field $\mathbb{K}$ is called a valued filed if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inaquality. If the tri triangle inequality is replaced by

$$|r + s| \leq \max\{|r| + |s|\} \forall r, s \in \mathbb{K}$$

then the function $|.|$ is called a norm -Archimedean valuational, and filed. Clearly $|1|$ =$|-1|$= 1 and $|n|$ ≤ 1, , $\forall n \in \mathbb{N}$. A trivial expamle of a non- Archimedean valuation is the function $|.|$ talking everything except for 0 into 1 and $|0|$ = 0 this paper, we assume that the base field is a non- Archimedean filed, $|\mathbf{2}| \neq \mathbf{1}$ hence call it simply a filed.

Definition 2.1. Let be a vecter space over a filed K with a non -Archimedean |.|. A function ||•|| : X → [0, ∞) is said a non -Archimedean norm if it satisfies the follwing conditions:

$$(1)\ \|x\| = 0 \ if \ and \ only \ if \ x = 0;$$

$$(2)\ \|rx\| = |r| \ \|x\| \ (r \in \mathbb{K}, x \in X);$$

$$(3)\ \|x + y\| \leq \max\{\|x\|, \|y\|\} x, y, \in X \ hold$$

Then $(X, \|\bullet\|)$ is called a norm -Archimedean norm space.

Definition 2.2. Let, $\{x_n\}$, be a sequence in a non --Archimedean normed space X is a Cauchy sequence if and only if $\{x_{n \to} x_m\} \to 0$

Definition 2.3. . Let, $\{x_n\}$, be a sequence in a norm--Archimedean normed space X

1. A sequence , $\{x_n\}_{n=1}^{\infty}$in a non -Archimedean space is a Cauchy sequence if the, $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.
2. The sequence $\{x_n\}$, is said to be convergent if, for any $\in > 0$, there are a positive integer **N** and $x \in X$ such that

$$\|x_n - x\| \leq \in \forall n \geq N,$$

for all $n, m \geq N$. Then the point $x \in X$ is called the limit of sequence x_n, which is denoted by $\lim_{n\to\infty} x_n = x$.

3. If every sequence Cauchy in X converges, then the norm -Archimedean normed space X is called a norm -Archimedean Branch space.

2.2. Solutions of the inequalities. The functional equation

$$f\ (x + y) + f\ (x - y) = 2f\ (x) + 2f\ (y)$$

is called the quadratic equation. In particular, every solution of the quadratic equation is said to be an $quadratic\ mapping$.

3. QUADRATIC FUNCTIONAL INEQUALITY IN BANACH SPACE

Now, we first study the solutions o f (1.1). Note that for this inequalitie, X_1 and X_2 is Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) is quadratic. These results are give in the following.

Lemma 3.1. A mapping $F : X_1 \to X_2$ satilies

$$\left\| F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\right) - 2\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - 2\sum_{j=1}^{k} F(x_j)\right\|_{X_2}$$

$$\leq \left\| F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - \frac{2}{k}\sum_{j=1}^{k} F(x_j)\right\|_{X_2} \quad (3.1)$$

$for\ all\ x_j\ ;\ x_{k+j} \in X_1\ for\ all\ j = 1 \to k\ if\ and\ only\ F : X_1 \to X_2\ is\ quadratic.$

$Proof$: *Assume that* $F : X_1 \to X_2\ satisfies\ (3.1)$

Letting $x_j = x_{k+j} = 0, j = 1 \to k\ in\ , we\ get$

$$(|2k - 1| - 1)\|F(0)\|_{X_2} \leq 0$$

So $F(0) = 0$.

Letting x_{k+j} = 0 and $x_j = x$ for all $j = 1 \to k$ in (3.1), we get

$$\|f(kx) - kf(x)\|_{X_2} \leq 0 \quad (3.2)$$

$$and\ so\ f\ (kx) = kf(x) or\ all\ x \in X_1.$$

$Thus$

$$f\left(\frac{x}{k}\right) = \frac{1}{k} f(x) \quad (3.3)$$

for all $x \in X_1$ It follows from (3.1) and (3.3) that:

$$\left\|f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}f(x_j)\right\|_{X_2}$$

$$\leq\left\|\frac{1}{k}f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+\frac{1}{k}f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}f\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}f(x_j)\right\|_{X_2}$$

$$=\frac{1}{k}\left\|f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+1}+\sum_{j=1}^{k}x_j\right)+f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+1}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}f(x_j)\right\|_{X_2} \tag{3.4}$$

and so

$$f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)=2\sum_{j=1}^{k}f\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}f(x_j)$$

for all $x_j\ ;\ x_{k+j}\ \in X_1\ for\ all\ j\ =\ 1\to k$. Hence $F\ :\ X_1\to X_2$ is quadratic.

The coverse is obviously true.

***Theorem* 3.2.** *Let* $\varphi:\ X_1^{2k}\ \to\ [0;\infty)$ *be a function and let* $X_1\to X_2$ *be mapping such that*

$$\varphi\,(x_1,x_2,\ldots..x_k,x_{k+1},x_{k+2},\ldots.x_2k)=\sum_{j=1}^{\infty}k^j\psi\left(\frac{x_1}{k^j},\frac{x_2}{k^j},\ldots.,\frac{x_k}{k^j},\frac{x_{k+1}}{k^j},\frac{x_{k+2}}{k^j}\ldots..\frac{x_{2k}}{k^j}\right)<\ \infty \tag{3.5}$$

$$\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$\leq\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$
$$+\varphi\left(x_1,x_2,\ldots..x_k,x_{k+1},x_{k+2},\ldots.x_{2k}\ \right) \tag{3.6}$$

for all $x_j\ ;\ x_{k+j}\ \in\ \mathbb{X}$, *for all* $j\ =\ 1\to k$. *Then there exists a unique quadratic mapping* $Q\ :\ X_1\to X_2$ *such that,*

$$\|F(x)-Q(x)\|_{X_2}\leq\frac{1}{k}\ \varphi\,(x,x,\ldots x,0,0\ldots\ldots,0) \tag{3.7}$$

for all $x\ \in\ X_1$
Proof. *Letting* $x_j\ =\ x_{k+j}=0$ *for all* $j\ =\ 1\to k$ *in (3.6),*
we get

$$(|2k-1|-1)\|F(0)\|_{X_2} \leq 0 \tag{3.8}$$

So

$F(0) = 0$

Letting $x_{k+j} = 0, x_j = x$ *for all* $j = 1 \to k$ *in (3.6), we get*

$$\|F(kx) - kF(x)\|_{X_2} \leq \frac{1}{k}\,\psi\,(x, x, \dots x, 0,0 \dots \dots ,0) \tag{3.9}$$

$$\left\|F(x) - kF(\frac{x}{k})\right\|_{X_2} \leq \frac{1}{k}\,\psi\,(x/k, x/k, \dots x/k, 0,0 \dots \dots ,0)$$

Hence

$$\begin{aligned}\left\|k^l F\left(\frac{x}{k^l}\right) - k^m F\left(\frac{x}{k^m}\right)\right\|_{X_2} \\ &\leq \sum_{j=1}^{m-1}\left\|k^j F\left(\frac{x}{k^j}\right) - k^{j+1}F\left(\frac{x}{k^{j+1}}\right)\right\|_{X_2} \\ &\leq \frac{1}{k}\sum_{j=l+1}^{m} k^j \psi\left(\frac{x}{k^j}, \frac{x}{k^j}, \dots, \frac{x}{k^j}, 0,0, \dots .0\right)\end{aligned} \tag{3.10}$$

for all nonnegative, integers m, and l with $m > l$ and all $x \in X_1$. It follows from (3.10) that the sequence $\{k^n F\,(\frac{x}{k^n})\}$ is a cauchy sequence for all $x \in X_1$. Since X_2 is complete space, the sequence $\{k^n F\,(\frac{x}{k^n})\}\ coverges$. So one can define the mapping $Q : X_1 \to X_2$ by

$$Q\,(x) := \lim_{n\to\infty} k^n F\,\left(\frac{x}{k^n}\right)$$

for all $x \in X_1$. Moreover, letting l = 0 and passing the limit m ! 1 in (3.10), we get (3.7).

for all $x \in X_1$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.7)

Now, It follows from (3.5) and (3.6) that

$$\begin{aligned}\left\|Q\left(\sum_{j=1}^{k} x_{\frac{k+j}{k}} + \sum_{j=1}^{k} x_j\right) + Q\left(\sum_{j=1}^{k} x_{\frac{k+j}{k}} - \sum_{j=1}^{k} x_j\right) - 2\sum_{j=1}^{k} Q\left(x_{\frac{k+j}{k}}\right) - 2\sum_{j=1}^{k} Q(x_j)\right\|_{X_2} \\ &\leq \lim_{n\to\infty} k^n \left\|F\left(\sum_{j=1}^{k} x_{\frac{k+j}{k^{n+1}}} + \frac{1}{k^n}\sum_{j=1}^{k} x_j\right) + F\left(\sum_{j=1}^{k} x_{\frac{k+j}{k^{n+1}}} - \frac{1}{k^n}\sum_{j=1}^{k} x_j\right)\right. \\ &\left. - 2\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k^{n+1}}}\right) - 2\sum_{j=1}^{k} F\left(x_{\frac{j}{k^n}}\right)\right\|_{X_2}\end{aligned}$$

$$\leq \lim_{n\to\infty} k^n \left\| F\left(\sum_{j=1}^{k} x_{\frac{k+j}{k^{n+2}}} + \frac{1}{k^n}\sum_{j=1}^{k} x_j\right) + F\left(\sum_{j=1}^{k} x_{\frac{k+j}{k^{n+2}}} - \frac{1}{k^n}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k^{n+2}}}\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(x_{\frac{j}{k^n}}\right)\right\|_{X_2} + \lim_{n\to\infty} k^n\ \psi\left(\frac{x_1}{k^j},\frac{x_2}{k^j},\dots,\frac{x_k}{k^j},\frac{x_{k+1}}{k^j},\frac{x_{k+2}}{k^j},\dots\frac{x_{2k}}{k^j}\right)$$

$$= \left\| F\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) + F\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - \frac{2}{k}\sum_{j=1}^{k} F(x_j)\right\|_{X_2} \quad (3.11)$$

for all $x_j\ ;\ x_{k+j} \in X_1$, *for all* $j = 1 \to k$.

So

$$\left\| Q\left(\sum_{j=1}^{k} x_{\frac{k+j}{k}} + \sum_{j=1}^{k} x_j\right) + Q\left(\sum_{j=1}^{k} x_{\frac{k+j}{k}} - \sum_{j=1}^{k} x_j\right) - 2\sum_{j=1}^{k} Q\left(x_{\frac{k+j}{k}}\right) - 2\sum_{j=1}^{k} Q(x_j)\right\|_{X_2}$$

$$\leq \left\| Q\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) + Q\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} Q\left(x_{\frac{k+j}{k}}\right) - \frac{2}{k}\sum_{j=1}^{k} Q(x_j)\right\|_{X_2} \quad (3.12)$$

for all $x_j\ ;\ x_{k+j} \in X_1$, *for all* $j = 1 \to k$. By Lemma (3.1), the mapping $Q : X_1 \to X_2$ is quadratic. Next, suppose that $T : X_1 \to X_2$ be another quadratic mapping satisfying (3.7). Then we have

$$\|Q(x) - T(x)\|_Y = k^n \left\| Q\left(\frac{x}{k^n}\right) - T\left(\frac{x}{k^n}\right)\right\|_Y$$

$$\leq k^n\left(\left\| Q\left(\frac{x}{k^n}\right) - F\left(\frac{x}{k^n}\right)\right\|_Y + \left\| T\left(\frac{x}{k^n}\right) - F\left(\frac{x}{k^n}\right)\right\|_Y\right)$$

$$\leq k^n\left(\frac{1}{k^n}\varphi\left(\frac{x}{k^n},\frac{x}{k^n},\dots,\frac{x}{k^n},0,0,\dots,0\right) + \frac{1}{k^n}\varphi\left(\frac{x}{k^n},\frac{x}{k^n},\dots,\frac{x}{k^n},0,0,\dots,0\right)\right)$$

$$= k^n.\frac{2}{k}\ \varphi\left(\frac{x}{k^n},\frac{x}{k^n},\dots,\frac{x}{k^n},0,0,\dots,0\right)$$

$$\leq k^n.\varphi\left(\frac{x}{k^n},\frac{x}{k^n},\dots,\frac{x}{k^n},0,0,\dots,0\right) \quad (3.13)$$

which tends to zero as $n \to \infty$ for all $x \in X_1$. So we can conclude that $Q(x) = T(x)$ for all $x \in X_1$. This proves the uniqueness of Q. Thus the mapping $Q : X_1 \to X_2$ is a unique quadratic mapping satisfying (3.7).

Corollary 3.3. *Let* $r > 1$ *and* θ *be nonnegative real numbers and* $F: X_1 \to X_2$ *be a mapping satisfying*

$$\left\| F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\right) - 2\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - 2\sum_{j=1}^{k} F(x_j)\right\|_{X_2}$$

$$\leq \left\| F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - \frac{2}{k}\sum_{j=1}^{k} F(x_j) \right\|_{X_2}$$

$$+ \ \theta\left(\sum_{j=1}^{k} \|x_i\|_{X_2}^r + \sum_{j=1}^{k} \|x_{k+i}\|_{X_1}^r\right) \qquad (3.14)$$

for all x_j ; $x_{k+j} \in X_1$, for all $j = 1 \to k$. Then there exists a unique quadratic mapping Q : $X_1 \to X_2$ such that

$$\|F(x) + Q\ (x)\|_{X_2} \leq \frac{2k\theta}{k^r - k} \|x\|^r \qquad 3.15)$$

for all $x \in X_1$

Theorem 3.4. Let ' $\varphi: X_1^{2k} \ \rightarrow \ [0;\infty)$ *be a function and let* $F : X_1 \to X_2$*be mapping such that*

$x_1, x_2 \ldots\ldots x_k, x_{k+1}, x_{k+2}, \ldots\ldots x_{2k}$

$$= \sum_{j=1}^{\infty} \frac{1}{k^j}\ \psi\left(k^j x_1, k^j x_2 \ldots\ldots k^j x_k, k^j x_{k+1}, k^j x_{k+2}, \ldots\ldots k^j x_{2k}\right) < \infty \qquad (3.16)$$

$$\left\| F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\right) - 2\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - 2\sum_{j=1}^{k} F(x_j) \right\|_{X_2}$$

$$\leq \left\| F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - \frac{2}{k}\sum_{j=1}^{k} F(x_j) \right\|_{X_2}$$

$$+ \ \ \psi\left(k^j x_1, k^j x_2 \ldots\ldots k^j x_k, k^j x_{k+1}, k^j x_{k+2}, \ldots\ldots k^j x_{2k}\right) \qquad 3.17)$$

for all all x_j ; $x_{k+j} \in X_1$, for all $j = 1 \to k$. Then there exists a unique additive mapping $Q : X_1 \to X_2$ such that,

$$\|F(x) + Q\ (x)\|_{X_2} \leq \frac{1}{k}\psi\ (x, x, \ldots\ldots. x, 0{,}0 \ldots.., 0) \qquad (3.18)$$

for all $x \in X_1$

Proof. Letting$x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (3.17), we get

$$(|2k - 1| - 1)\|F(0)\|_{X_2} \qquad (3.19)$$

So,

$$F(0) = 0$$

Letting $x_{k+j} = 0$, $x_j = x$ for all $j = 1 \to k$ in (3.17), we get

$$\|F(kx) + KF(x)\|_{X_2} \leq \frac{1}{k}\psi\ (x, x, \ldots\ldots. x, 0{,}0 \ldots.., 0) \qquad (3.20)$$

Thus

$$\left\| F(x) + \frac{1}{k}F(kx) \right\|_{X_2} \leq \frac{1}{k}\psi\ (x, x, \ldots\ldots. x, 0{,}0 \ldots.., 0)$$

Hence

$$\left\|\frac{1}{k^l}F(k^l x)+\frac{1}{k^m}F(k^m x)\right\|_{X_2}$$

$$\leq \sum_{j=1}^{m-1}\left\|\frac{1}{k^j}F(k^j x)-\frac{1}{k^{j+1}}F(k^{j+1}x)\right\|_{X_2}$$

$$\leq \frac{1}{k}\sum_{j=1}^{m}\frac{1}{k^{j+1}}\psi(k^j x, k^j x \ldots\ldots k^j x, 0,0,\ldots\ldots 0) \tag{3.21}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X_1$. It follows from (3.21) that the sequence $\left\{\frac{1}{k^n}F(k^n x)\right\}$ is a cauchy sequence for all $x \in X_1$. Since$X_2$is complete

space, the sequence$\left\{\frac{1}{k^n}F(k^n x)\right\}$coverges.

So one can define the mapping$Q : X_1 \to X_2$by

$$Q(x) := \lim_{n\to\infty}\frac{1}{k^n}f(k^n x)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.21), we get (3.18).

We use the similar manner to the proof of Theorem 3.2 for the rest of the proof.

Corollary 3.5. Let $r < 1$ and θ be nonnegative real numbers and $F : X_1 \to X_2$ be a mapping satisfying

$$\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$\leq \left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$+\ \theta\left(\sum_{j=1}^{k}\|x_i\|^r+\sum_{j=1}^{k}\|x_{k+i}\|^r\right) \tag{3.22}$$

for all x_j ; $x_{k+j} \in X_1$, for all $j = 1 \to k$. Then there exists a unique additive mapping $Q : X_1 \to X_2$ such that,

$$\|F(x)+Q(x)\|_{X_2} \leq \frac{2k\theta}{k-k^r}\|x\|^r \tag{3.23}$$

for all $x \in X_1$

4. QUADRATIC FUNCTIONAL INEQUALITY IN NON-ARCHIMEDEAN BANACH SPACE

Now, we study the solutions of (1.2). Note that for these inequalitie, X_1 is a non-Archimedean normed space and X_2 is a non-Archimedean Banach spaces. Under this setting, we can show that the mapping satisfying (1.2) is quadratic. These results are give in the following. Assume that where k is a fixed positive integer with intiger with $|k| \neq 1$.

Lemma 4.1. A mapping $F : X_1 \to X_2$ satilies

$$\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$\leq\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2} \quad (4.1)$$

for all x_j ; $x_{k+j} \in X_1$,for all $j = 1 \to k$. if and only if $F : X_1 \to X_2$ is quadratic.

Proof. Assume that $F : X_1 \to X_2$satisfies (4.1).

Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (4.1), we get

$$(|2k-1|-1)\|F(0)\|_{X_2} \leq 0$$

So

$$F(0)=0$$

Other face

Letting $x_1 = x; x_{j+1} = x_{k+j} = 0; j = 1 \to k$ in (4.1), we obtain

$$\left\|F\left(\frac{x}{k}\right)+\frac{1}{k}F(x)\right\|_{X_2} \leq 0$$

$$\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$=\left\|F\left(\frac{1}{k}\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)\right)+F\left(\frac{1}{k}\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$=\left|\frac{1}{k}\right|\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$\leq\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2} \quad (4.2)$$

for all x_j ; $x_{k+j} \in X_1$,for all $j = 1 \to k$. Since $|k| < 1$

$$F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)=0$$

for all x_j ; $x_{k+j} \in X_1$,for all $j = 1 \to k$. On the other hand the converse is obviously true.

Theorem 4.2 Let $\varphi: X_1^{2k} \to [0;\infty)$ be a function and let $F : X_1 \to X_2$ be mapping with $\varphi(0,\ldots\ldots,0,0,\ldots\ldots,0) = 0$

$$\varphi(x_1,x_2\ldots\ldots x_k,x_{k+1},x_{k+2},\ldots\ldots x_{2k})=\sum_{j=1}^{\infty}|k^j|\ \psi\left(\frac{x_1{}^j}{k},\frac{x_2}{k^j}\ldots\frac{x_k}{k^j},x_{k+\frac{1}{kj}},x_{k+\frac{2}{kj}},\ldots\ldots\frac{x_{2k}}{k^j}\right)$$
$$<\infty \tag{4.3}$$

$$\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$
$$\leq\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$
$$+\varphi(x_1,x_2\ldots\ldots x_k,x_{k+1},x_{k+2},\ldots\ldots x_{2k}) \tag{4.4}$$

for all x_j ; $x_{k+j} \in X_1$,for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X_1 \to X_2$ such that

$$\|F(x)+H\,(x)\|_{X_2}\leq|k|\varphi(0,\ldots\ldots,0,0,\ldots\ldots,0)=0 \tag{4.5}$$

for all $x \in X_1$

Proof. Letting x_j ; $x_{k+j} \in X_1$,for all $j = 1 \to k$ in (4.4), we get

$$(|2k-1|-1)\|F(0)\|_{X_2}\leq 0$$

So

$$F(0)=0$$

Other face

Letting $x_1 = x; x_{j+1} = x_{k+j} = 0$ for all $j = 1 \to k$ in (4.4), we get

$$\left\|F(x)+KF\left(\frac{x}{k}\right)\right\|_{X_2}\leq|k|\psi\,(x,0,\ldots0,0,\ldots\ldots,0) \tag{4.6}$$

for all $x \in X_1$

Hence

$$\left\|k^lF\left(\frac{x}{k^l}\right)+k^mF\left(\frac{x}{k^m}\right)\right\|_{X_2}$$
$$\leq max\{\left\|k^lF\left(\frac{x}{k^l}\right)-k^{l+1}F\left(\frac{x}{k^{l+1}}\right)\right\|_{X_2},\ldots\ldots,\left\|k^{m-1}F\left(\frac{x}{k^{m-1}}\right)+k^mF\left(\frac{x}{k^m}\right)\right\|_{X_2}$$
$$\leq max\{\left\||k^l|\left(\frac{x}{k^j}\right)-kF\left(\frac{x}{k^{j+1}}\right)\right\|_{X_2},\ldots\ldots,k^{m-1}\left\|F\left(\frac{x}{k^{m-1}}\right)+kF\left(\frac{x}{k^m}\right)\right\|_{X_2}$$

$$\leq \sum_{j=1}^{m} |k|^{j+1}\psi\left(x/k^{j}, 0,0,0, \dots\dots 0\right) \tag{4.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X_1$. It follows from (4.7) that the sequence $\{k^n F(x/k^n)\}$ is a cauchy sequence for all $x \in X_1$. Since$X_2$is complete space, the sequence$\{k^n F(x/k^n)\}$ coverges.

So one can define the mapping $H : X_1 \to X_2$by

$$H(x) \coloneqq \lim_{n\to\infty} k^n f(x/k^n)$$

for all $x \in X_1$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.7), we get (4.5).

Now, It follows from (4.3) and (4.4) that

$$\left\|H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}H\left(x_{\frac{k+j}{k}}\right)-\frac{2}{k}\sum_{j=1}^{k}H(x_j)\right\|_{X_2}$$

$$= \lim_{n\to\infty}|k|^n\left\|F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}}+\frac{1}{k^{n+1}}\sum_{j=1}^{k}x_j\right)+F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}}-\frac{1}{k^{n+1}}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k^{n+1}}}\right)\right.$$
$$\left.-2\sum_{j=1}^{k}F\left(\frac{x_j}{k^n}\right)\right\|_{X_2}$$

$$= \lim_{n\to\infty}|k|^n\left\|F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}}+\frac{1}{k^{n}}\sum_{j=1}^{k}x_j\right)+F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}}-\frac{1}{k^{n}}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k^{n+1}}}\right)\right.$$
$$\left.-2\sum_{j=1}^{k}F\left(\frac{x_j}{k^n}\right)\right\|_{X_2} + \lim_{n\to\infty}|k|^n\psi\left(\frac{x_1}{k^n},\frac{x_2}{k^n},\dots,\frac{x_k}{k^n},\frac{x_{k+1}}{k^n},\frac{x_{k+2}}{k^n},\dots\frac{x_{2k}}{k^n}\right)$$

$$= \left\|F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)\right.$$
$$\left.-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2} \tag{4.8}$$

for all x_j ; $x_{k+j} \in X_1$,for all $j = 1 \to k$.

So

$$\left\|H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right)-\frac{2}{k}\sum_{j=1}^{k}H(x_j)\right\|_{X_2}$$

$$\left\| H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}H(x_j)\right\|_{X_2} \qquad (4.9)$$

for all x_j ; $x_{k+j} \in \boldsymbol{X}_1$,for all $j = 1 \to k$. Lemma 4.1, the mapping $H : X_1 \to X_2$ is quadratic.

Next, suppose that $T : X_1 \to X_2$ be another quadratic mapping satisfying (4.5). Then we have

$$\|H(x)+T(x)\|_{X_2} = \left\|k^n H\left(\frac{x}{k^n}\right)+k^n T\left(\frac{x}{k^n}\right)\right\|_{X_2}$$

$$\le \max\{\left\|k^n H\left(\frac{x}{k^n}\right)+k^n F\left(\frac{x}{k^n}\right)\right\|_{X_2}, \left\|k^n T\left(\frac{x}{k^n}\right)+k^n F\left(\frac{x}{k^n}\right)\right\|_{X_2}\}$$

$$\le \frac{1}{|k|}|k|^n\psi\left(\frac{x}{k^n},0,\ldots 0,0,\ldots\ldots,0\right) \qquad (4.10)$$

which tends to zero as $n \to \infty$ for all $x \in X_1$. So we can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. Thus the mapping $H : X \to X_2$ is a unique quadratic mapping satisfying (4.5).

Corollary 4.3. Let $r < 1$ and θ be nonnegative real numbers and $F: X_1 \to X_2$ be a mapping with F(0)=0 satisfying

$$\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{2}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$\le \left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)+F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}F\left(x_{\frac{k+j}{k}}\right)-2\sum_{j=1}^{k}F(x_j)\right\|_{X_2}$$

$$+\theta\left(\sum_{j=1}^{k}\|x_i\|_{X_1}^r+\sum_{j=1}^{k}\|x_{k+i}\|_{X_2}^r\right) \qquad (4.11)$$

for all x_j ; $x_{k+j} \in \boldsymbol{X}_1$,for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X_1 \to X_2$ such that

$$\|F(x)+H(x)\|_{X_2} \le \frac{|k|^{r+1}\theta}{|k|^r-|k|}\|x\|^r \qquad (4.12)$$

for all $x \in \boldsymbol{X}_1$.

Theorem 4.4.

Let $\varphi: X_1^{2k} \to [0;\infty)$ be a function and let $F : X_1 \to X_2$ be mapping with $\varphi(0,\ldots\ldots,0,0,\ldots\ldots,0) = 0$ sattisfying

$$\varphi(x_1, x_2 \dots\dots x_k, x_{k+1}, x_{k+2}, \dots\dots x_{2k}) = \sum_{j=1}^{\infty} \frac{1}{k^j} \psi(k^j x_1, k^j x_2 \dots\dots k^j x_k, k^j x_{k+1}, k^j x_{k+2}, \dots\dots k^j x_{2k}) < \infty \quad (4.13)$$

Let $r > 1$ and θ be nonnegative real numbers and $F: X_1 \to X_2$ be a mapping satisfying

$$\left\| F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - \frac{2}{k}\sum_{j=1}^{k} F(x_j) \right\|_{X_2}$$
$$\leq \left\| F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\right) - 2\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - 2\sum_{j=1}^{k} F(x_j) \right\|_{X_2}$$
$$+ \psi(x_1, x_2 \dots\dots x_k, x_{k+1}, x_{k+2}, \dots\dots x_{2k}) \quad (4.14)$$

for all x_j ; $x_{k+j} \in \boldsymbol{X}_1$,for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X \to X_2$ such that

$$\|F(x) + H(x)\|_{X_2} \leq |k|\psi\,(x, 0, \dots.0,0 \dots..,0) \quad (4.15)$$

for all $x \in X_1$.

Proof. Letting $x_1 = x, x_{j+1} = x_{k+j} = 0$ for all $j = 1 \to k$ *in (4.14),* we get

$$\left\|F(x) + \frac{1}{k}F(kx)\right\|_{X_2} \leq \psi\,(kx, 0, \dots.0,0 \dots..,0) \quad (4.16)$$

for all $x \in X_1$ the rest of the proof is similar to the proof of theorem 4.2.

Corollary 4.5. Let $r > 1$ and θ be nonnegative real numbers and $F: X_1 \to X_2$ be a mapping with F(0)=0 satisfying

$$\left\| F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{2}{k}\sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \frac{2}{k}\sum_{j=1}^{k} F(x_j) \right\|_{X_2}$$
$$\leq \left\| F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) + F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\right) - 2\sum_{j=1}^{k} F\left(x_{\frac{k+j}{k}}\right) - 2\sum_{j=1}^{k} F(x_j) \right\|_{X_2}$$
$$+ \theta\left(\sum_{j=1}^{k} \|x_i\|_{X_1}^r + \sum_{j=1}^{k} \|x_{k+i}\|_{X_2}^r\right) \quad (4.17)$$

for all x_j ; $x_{k+j} \in \boldsymbol{X}_1$,for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X_1 \to X_2$ such that

$$\|F(x) + H(x)\|_{X_2} \leq \frac{|k|^{r+1}\theta}{|k| - |k|^r}\|x\|^r \quad (4.18)$$

for all $x \in \boldsymbol{X}_1$

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