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RESEARCH ARTICLE

REMARKS ON FROBENIUS GROUPS

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ABSTRACT

Let the finite group G act transitively and non-regularly on a finite set Ω whose cardinality $|\Omega|$ is greater than one. Use N to denote the full set of fixed-point-free elements of G acting on Ω along with the identity element. Write H to denote the stabilizer of some $\alpha \in \Omega$ in G. In the note, it is proved that the subset N is a subgroup of G if and only if G is a Frobenius group. It is also proved G = <N> H, where <N> is the subgroup of G generated by N.

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1. INTRODUCTION

Finite group G is a transitive permutation group acting on a set Ω, where $|\Omega| > 1$. We say that an element g of G is a *derangement* if g acts fixed-point-freely on Ω. Let N be the subset of G consisting of all derangements together with the identity, so N is clearly a normal subset of G, but it need not be a subgroup in general. We refer to N as the derangement kernel of G. Observe that G is the union of the derangement kernel N together with all of point stabilizers, which are conjugate in G, hence $|N| > 1$. Recall that a transitive action of G on Ω is said to be a Frobenius action if every point stabilizer is nontrivial but the intersection of any two point stabilizers is trivial. A group G is called a Frobenius group when it has a Frobenius action on some set Ω whose cardinality is greater than one. A celebrated theorem of Frobenius asserts that if G is a Frobenius group, then its derangement kernel N is a proper subgroup of G ([5, Theorem 7.2]), and in that case N is called the Frobenius kernel. In [6], it is proved that if all elements in N are involutions, then N is an elementary abelian 2-group such that either G = N or G is a Frobenius group with kernel N. In this note, we show that if the derangement kernel N is a proper subgroup, then the action of G on Ω is of Frobenius. When N is a subgroup, it is easy to prove G = NH, where H is a point stabilizer in G. In fact, there are other conditions to guarantee G = NH. For example, we show that G = NH when G is 2-transitive on Ω (Proposition 2.4). Also we prove that it is always true that G = <N>H, where <N> is the subgroup of G generated by N (Theorem 2.3). We even guess that G = NH whenever G has a transitive action on Ω. However, we can neither prove the claim nor give a counterexample. Under only the hypothesis that G acts transitively and non-regularly on Ω, the subset N is not generally a subgroup of G. We prove that N is a group if and only if G is Frobenius group (Theorem2.1).

We mention that Frobenius groups play a prominent role in the theory of finite groups, they usually act as either a starting point or a reduced goal (by the minimal counterexample argument) when investigating some problems of group theory, for example, see[1, 2, 8].

Unless otherwise stated, the notation and terminology is standard, as presented in [5].

2. RESULTS

The following result indeed shows that the derangement kernel N is a group exactly when G is a Frobenius group or a regular group. If G is a regular group on Ω, it is clear that |G|= |Ω|= |N| so N is a group.

Theorem 2.1 Let G be a transitive and non-regular group acting on Ω with the derangement kernel N, then |Ω| ≤ |N|. Furthermore, the following statements are equivalent.

1. The action of G on Ω is Frobenius.
2. The set N is a subgroup of G.
3. The equality |Ω| = |N| holds.

Proof. Write H for $C_G(\alpha_1)$. Since G acts transitively on Ω, it follows that |Ω| = n = |G:H| and G= $N \bigcup C_G(\alpha_1) \bigcup \cdots \bigcup C_G(\alpha_n)$, then we may deduce that |G| ≤ |N| + n（|H|-1. Thus we get that |G| ≤ |N| + n|H| - n = |N| + |G:H||H| - n, hence |Ω| = n ≤ |N|, as desired.

Now assume part 1. Then the derangement kernel N is just the Frobenius kernel, and Frobenius' theorem([4, Satz V.7.6] or [5, Theorem 7.2]) yields that N is a group, part 2 follows.

Assuming part 2, we may deduce that |Ω| =|α_1^G| ≥ |α_1^N| =|N:$C_N(\alpha_1)$| =|N|. Also |Ω| ≤ |N|, we have |Ω| = |N|, yielding part 3.

Finally assume part 3. We have |Ω| = |G:H|, so |G| = |Ω||H|. We are assuming that |N| = |Ω| = n, and thus |G| = |N||H|. By the definition of N, we have G= $N \bigcup C_G(\alpha_1) \bigcup \cdots \bigcup C_G(\alpha_n)$, by writing $C_i = C_G(\alpha_i) - \{1\}$, i = 1,..., n, we further have G= $N \bigcup C_1 \bigcup \cdots \bigcup C_n$. Since |$C_i$| = |H| - 1 and |N| = n, we derive that |N||H| = |G| = |$N \bigcup C_G(\alpha_1) \bigcup \cdots \bigcup C_G(\alpha_n)$| ≤ |N| + |$C_1$| + ... + |$C_n$| = |N |+ n(|H| - 1) = |N| + |N|(|H| - 1) = |N||H|. Equality holds and so the unions are pairwise disjoint, which means $C_G(\alpha_i) \bigcap C_G(\alpha_j) = 1$ whenever i and j are different, and hence by definition, the action of G on Ω is Frobenius. The proof is finished.

Lemma 2.2. Let G act transitively on the set Ω where |Ω| > 1, H = $C_G(\alpha)$ for $\alpha \in \Omega$ and N a subset of G. Then G = HN if and only if $\alpha^N = \Omega$, where $\alpha^N = \{\alpha^n \mid n \in N\}$.

Proof. If G = HN, then Ω = $\alpha^G = \alpha^{HN} = \alpha^N$, as wanted. Conversely, for g $\in$ G, let $\alpha^g = \beta$ for $\beta \in \Omega$, then since $\alpha^N = \Omega$, there exists some n$\in$N such that $\alpha^n = \beta$, thus $\alpha^g = \alpha^n$, so $gn^{-1} \in$ H, hence g$\in$HN, and so G = HN, as desired.

Theorem 2.3. Let G be a transitive group acting on Ω with the derangement kernel N and H = $C_G(\alpha)$. Then the subgroup <N> is transitive on Ω and < N>H = G. Furthermore, if NH is subgroup, then NH = G.

Proof. Since N is a normal subset, it follows that <N> is a normal subgroup, and thus <N> H is a subgroup that contains N. Now $\bigcup_{g\in G}(<N>H)^g$ contains N and all conjugates of H, and since G is the union of N and the conjugates of H, it follows that $\bigcup_{g\in G}(<N>H)^g$ = G. But it is a fact that if the union of all conjugates of some subgroup of a group is the whole group, then the subgroup must be the whole group. We have G = <N>H = H<N>. By Lemma 2.2, all α_i are in the <N>-orbit containing α_1, and thus <N> acts transitively on Ω. Finally, suppose NH is a subgroup. Then NH contains both <N > and H, so it contains <N > H = G, and thus NH = G. The proof is complete.

Observe that G may be expressible as G = NH even though N is not a subgroup, as shown in the following consequence.

Proposition2.4. Let G act on the set $\Omega = \{\alpha_1, \alpha_2, \cdots, \partial_n\}$ with the derangement kernel N and H = $C_G(\alpha_1)$, n > 1. If the action is 2-transitive, then G = NH.

Proof. Pick g$\in$G - H and let $\alpha_1^g = \alpha_i$. Choose $1 \neq z \in N$ and let $\alpha_1^z = \alpha_j$. By the 2-transitivity, we know that H acts transitively on the difference set Ω - $\{\alpha_1\}$, and so there exists h $\in$ H such that $\alpha_j^h = \alpha_i$, then $zhg^{-1} \in H$ and so g$\in$HNH. Because HNH = NHH = NH, it follows g $\in$NH. We therefore conclude G = NH, as desired.

It is known that Symmetric group S_n and Alternating group A_n are 2-transitive when n $\geq$ 4. Thus they have the above product form.

For the alternating group A_5 of degree 5, we may get via GAP ([3]) that

N = {(), (1,5,4,3,2), (1,4,2,5,3), (1,3,5,2,4), (1,2,3,4,5), (1,4,5,3,2), (1,2,4,3,5), (1,5,3,2,4), (1,4,5,2,3), (1,5,4,2,3), (1,3,4,5,2), (1,5,3,4,2), (1,3,2,4,5), (1,3,2,5,4), (1,2,4,5,3), (1,5,2,3,4), (1,2,5,4,3), (1,4,3,2,5), (1,2,3,5,4), (1,4,3,5,2), (1,3,4,2,5), (1,5,2,4,3), (1,4,2,3,5), (1,3,5,4,2), (1,2,5,3,4)}, and the derangement kernel is N ={ (), (1,5,4,3,2), (1,4,2,5,3), (1,3,5,2,4),(1,2,3,4,5)}. It is actually a right transversal for A_4 in A_5, thus we achieve that $A_5 = A_4N = NA_4$ (as N is a normal subset). As (1,4,5,3,2)*(1,5,4,3,2) = (1,3)(2,5)$\notin$N, we see that N is not a group. (The nonabelian simple group A_5 has a proper normal subset N and a nontrivial factorization form $A_5 = NA_4$. This is really an interesting thing!) For A_6, we may also verify via GAP ([3]) that $A_6 = A_5N = NA_5$.

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