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ON THE HOMOGENEOUS QUADRATIC EQUATION WITH THREE UNKNOWNS $x^2 - xy + y^2 = (k^2 + 3)z^2$

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ABSTRACT

The ternary quadratic Equation with 3 unknown given by $x^2 - xy + y^2 = (k^2 + 3)z^2$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEY WORDS: Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

INTRODUCTION

Ternary quadratic equations are rich in variety [1-4].For an extensive review of sizable literature and various problems, one may refer [5-13]In this communication, we consider yet another interesting ternary quadratic equation $x^2 - xy + y^2 = (k^2 + 3)z^2$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

METHOD OF ANALYSIS

The Homogeneous Quadratic Diophantine equation with three unknowns to be solved for its non zero distinct integral solutions is

$$x^2 - xy + y^2 = (k^2 + 3)z^2 \qquad (1)$$

Different patterns of solutions of (1) are presented below.

Pattern -1

Treating (1) as quadratic in x, solving for x, we have

$$x = \frac{1}{2}[y \pm \sqrt{y^2 - 4(y^2 - (k^2 + 3)z^2}] \qquad (2)$$

Replace y by 2Y,we get

$$x = Y \pm \sqrt{(k^2 + 3)z^2 - 3Y^2} \qquad (3)$$

Let $\alpha^2 = (k^2 + 3)z^2 - 3Y^2 \qquad (4)$

Substitution of linear transformation

$$z = X \pm 3T\,, \quad Y = X \pm (k^2+3)T \tag{5}$$

In (4), leads to $\alpha^2 = k^2X^2 - 3k^2(k^2+3)T^2$

Taking $\alpha = k\beta$ we get

$$X^2 = \beta^2 + 3(k^2+3)T^2 \tag{6}$$

In particular,when $3(k^2+3) \neq$ a perfect square.

(6) is of the form $z^2 = Dx^2 + y^2$

Hence the solution of (6) is

$$T = 2rs$$
$$\beta = 3(k^2+3)r^2 - s^2$$
$$X = 3(k^2+3)r^2 + s^2$$

Hence the corresponding nonzero distinct integral solutions of (1) are given by

$$x = 3(k^2+3)r^2 + s^2 + 2(k^2+3)rs + 3k(k^2+3)r^2 - ks^2$$
$$y = 2[3(k^2+3)r^2 + s^2 + 2(k^2+3)rs]$$
$$z = 3(k^2+3)r^2 + s^2 + 6rs$$

Pattern-2:

If $3(k^2+3) = \gamma^2$

(6) Leads to

$$X^2 = \beta^2 + \gamma^2T^2$$

Which is satisfied by

$$X = r^2 + s^2$$
$$\beta = r^2 - s^2 \qquad r > s > 0$$
$$T = \frac{2rs}{\gamma_n}$$

Our interest is on finding integer solutions, so take $r = \gamma_n R$

Hence the corresponding nonzero distinct integral solutions of (1) are given by

$$x = (\gamma_n R)^2 + s^2 + 2(k_n^2+3)Rs + k_n(\gamma_n R)^2 - k_n s^2$$
$$y = 2[(\gamma_n R)^2 + s^2 + 2(k_n^2+3)Rs]$$
$$z = (\gamma_n R)^2 + s^2 + 6Rs$$

Also the solution of (6) is

$$X = r^2 + s^2$$
$$\beta = 2rs \qquad r > s > 0$$
$$T = \frac{r^2 - s^2}{\gamma_n}$$

Let us assume that $r = \gamma_n R$, $s = \gamma_n S$

Thus, the corresponding nonzero distinct integral solutions of (1) are

$$x=(\gamma_n R)^2+(\gamma_n s)^2+2(k_n^2+3)\gamma_n(R^2-S^2)+2k_n(\gamma_n)^2 RS$$
$$y=2[(\gamma_n R)^2+(\gamma_n s)^2+(k_n^2+3)(R^2-S^2)]$$
$$z=(\gamma_n R)^2+(\gamma_n s)^2+3\gamma_n(R^2-S^2)$$

where

$$\gamma_n=\frac{3}{2}[(2+\sqrt{3})^{n+1}+(2-\sqrt{3})^{n+1}]$$

$$k_n=\frac{3}{2\sqrt{3}}[(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}]$$

REMARKABLE OBSERVATIONS:

I: If the non-zero integer triple (x_0,y_0,z_0) is any solution of (1) then the triple (x_n,y_n,z_n)

Where $\begin{pmatrix}x_{2n-1}\\ y_{2n-1}\end{pmatrix}=9^{n-1}M\begin{pmatrix}x_0\\ y_0\end{pmatrix}$, $M=\begin{pmatrix}3 & -3\\ 0 & 3\end{pmatrix}$

$$z_n=3^n z_0$$

and

$$\begin{pmatrix}x_{2n}\\ y_{2n}\end{pmatrix}=9^{n-1}I\begin{pmatrix}x_0\\ y_0\end{pmatrix},\qquad I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$z_n=3^n z_0$ also satisfies (1).

II: Let x, y be any two non zero distinct positive integer solutions of (1) that x>y. Let (α,β,γ) be the Pythagorean triangle with x and y as its generators such that $\alpha=2xy,\beta=x^2-y^2,\gamma=x^2+y^2$ .Hence the relation betwee the sides of the phthagorean triangle are exhibited below.

1. $\gamma+\beta-\frac{4A}{P}\equiv 0(\mathrm{mod}(k^2+3))$
2. $3(k^2+3)(2\gamma-\alpha)$ is a nasty number.
3. $(k^2+3)(\gamma-2A)$ is a perfect square.

Where A and P are the Area and perimeter of the Pythagorean triangle respectively.

III: Consider x and y to be the length and breath of a Rectangle R, where A= Area, P= Perimeter, L= Length of the diagonal

Then it is noted that

1. $L^2-A\equiv 0\,\mathrm{mod}(k^2+3)$
2. $(k^2+3)(P^2-12A)$ is a perfect square.
3. $6(k^2+3)(L^2-A)$ is a Nasty number.

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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