BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

A COMMON FIXED POINT THEOREM IN COMPLEX VALUED B-METRIC SPACES

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ABSTRACT

In this paper, we give a common fixed point theorem for four maps in complex valued b-metric spaces and obtain a generalization of theorem of R.K.Verma and H.K.Pathak [4].

Mathematics Subject Classification: 47H10, 54H25.

Keywords: Complex valued b-metric space, weakly compatible

Article Info:
Article received: 15/07/2013
Revised on:26/08/2013
Accepted on:30/08/2013

Introduction

The study of metric spaces expressed the most important role to many fields both in pure and applied science such as biology, medicine, physics, and computer science. Many authors generalized and extended the notion of a metric space such as a vector valued metric spaces, G-metric spaces, a cone metric space and a modular metric spaces and etc.

The concept of b-metric space was introduced by Czerwik[2]. Several papers deal with fixed point theory for single valued and multivalued operators in b-metric spaces.

Recently, Azam et al.[1] first introduced the complex valued metric spaces which is more general than well-known metric spaces and also gave common fixed point theorems for mappings satisfying generalized contraction condition. This new idea can be utilized to define complex valued normed spaces and complex valued inner product spaces. Several authors studied many common fixed point results on complex valued metric spaces (see[5-7]).

In this paper, we give a common fixed point theorem for four maps in complex valued b-metric spaces and obtain a generalization of theorem of R.K.Verma and H.K.Pathak [4].

2. Preliminaries

Let C be the set of complex numbers and $z_1, z_2 \in C,$ Define a partial order $\prec$ and $\precsim$ on C as follows:

(i) $z_1 \prec z_2$ if and only if $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(ii) $z_1 \precsim z_2$ if and only if $Re(z_1) \le Re(z_2)$ and $Im(z_1) \le Im(z_2)$.

It follows that $z_1 \preceq z_2$, if one of the following conditions is satisfied

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In (i), (ii) and (iii), we have $|z_1| < |z_2|$.In (iv), we have $|z_1| = |z_2|$. So $|z_1| \le |z_2|$.
In particular, $z_1 \precnsim z_2$ if $z_1 \ne z_2$ and one of (i), (ii) and (iii) is satisfied, In this case $|z_1| < |z_2|$.
We will write $z_1 \prec z_2$ if only (iii) satisfied.

It follows
(i) $0 \precsim z_1 \precnsim z_2$ implies $|z_1| < |z_2|$,
(ii) $z_1 \precsim z_2$ and $z_2 \prec z_3$ implies $z_1 \prec z_3$,
(iii) $0 \precsim z_1 \precsim z_2$ implies $|z_1| \le |z_2|$,
(iv) $a, b \in R, 0 \le a \le b$ *and* $z_1 \precsim z_2$ implies $az_1 \precsim bz_2$.

Now, we briefly review the notation about complex valued b-metric space.

Definition 2.1: Let X be a nonempty set and $s \ge 1$ a given real number. A function

d: XxX→C is called a complex valued b-metric (cvbm) if it satisfies the following

(cvbm-1) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(cvbm-2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(cvbm-3) $d(x,y) \precsim s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

The pair (X,d) is called a complex valued b-metric space.

Definition 2.2: Let (X,d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in C$ such that B(x,r) = $\{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called limit point of a set $A \subseteq X$ whenever for every $0 \prec r \in C$ such that B(x,r) $\cap (X - A) \neq \phi$.

(iii) A subset $B \subseteq X$ is called open whenever each element of B is an interior point of B.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.

(v) The family F = {B(x,r) : $x \in X, and\, 0 \prec r$} is a sub basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.3 : Let (X,d) be a complex valued b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in C$ with $0 \prec c$ there is $n_0 \in N$ such that for all n > n_0 , $d(x_n,x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is limit point of $\{x_n\}$. We denote this by $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

(ii) If for every $c \in C$ with $0 \prec c$ there is $n_0 \in N$ such that for all n > n_0, $d(x_n,x_{n+m}) \prec c$, where $m \in N$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex valued b-metric space.

One can prove the following lemmas in similar lines as in [1].

Lemma 2.4 : Let (X,d) be a complex valued b-metric space, and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0\, as\, n \to \infty$.

Lemma 2.5 : Let (X,d) be a complex valued b-metric space, and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0\, as\, n \to \infty, where\, m \in N$.

Definition 2.6([3]): Let S and T be two self-maps defined on a set X. The mappings S and T are weakly compatible if STx = TSx whenever Sx = Tx.

3. MAIN RESULTS

Theorem 3.1: Let (X,d) be a complex valued b-metric space and let S,T,A and B are four self maps on X such that

(i) S(X) $\subseteq$ B(X) and T(X) $\subseteq$ A(X)

(ii) $d(Sx,Ty) \precsim q \max\{d(Ax,By), d(Ax,Sx), d(By,Ty), d(Ax,Ty), d(By,Sx)\}$ for all $x, y \in X$.

Suppose that the pairs (S, A) and (T, B) are weakly compatible and T(X) is closed.

Assume that $0 \le q < \frac{1}{s^2 + s}$.

Then S, T, A and B have unique common fixed point in X.

Proof: Since $q < \frac{1}{s^2 + s}$, we have $0 \le q < 1$.

Suppose x_0 is an arbitrary point of X and define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1} \text{ and } y_{2n+1} = Tx_{2n+1} = Ax_{2n+2} \text{ for all } n = 0,1,2,3,\ldots.$$

$$\begin{aligned} d(y_{2n},y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\precsim q \max \{d(Ax_{2n},Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1},Tx_{2n+1}), d(Ax_{2n},Tx_{2n+1}), \\ &\qquad d(Bx_{2n+1},Sx_{2n})\} \\ &\precsim q \max \{d(y_{2n-1},y_{2n}), d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}), d(y_{2n-1},y_{2n+1}), d(y_{2n},y_{2n})\} \\ &\precsim q \max \{d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}), s[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})]\} \qquad (3.1) \end{aligned}$$

If $y_{2n-1} = y_{2n}$, for some n then from (3.1), $d(y_{2n},y_{2n+1}) \precsim qs\, d(y_{2n},y_{2n+1})$.

Hence $d(y_{2n},y_{2n+1}) = 0$ so that $y_{2n} = y_{2n+1}$.

Continuing in this way we can show that

$y_{2n-1} = y_{2n} = y_{2n+1} = \ldots\ldots\ldots$

Hence $\{y_n\}$ is a Cauchy sequence.

Now assume that $y_n \ne y_{n+1}$ for all n.

For simplicity, write $d_{2n} = d(y_{2n},y_{2n+1})$. From (3.1), we have

$$d_{2n} \precsim q \max \{ d_{2n-1}, d_{2n}, s(d_{2n-1} + d_{2n}) \} \qquad (3.2)$$

If $\max \{ d_{2n-1}, d_{2n}, s(d_{2n-1} + d_{2n}) \} = d_{2n}$ then from (3.2) $d_{2n} \precsim q\, d_{2n}$, which is a contradiction since $0 \le q < 1$.

Therefore, (3.2) becomes $d_{2n} \precsim q \max \{ d_{2n-1}, s(d_{2n-1} + d_{2n}) \}$.

$$\text{Thus } d_{2n} \precsim \gamma\, d_{2n-1}, \text{ for all } n \in N \qquad (3.3)$$

where $\gamma = \max\left\{q, \frac{sq}{1-sq}\right\}$.

$$\text{Similarly we can show that } d_{2n-1} \precsim \gamma\, d_{2n-2}, \text{ for all } n \in N. \qquad (3.4)$$

Thus from (3.3) and (3.4), we have $d_n \precsim \gamma\, d_{n-1}$ which in turn yields that

$$d_n \precsim \gamma^n d_0 \text{ for all } n=1,2,3,\ldots\ldots\ldots \qquad (3.5)$$

If $\gamma = q$ then $s\gamma < s\left[\frac{1}{(s^2 + s)}\right] = \frac{1}{1+s} \le \frac{1}{2} < 1$.

If $\gamma = \frac{sq}{1-sq}$ then $s\gamma = s\left[\frac{sq}{1-sq}\right] < s\left[\frac{\frac{1}{1+s}}{1-\frac{1}{1+s}}\right] = 1$.

$$\text{Thus } \gamma s < 1. \qquad (3.6)$$

Now for $m, n \in N$ with $n < m$, we have

$$\begin{aligned} d(y_n,y_m) &\precsim s\, d(y_n,y_{n+1}) + s^2 d(y_{n+1},y_{n+2}) + s^3 d(y_{n+2},y_{n+3}) + \text{----------} + s^{m-n} d(y_{m-1},y_m) \\ &\precsim \gamma^n s\, d(y_0, y_1) + \gamma^{n+1} s^2 d(y_0, y_1) + \text{----------} + \gamma^{m-1} s^{m-n} d(y_0, y_1), \text{ from (3.5)} \end{aligned}$$

$\precsim \gamma^n s\,(1 + \gamma s + \gamma^2 s^2 + \ldots\ldots\ldots\ldots)\, d(y_0, y_1)$

$|d(y_n, y_m)| \leq \frac{\gamma^n s}{1-\gamma s}|d(y_0, y_1)| \quad \rightarrow 0$ as n,m $\rightarrow \infty$.

Hence $\{y_n\}$ is a Cauchy sequence . Since X is complete, there exists $z \in X$ such that $y_n \rightarrow z$.
Since T(X) is closed, so $z \in T(X)$.

Since T(X) $\subseteq$ A(X) then there exists $u \in X$ such that z = Au.

Now we show that Su = Au = z.

$d(Su, z) \precsim s[d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, z)]$

$\frac{1}{s}d(Su, z) \precsim q \max\{d(Au, Bx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}), d(Au, Tx_{2n+1}), d(Bx_{2n+1}, Su)\} + d(Tx_{2n+1}, z)$

$\frac{1}{s}d(Su, z) \precsim q \max\{d(z, y_{2n}), d(z, Su), d(y_{2n}, y_{2n+1}), d(z, y_{2n+1}), d(y_{2n}, Su)\} + d(y_{2n+1}, z)$

$\frac{1}{s}|d(Su, z)| \leq q \max\{|d(z, y_{2n})|, |d(z, Su)|, |d(y_{2n}, y_{2n+1})|, |d(z, y_{2n+1})|, |d(y_{2n}, Su)|\} + |d(y_{2n+1}, z)|$

Taking $n \rightarrow \infty$ *we have*

$\frac{1}{s}|d(Su, z)| \leq q \max\{|d(z, Su)|, s|d(z, Su)|\}$

$\frac{1}{s}|d(Su, z)| \leq qs|d(z, Su)|$

$|d(Su, z)| \leq s^2 q\,|d(Su, z)|.$

Since $s^2 q < 1$, *we have* $|d(Su, z)| = 0$ *implies* $z = Su$. Therefore Su = Au = z.

Since z = Su $\in$ S(X) $\subseteq$ B(X) then there exists v in X such that z = Bv.

Now we show that Tv = Bv = z.

$d(Tv, z) \precsim s[d(Tv, Sx_{2n}) + d(Sx_{2n}, z)]$

$d(Tv, z) \precsim s[d(Sx_{2n}, Tv) + d(Sx_{2n}, z)]$

$\frac{1}{s}d(Tv, z) \precsim q \max\{d(Ax_{2n}, Bv), d(Ax_{2n}, Sx_{2n}), d(Bv, Tv), d(Ax_{2n}, Tv), d(Bv, Sx_{2n})\} + d(Sx_{2n}, z)$

$\frac{1}{s}d(Tv, z) \precsim q \max\{d(y_{2n-1}, z), d(y_{2n-1}, y_{2n}), d(z, Tv), d(y_{2n-1}, Tv), d(z, y_{2n})\} + d(y_{2n}, z)$

$\frac{1}{s}|d(Tv, z)| \leq q \max\{|d(y_{2n-1}, z)|, |d(y_{2n-1}, y_{2n})|, |d(z, Tv)|, |d(y_{2n-1}, Tv)|, |d(z, y_{2n})|\} + |d(y_{2n}, z)|$

Taking $n \rightarrow \infty$ *we have*

$\frac{1}{s}|d(Tv, z)| \leq q \max\{|d(z, Tv)|, s|d(z, Tv)|\}$

$\frac{1}{s}|d(Tv, z)| \leq q\,s|d(Tv, z)|$

$|d(Tv, z)| \leq s^2 q\,|d(Tv, z)|.$

Since $s^2 q < 1$, *we have* $|d(Tv, z)| = 0$ *implies* $z = Tv$. Therefore Tv = Bv = z.

Since S and A are weakly compatible and Au =Su= z, we have Sz = Az. Now

$d(Sz, z) \precsim s[d(Sz, Tx_{2n+1}) + d(Tx_{2n+1}, z)]$

$\frac{1}{s}d(Sz,z) \precsim$ q max{d(Az,Bx_{2n+1}), d(Az,Sz), d(Bx_{2n+1},Tx_{2n+1}), d(Az, Tx_{2n+1}),

d(Bx_{2n+1},Sz)}+d(Tx_{2n+1},z)

$\frac{1}{s}d(Sz,z) \precsim$ q max{d(Sz,y_{2n}), d(Sz,Sz), d(y_{2n},y_{2n+1}), d(Sz,y_{2n+1}),d(y_{2n},Sz)} + d(y_{2n+1},z)

$$\frac{1}{s}|d(Sz,z)| \le q\max\{|d(Sz,y_{2n})|,|d(Sz,Sz)|,|d(y_{2n},y_{2n+1})|,|d(Sz,y_{2n+1})|,|d(y_{2n},Sz)|\}+|d(y_{2n+1},z)|$$

Taking $n \to \infty$ *we have*

$$\frac{1}{s}|d(Sz,z)| \le q\max\{s|d(Sz,z)|,s|d(Sz,z)|,s|d(z,Sz)|\}$$

$$|d(Sz,z)| \le s^2 q\,|d(Sz,z)|.$$

Since $s^2q<1,$ *we have* $|d(Sz,z)|=0$ *implies* $z=Sz$. Therefore Sz = Az = z.

Since T and B are weakly compatible and Tv = Bv = z ,we have Tz = Bz.Now

$d(Tz,z) \precsim s[d(Tz,Sx_{2n}) + d(Sx_{2n},z)]$

$d(Tz,z) \precsim s[d(Sx_{2n},Tz) + d(Sx_{2n},z)]$

$\frac{1}{s}d(Tz,z) \precsim$ q max{d(Ax_{2n},Bz), d(Ax_{2n},Sx_{2n}), d(Bz,Tz), d(Ax_{2n}, Tz),

d(Bz,Sx_{2n})}+d(Sx_{2n},z)

$\frac{1}{s}d(Tz,z) \precsim$ q max {d(y_{2n-1},Tz), d(y_{2n-1}, y_{2n}), d(Tz,Tz), d(y_{2n-1},Tz),d(Tz, y_{2n})} + d(y_{2n},z)

$$\frac{1}{s}|d(Tz,z)| \le q\max\{|d(y_{2n-1},Tz)|,|d(y_{2n-1},y_{2n})|,|d(y_{2n-1},Tz)|,|d(Tz,y_{2n})|\}+|d(y_{2n},z)|$$

Taking $n \to \infty$ *we have*

$$\frac{1}{s}|d(Tz,z)| \le q\max\{s|d(Tz,z)|,s|d(Tz,z)|,s|d(Tz,z)|\}$$

$$|d(Tz,z)| \le s^2 q\,|d(Tz,z)|.$$

Since $s^2q<1,$ *we have* $|d(Tz,z)|=0$ *implies* $z=Tz$. Therefore Tz = Bz = z.

Therefore z is common fixed point of S,T,A and B.

Uniqueness

Let w be another fixed point of S, T, A and B.

Then Sw = Tw = Aw = Bw = w

d(z,w) = d(Sz,Tw) $\precsim$ q max{d(Az, Bw), d(Az, Sz), d(Bw, Tw), d(Az, Tw), d(Bw, Sz)}

= q d(z,w)

Thus we have $|d(z,w)| \precsim$ q $|d(z,w)|$,which in turn yields that z = w .

Hence z is the unique common fixed point of S,T,A and B.

Remark 3.2: By taking s = 1, A = B = Identity map in Theorem 3.1, we get Theorem 2.1 of R.K.Verma and H.K.Pathak [4].

Corollary 3.3: Let (X,d) be a complex valued metric space and let S,T,A and B are four self maps on X such that

(i)S(X) $\subseteq$ B(X) and T(X) $\subseteq$ A(X)

(ii)$d(Sx,Ty) \precsim q \max\{d(Ax,By), d(Ax,Sx), d(By,Ty), d(Ax,Ty), d(By,Sx)\}$ for all $x, y \in X$.
Suppose that the pairs (S, A) and (T, B) are weakly compatible and T(X) is closed.

Assume that $0 \le q < \frac{1}{2}$. Then S, T, A and B have unique common fixed point in X.

Now we give an example to illustrate Theorem 3.1

EXAMPLE 3.4: Let (X,d) be a Complex Valued b-metric space, where X = [0,1] and
d:XxX→C by d(x,y) = $|x-y|^2 + i|x-y|^2$.

$$d(x,y) = |x-y|^2 + i|x-y|^2 \le |(x-z)+(z-y)|^2 + i\,|(x-z)+(z-y)|^2$$
$$\le \left[|x-z|^2 + |z-y|^2 + 2|x-z||z-y|\right] + i\left[|x-z|^2 + |z-y|^2 + 2|x-z||z-y|\right]$$

$$\le \left[|x-z|^2 + |z-y|^2 + |x-z|^2 + |z-y|^2\right] + i\left[|x-z|^2 + |z-y|^2 + |x-z|^2 + |z-y|^2\right]$$
$$\le 2\left\{\left[|x-z|^2 + i|x-z|^2\right] + \left[|z-y|^2 + i|z-y|^2\right]\right\}$$
$$d(x,y) \le 2\left[d(x,z) + d(z,y)\right]$$

Here s = 2.

Define S, T, A and B: X→X by Sx = $\frac{x}{18}$, Tx = $\frac{x^2}{27}$, Ax = $\frac{x}{2}$ and Bx = $\frac{x^2}{3}$

d(Sx, Ty) = $\left[\left|\frac{x}{18} - \frac{y^2}{27}\right|^2 + i\left|\frac{x}{18} - \frac{y^2}{27}\right|^2\right] = \frac{1}{81}\left[\left|\frac{x}{2} - \frac{y^2}{3}\right|^2 + i\left|\frac{x}{2} - \frac{y^2}{3}\right|^2\right]$

d(Ax, By) = $= \left[\left|\frac{x}{2} - \frac{y^2}{3}\right|^2 + i\left|\frac{x}{2} - \frac{y^2}{3}\right|^2\right]$

d(Sx, Ty) = $\frac{1}{81}$ d(Ax, By)

Here q = $\frac{1}{81} < \frac{1}{s^2+s} = \frac{1}{6}$

All conditions of the Theorem 3.1 are satisfied. Clearly '0' is the unique common fixed point of S, T, A and B .

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